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WHEN DOES A CLOSED IDEAL OF A COMMUTATIVE UNITAL BANACH ALGEBRA CONTAIN A DENSE SUBIDEAL?

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Dedicated to the memory of Susanne Dierolf

Abstract: The question formulated in the title is answered in the case of a separable algebra. The necessary and sufficient condition in this case is that the ideal in question is not finitely (algebraically) generated. We conjecture that this result is true in the general case.

Keywords: Banach algebra, dense subideal.

Let A be a commutative complex unital Banach algebra and I its proper closed ideal. We say that a subset $S \subset A$ generates I algebraically if I is the smallest ideal containing the set S. In this case we write $I = [S]_a$. Similarly, we say that S generates I topologically, if it is the smallest closed ideal containing the set S. In this case we write $I = [S]_t$. Clearly, every set generating I algebraically generates it also topologically.

An easy, but not very satisfactory, answer to the question asked in the title is given in the following proposition (the term "subideal" of I means any ideal of A contained in I).

Proposition.[1] *An ideal $I \subset A$ has no proper dense subideal if and only if every set generating I topologically generates it also algebraically.*

Proof. Let $S \subset I$ and $I = [S]_t$. If I contains no proper dense subideal, we have also $I = [S]_a$ since $[S]_a$ is dense in I. Conversely, if I is the closure of a non-closed ideal $J \subset I$, then setting $S = J$ we have $[S]_a = J$ and $[S]_t = I$. Thus I is not algebraically generated by S and the conclusion follows. ■

Our conjecture, which will be proved only in the separable case is formulated as follows. Here we say that A is finitely generated (algebraically or topologically) if it is generated by a finite set S.

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[1] This Proposition extends to arbitrary semitopological algebras, i.e. topological vector spaces equipped with a separately continuous (associative) multiplication. Its proof will appear elsewhere (added in proof).

Conjecture. *Let I be a closed ideal in a commutative unital complex Banach algebra. Then I has a proper dense subideal if and only if it is not finitely algebraically generated.*

This conjecture is suggested by a result of Grauert and Remmert, which says that a commutative Banach algebra is Noetherian if and only if it is finite dimensional (see Appendix to §5 in [4], for various generalizations of this result see [1]–[3], [5]-[7]). Their basic lemma ([4] Bemerkung 2, p. 54) says that if A is as above, and the closure $\overline{J}$ of an ideal J in A is finitely generated, then $J = \overline{J}$. Thus a finitely generated closed ideal has no proper dense subideal. Our conjecture says that the converse result also holds true. We prove it under an additional assumption that A is separable.

Theorem. *Let I be a closed ideal in a commutative complex unital separable Banach algebra A. Then I has no proper dense subideal if and only if it is algebraically finitely generated.*

Proof. In view of the, above formulated, lemma of Grauert and Remmert, it is sufficient to show that if I is not finitely algebraically generated, then it contains a dense proper subideal. Since A is separable, the ideal I is also a separable space. Let $S = \{x_1, x_2, \dots\}$ be a dense sequence of elements of I. Consider the ideals

$$I_n = Ax_1 + \cdots + Ax_n,$$

and put $J_n = \overline{I}_n$. Clearly, J_n is a closed subideal of I for all natural n. If $I = J_n$ for some n, then I_n is dense in I and $I_n \neq I$, since I is not finitely algebraically generated. Thus, in this case, the conclusion follows. Otherwise, we have

$$J_1 \subset J_2 \subset \dots,$$

and $J_n \neq I$ for all n. Put

$$J = \bigcup_n J_n,$$

it is a dense subideal of I since it contains all elements x_i of a dense subset of I. Moreover, J is non-closed as a countable union of proper closed subspaces J_n of I. The conclusion follows. ■

It is easy to see examples of ideals having or not having proper dense subideals. For instance, if $A = \mathcal{A}(\Delta)$ is the disc algebra, i.e. the uniform algebra of all continuous functions in the closed unit disc Δ, holomorphic in its interior, then each maximal ideal corresponding to a point ξ of the interior of Δ is algebraically generated by the function $f(\zeta) = \zeta - \xi$, so it has no proper dense subideal. On the other hand, every maximal ideal corresponding to ξ with $|\xi| = 1$ cannot be finitely algebraically generated. In fact, it is topologically generated by the function $f(\zeta) = \zeta - \xi$, but it is not algebraically generated by this function, since the function $g(\zeta) = [f(\zeta)]^{1/2}$ is in A but cannot be written as $g = fh$ with h in A. Our Proposition shows that this ideal contains proper dense subideal and so cannot be finitely generated.

References

[1] R. Choukri and A. El Kinani, *Topological algebras with ascending and descending chain conditions*, Acta Math. (Basel) **72** (1999), 438–443.

[2] R. Choukri, A. El Kinani and M. Oudadess, *Algèbres topologiques à idéaux à gauche fermés*, Studia Math. **168** (2005), 159–164.

[3] A.V. Ferreira and G. Tomassini, *Finitness properties of topological algebras*, Ann. Scuola Norm. Sup. Pisa **5** (1978), 471–488.

[4] H. Grauert and R. Remmert, *Analytische Stellenalgebren*, Springer, 1971.

[5] A.M. Sinclair and A.W. Tullo, *Noetherian Banach algebras are finite dimensional*, Math. Ann. **211** (1974), 151–153.

[6] W. Żelazko, *A characterization of commutative Fréchet algebras with all ideals closed*, Studia Math. **138** (2000), 293–300.

[7] W. Żelazko, *A characterization of F-algebras with all one-sided ideals closed*, Studia Math. **168** (2005), 135–145.

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