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HYPOTHESIS H AND THE PRIME NUMBER THEOREM FOR AUTOMORPHIC REPRESENTATIONS

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Dedicated to Jean-Marc Deshouillers
on the occasion of his sixtieth birthday

Abstract: For any unitary cuspidal representations π_n of $GL_n(\mathbb{Q}_{\mathbb{A}})$, $n = 2, 3, 4$, respectively, consider two automorphic representations Π and Π' of $GL_6(\mathbb{Q}_{\mathbb{A}})$, where $\Pi_p \cong \wedge^2\pi_{4,p}$ for $p \neq 2, 3$ and $\pi_{4,p}$ not supercuspidal ($\pi_{4,p}$ denotes the local component of π_4), and $\Pi' = \pi_2 \boxtimes \pi_3$. First, Hypothesis H for Π and Π' is proved. Then contributions from prime powers are removed from the prime number theorem for cuspidal representations π and π' of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. The resulting prime number theorem is unconditional when $m, m' \leqslant 4$ and is under Hypothesis H otherwise.
Keywords: Hypothesis H, functoriality, prime number theorem.

1. Introduction

Recent developments in functoriality by the Langlands-Shahidi method have many profound applications in prime distribution. To name a few, we recall a recent proof of Hypothesis H for any cuspidal representation of $GL_4(\mathbb{Q}_{\mathbb{A}})$ and for $\mathrm{Sym}^4(\pi)$ by Kim [2], where π is an automorphic cuspidal representation of $GL_2(\mathbb{Q}_{\mathbb{A}})$. Here Hypothesis H predicts the convergence of a certain Dirichlet series associated with $(L'/L)'(s, \pi \times \widetilde{\pi})$ taken over prime powers.

More precisely, let $\pi = \otimes_p \pi_p$ be a unitary automorphic cuspidal representation of $GL_m(\mathbb{Q}_{\mathbb{A}})$. Or more generally, let π be an automorphic representation irreducibly induced from unitary cuspidal representations, i.e., $\pi = \mathrm{Ind}\ \sigma_1 \otimes \cdots \otimes \sigma_k$, where σ_j is a cuspidal representation of $GL_{m_j}(\mathbb{Q}_{\mathbb{A}})$, with $m_1 + \cdots + m_k = m$. The local component π_p with $p < \infty$ can be parameterized by the Satake parameters $\mathrm{diag}[\alpha_\pi(p,1), \ldots, \alpha_\pi(p,m)]$. For $\nu \geqslant 1$ define

$$a_\pi(p^\nu) = \sum_{j=1}^{m} \alpha_\pi(p,j)^\nu. \tag{1.1}$$

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Let $\widetilde{\pi}$ be the contragredient representation of π, and $L(s,\pi\times\widetilde{\pi})$ the Rankin-Selberg L-function. Then for $\Re e\, s>1$, we have (see [10], **RS 1**)

$$\Big(\frac{L'}{L}\Big)'(s,\pi\times\widetilde{\pi})=\sum_{n=1}^{\infty}\frac{(\log n)\Lambda(n)|a_\pi(n)|^2}{n^s}. \tag{1.2}$$

Here $\Lambda(n)=\log p$ if $n=p^\nu$ and $\Lambda(n)=0$ otherwise, so that the series in (1.2) is taken over primes and prime powers.

Hypothesis H. (Rudnick and Sarnak [10]) *For any fixed* $\nu\geqslant 2$,

$$\sum_p\frac{(\log p)^2|a_\pi(p^\nu)|^2}{p^\nu}<\infty.$$

Hypothesis H is trivial for $m=1$. For $m=2,3$, Hypothesis H follows from the Rankin-Selberg theory 10]. The GL_4 case was proved by Kim [2] based on his proof of the (weak) functoriality of the exterior square $\wedge^2\pi$ from a cuspidal representation π of $GL_4(\mathbb{Q}_\mathbb{A})$ (see [1]). Beyond GL_4, the only known special case for Hypothesis H is the symmetric fourth power $\mathrm{Sym}^4(\pi)$ of a cuspidal representation π of $GL_2(\mathbb{Q}_\mathbb{A})$, which is an automorphic representation of $GL_5(\mathbb{Q}_\mathbb{A})$.

The first goal of the present paper is to prove Hypothesis H for two types of automorphic representations of $GL_6(\mathbb{Q}_\mathbb{A})$.

Theorem 1. *Let π be a cuspidal representation of $GL_4(\mathbb{Q}_\mathbb{A})$. Denote by T the set of places consisting of $p=2,3$ and those p at which π_p is supercuspidal. Let Π be the automorphic representation of $GL_6(\mathbb{Q}_\mathbb{A})$ such that $\Pi_p\cong\wedge^2\pi_p$ if $p\notin T$, according to [1]. Then Hypothesis H holds for Π.*

Theorem 2. *Let π_1 (resp. π_2) be a cuspidal representation of $GL_2(\mathbb{Q}_\mathbb{A})$ (resp. $GL_3(\mathbb{Q}_\mathbb{A})$). Let Π' be the automorphic representation of $GL_6(\mathbb{Q}_\mathbb{A})$ equal to $\pi_1\boxtimes\pi_2$ according to [3]. Then Hypothesis H holds for Π'.*

As an application, one can use Hypothesis H to deduce the following Mertens' theorem for automorphic representations, or the so-called Selberg orthogonality conjecture, from unconditional results on similar sums taken over primes and prime powers:

$$\sum_{p\leqslant x}\frac{|a_\pi(p)|^2}{p}=\log\log x+O(1); \tag{1.3}$$

$$\sum_{p\leqslant x}\frac{a_\pi(p)\overline{a}_{\pi'}(p)}{p}=O(1), \tag{1.4}$$

when $\pi\not\cong\pi'$. Here (1.3) was proved by Rudnick and Sarnak [10], while (1.4) was proved by Liu, Wang and Ye ([6], [4]). Results in (1.3) and (1.4) played crucial roles in the n-level correlation of nontrivial zeros of automorphic L-functions and random matrix theory ([10], [5], [7]).

Another application of Hypothesis H is on the prime number theorem for automorphic representations. For any self-dual cuspidal representation π of $GL_m(\mathbb{Q}_\mathbb{A})$, Liu, Wang and Ye [4] showed that there is a constant $c > 0$ such that

$$\sum_{n\leqslant x} \Lambda(n)|a_\pi(n)|^2 = x + O\big(xe^{-c\sqrt{\log x}}\big). \tag{1.5}$$

More generally, Liu and Ye [8] proved that

$$\sum_{n\leqslant x} \Lambda(n)a_\pi(n)\overline{a}_{\pi'}(n) = \begin{cases} \dfrac{x^{1+i\tau_0}}{1+i\tau_0} + O\big(xe^{-c\sqrt{\log x}}\big) & \text{if } \quad \pi' \cong \pi\otimes|\det|^{i\tau_0} \quad \text{for some } \tau_0\in\mathbb{R}; \\ O\big(xe^{-c\sqrt{\log x}}\big) & \text{if } \quad \pi' \ncong \pi\otimes|\det|^{i\tau} \quad \text{for any } \tau\in\mathbb{R}, \end{cases} \tag{1.6}$$

where π and π' are cuspidal representations of $GL_m(\mathbb{Q}_\mathbb{A})$ and $GL_{m'}(\mathbb{Q}_\mathbb{A})$, respectively, such that at least one of them is self-dual.

The second goal of the present paper is to use Hypothesis H to remove terms on prime powers from the left side of (1.6) and deduce a prime number theorem over primes.

Theorem 3. *Let π and π' be as above.* (i) *If $m, m' \leqslant 4$, then*

$$\sum_{p\leqslant x} (\log p)a_\pi(p)\overline{a}_{\pi'}(p) = \begin{cases} \dfrac{x^{1+i\tau_0}}{1+i\tau_0} + O\big(xe^{-c\sqrt{\log x}}\big) & \textit{if} \quad \pi' \cong \pi\otimes|\det|^{i\tau_0} \quad \textit{for some } \tau_0\in\mathbb{R}, \\ O\big(xe^{-c\sqrt{\log x}}\big) & \textit{if} \quad \pi' \ncong \pi\otimes|\det|^{i\tau} \quad \textit{for any } \tau\in\mathbb{R}. \end{cases} \tag{1.7}$$

(ii) *If $\max(m, m') \geqslant 5$, asymptotic relation (1.7) is true under Hypothesis H with error terms replaced by $O(x/\log x)$.*

We remark that (i) is an unconditional result.

2. Proof of Theorems 1 and 2

Lemma 2.1. *Let π be a unitary cuspidal representation for $GL_m(\mathbb{Q}_\mathbb{A})$, or an automorphic representation irreducibly induced from unitary cuspidal representations. Then for any $\nu_0 \geqslant (m^2+1)/2+1$, $\varepsilon > 0$, and integer $\ell \geqslant 0$,*

$$\sum_{\nu\geqslant\nu_0,\ p^\nu\leqslant x} (\log p)|a_\pi(p^\nu)|^2 \ll x^{1-2/(m^2+1)+1/\nu_0}\log x, \tag{2.1}$$

$$\sum_p \frac{(\log p)^\ell |a_\pi(p)|^2}{p^{1+\varepsilon}} < \infty. \tag{2.2}$$

Proof. From (1.1) and the bound toward the Ramanujan conjecture ([10])

$$|\alpha_\pi(p,j)| \leqslant p^{1/2-1/(m^2+1)} \qquad (j=1,\dots,m), \tag{2.3}$$

we know that

$$|a_\pi(p^\nu)|^2 \leqslant m^2 p^{\{1-2/(m^2+1)\}\nu}.$$

Then

$$\begin{aligned}\sum_{\nu\geqslant\nu_0,\ p^\nu\leqslant x}(\log p)|a_\pi(p^\nu)|^2 &\leqslant m^2 \sum_{\nu_0\leqslant\nu\leqslant 2\log x}\ \sum_{p\leqslant x^{1/\nu}}(\log p)p^{\{1-2/(m^2+1)\}\nu}\\ &\ll_m x^{1-2/(m^2+1)+1/\nu_0}\log x.\end{aligned}$$

Inequality (2.2) follows from the fact that the ℓth-derivation of $\log L(s,\pi\times\widetilde{\pi})$ converges absolutely for $\Re e\, s>1$. ∎

Lemma 2.2. *Let π' (resp. π'') be a unitary cuspidal representation, or an automorphic representation irreducibly induced from unitary cuspidal representations, for $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$ (resp. $GL_{m''}(\mathbb{Q}_{\mathbb{A}})$). Let $\nu\geqslant 2$ be an integer and $\mathcal{P}$ a set of prime numbers. If there are fixed constants $\delta'\in(0,1]$ and $\delta''\in(0,\frac12]$ such that*

$$|a_{\pi'}(p^\nu)|^2 \ll_\nu |a_{\pi''}(p)|^2 p^{(1-\delta')(\nu-1)} + p^{(1/2-\delta'')\nu} \tag{2.4}$$

for all $p\in\mathcal{P}$, then for any $\varepsilon>0$ we have

$$\sum_{p^\nu\leqslant x,\, p\in\mathcal{P}}(\log p)|a_{\pi'}(p^\nu)|^2 \ll_{\nu,\varepsilon} x^{1-\delta} \tag{2.5}$$

with $\delta:=\min\{\delta'/(2+\delta')-\varepsilon,\ \delta''\}$.

Proof. By (2.4) and the Rankin-Selberg theory, for any $\eta>0$ we can write

$$\begin{aligned}\sum_{\substack{p^\nu\leqslant x\\ p\in\mathcal{P}}}(\log p)|a_{\pi'}(p^\nu)|^2 &\ll_\nu \sum_{\substack{p^\nu\leqslant x\\ p\in\mathcal{P}}}(\log p)|a_{\pi''}(p)|^2 p^{(1-\delta')(\nu-1)} + x^{1/2+1/\nu-\delta''}\\ &\ll_\nu x^\eta \sum_{\substack{p^\nu\leqslant x^\eta\\ p\in\mathcal{P}}}\frac{(\log p)|a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} + x\sum_{\substack{x^\eta<p^\nu\leqslant x\\ p\in\mathcal{P}}}\frac{(\log p)|a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} + x^{1-\delta''}.\end{aligned}$$

By (2.2) with $\pi=\pi''$ and $\ell=1$, it follows that

$$\sum_{\substack{p^\nu\leqslant x^\eta\\ p\in\mathcal{P}}}\frac{(\log p)|a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} \ll 1$$

and

$$\sum_{\substack{x^\eta < p^\nu \leqslant x \\ p \in \mathcal{P}}} \frac{(\log p)|a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} \leqslant \frac{1}{(x^{\eta/\nu})^{\delta'(\nu-1)-\varepsilon}} \sum_{\substack{x^\eta < p^\nu \leqslant x \\ p \in \mathcal{P}}} \frac{(\log p)|a_{\pi''}(p)|^2}{p^{1+\varepsilon}}$$
$$\leqslant x^{-\eta[\delta'(\nu-1)-\varepsilon]/\nu}.$$

Inserting these two estimates into the preceeding inequality, we find

$$\sum_{\substack{p^\nu \leqslant x \\ p \in \mathcal{P}}} (\log p)|a_{\pi'}(p^\nu)|^2 \ll_{\nu,\varepsilon} x^\eta + x^{1-\eta[\delta'(\nu-1)-\varepsilon]/\nu} + x^{1-\delta''}.$$

Taking $\eta = \nu/\{(1+\delta')\nu - \delta'\} + \varepsilon$, we obtain

$$\begin{aligned}\sum_{p^\nu \leqslant x,\, p \in \mathcal{P}} (\log p)|a_{\pi'}(p^\nu)|^2 &\ll_{\nu,\varepsilon} x^{\nu/\{(1+\delta')\nu-\delta'\}+\varepsilon} + x^{1-\delta''} \\ &\ll_{\nu,\varepsilon} x^{1-\delta'/(2+\delta')+\varepsilon} + x^{1-\delta''} \\ &\ll_{\nu,\varepsilon} x^{1-\delta}.\end{aligned}$$

In the second inequality, we have used the fact that $\nu \geqslant 2$. ■

Remark. In proving Hypothesis H, an inequality of the form of (2.4) plays a crucial role. Lemma 2.2 has more flexibility as π'' is allowed to be different from π'.

Lemma 2.3. *Let Π'' be either Π or Π' as in Theorems 1 and 2. Then for any $\varepsilon > 0$, we have*

$$\sum_{\nu \geqslant 2,\ p^\nu \leqslant x} (\log p)|a_{\Pi''}(p^\nu)|^2 \ll_\varepsilon x^{1-1/38+\varepsilon}. \tag{2.6}$$

Proof. In view of (2.1) with the choice of $m = 6$ and $\nu_0 = [37 \times 38/39] + 1$, it suffices to show that for any fixed $\varepsilon > 0$ and $\nu \geqslant 2$ we have

$$\sum_{p^\nu \leqslant x} (\log p)|a_\Pi(p^\nu)|^2 \ll_{\nu,\varepsilon} x^{1-1/38+\varepsilon}, \tag{2.7}$$

$$\sum_{p^\nu \leqslant x} (\log p)|a_{\Pi'}(p^\nu)|^2 \ll_{\nu,\varepsilon} x^{1-1/38+\varepsilon}. \tag{2.8}$$

First let us consider the case of Π. Let $\pi = \otimes \pi_p$ be a cuspidal automorphic representation for $GL_4(\mathbb{A}_\mathbb{Q})$. Recall that Π is irreducibly induced from unitary cuspidal representations. Let S_0 be the set of places where Π_p is tempered. Then

$$\sum_{p \in S_0} (\log p)^2 |a_\Pi(p^\nu)|^2 < \infty. \tag{2.9}$$

Inequality (2.9) is also true if we replace S_0 by T, which is given in Theorem 1, because at most two terms for $p = 2, 3$ will then be added to (2.9).

If $p \notin S_0 \cup T$, we want to determine the Satake parameters of π_p. Recall that the general non-tempered representation π_p can be described as a Langlands quotient based on a standard parabolic subgroup P of type $(m_1, \ldots, m_r) = (4)$, $(3,1)$, $(2,2)$, or $(2,1,1)$:

$$\pi_p = J(G, P; \sigma_1[t_1], \ldots, \sigma_r[t_r]).$$

Here σ_j is a tempered representation of $GL(m_j)$, $t_j \in \mathbb{C}$, and $\sigma_j[t_j] = \sigma_j \otimes |\det|^{t_j}$, with $\{\sigma_j[t_j]\} = \{\tilde{\sigma}_k[-t_k]\}$. Consequently, the Satake parameters of π_p are in one of the following forms in view of (2.3):

$$\begin{aligned}
&S_1 : \mathrm{diag}\big[u_1 p^{a},\, u_2 p^{a},\, u_1 p^{-a},\, u_2 p^{-a}\big], && \text{where} \quad 0 < a \leqslant \tfrac{1}{2} - \tfrac{1}{17},\\
&S_2 : \mathrm{diag}\big[u_1 p^{a},\, u_2,\, u_3,\, u_1 p^{-a}\big], && \text{where} \quad 0 < a \leqslant \tfrac{1}{2} - \tfrac{1}{17},\\
&S_3 : \mathrm{diag}\big[u_1 p^{a_1},\, u_2 p^{a_2},\, u_1 p^{-a_1},\, u_2 p^{-a_2}\big], && \text{where} \quad 0 < a_2 < a_1 \leqslant \tfrac{1}{2} - \tfrac{1}{17},
\end{aligned} \tag{2.10}$$

where u_1, u_2, u_3 are complex numbers of absolute value 1 and we have suppressed their dependence on p for the simplicity of notation. As in [1], the corresponding Satake parameters of $\Pi_p \simeq \wedge^2 \pi_p$ are as follows:

$$\begin{aligned}
&S_1 : \mathrm{diag}\big[u_1u_2 p^{2a},\, u_1u_2, u_1^2,\, u_2^2,\, u_1u_2,\, u_1u_2 p^{-2a}\big],\\
&S_2 : \mathrm{diag}\big[u_1u_2 p^{a},\, u_1u_3 p^{a},\, u_1^2,\, u_2u_3,\, u_1u_2 p^{-a},\, u_1u_3 p^{-a}\big],\\
&S_3 : \mathrm{diag}\big[u_1u_2 p^{a_1+a_2},\, u_1u_2 p^{a_1-a_2},\, u_1^2,\, u_2^2,\, u_1u_2 p^{-(a_1-a_2)},\, u_1u_2 p^{-(a_1+a_2)}\big].
\end{aligned}$$

Since Π is a automorphic representation for $GL_6(\mathbb{A}_{\mathbb{Q}})$ which is irreducibly induced from unitary cuspidal, (2.3) gives

$$\begin{cases}
0 < 2a \leqslant \frac{1}{2} - \frac{1}{37} & \text{if} \quad p \in S_1,\\
0 < a \leqslant \frac{1}{2} - \frac{1}{17} & \text{if} \quad p \in S_2,\\
0 < a_2 < a_1 \leqslant \frac{1}{2} - \frac{1}{17} \quad \text{and} \quad a_1 + a_2 \leqslant \frac{1}{2} - \frac{1}{37} & \text{if} \quad p \in S_3.
\end{cases} \tag{2.11}$$

If $p \in S_1$, then

$$\begin{aligned}
\big|a_\Pi(p^\nu)\big| &= \big|(u_1u_2)^\nu (p^{2a\nu} + p^{-2a\nu} + 2) + u_1^{2\nu} + u_2^{2\nu}\big| \leqslant p^{2a\nu} + 5,\\
\big|a_\Pi(p)\big| &= \big|u_1u_2(p^{2a} + p^{-2a} + 2) + u_1^2 + u_2^2\big| \geqslant p^{2a}.
\end{aligned}$$

From these and (2.3) with $m = 6$, we deduce that

$$\begin{aligned}
|a_\Pi(p^\nu)|^2 &\leqslant (|a_\Pi(p)|^\nu + 5)^2\\
&\ll_\nu |a_\Pi(p)|^{2\nu} + 1\\
&\ll_\nu |a_\Pi(p)|^2 p^{(1-2/37)(\nu-1)} + 1,
\end{aligned}$$

where the implied constants are all independent of p.

Similarly if $p \in S_2$, then

$$\begin{aligned}\left|a_\Pi(p^\nu)\right| &= \left|u_1^\nu(u_2^\nu + u_3^\nu)(p^{a\nu} + p^{-a\nu}) + u_1^{2\nu} + (u_2u_3)^\nu\right| \leqslant 2p^{a\nu} + 4,\\ \left|a_\pi(p)\right| &= \left|u_1(p^a + p^{-a}) + u_2 + u_3\right| \geqslant p^a - 2.\end{aligned}$$

These and (2.3) with $m = 4$ imply

$$\begin{aligned}|a_\Pi(p^\nu)|^2 &\leqslant \{2(|a_\pi(p)| + 2)^\nu + 4\}^2\\ &\ll_\nu |a_\pi(p)|^{2\nu} + 1\\ &\ll_\nu |a_\pi(p)|^2 p^{(1-2/17)(\nu-1)} + 1.\end{aligned} \tag{2.12}$$

Finally if $p \in S_3$, then

$$\left|a_\Pi(p^\nu)\right| \leqslant 2p^{(a_1+a_2)\nu} + 4, \qquad \left|a_\Pi(p)\right| \geqslant p^{a_1+a_2} - 1,$$

from which we deduce, as before,

$$\begin{aligned}|a_\Pi(p^\nu)|^2 &\leqslant \{2(|a_\Pi(p)| + 1)^\nu + 4\}^2\\ &\ll_\nu |a_\Pi(p)|^{2\nu} + 1\\ &\ll_\nu |a_\Pi(p)|^2 p^{(1-2/37)(\nu-1)} + 1.\end{aligned} \tag{2.13}$$

Now we apply Lemma 2.2 with the choice of parameters

$$(\pi', \pi'', \delta', \delta'') = \begin{cases}(\Pi, \Pi, \frac{2}{37}, \frac{1}{2}) & \text{if } \ \mathcal{P} = S_1 \text{ or } S_3\\ (\Pi, \pi, \frac{2}{17}, \frac{1}{2}) & \text{if } \ \mathcal{P} = S_2\end{cases}$$

to write

$$\sum_{p^\nu \leqslant x,\, p \in S_j} (\log p)|a_\Pi(p^\nu)|^2 \ll_\nu \begin{cases} x^{1-1/38+\varepsilon} & \text{if } \ j = 1, 3.\\ x^{1-1/19+\varepsilon} & \text{if } \ j = 2,\end{cases} \tag{2.14}$$

Now the required estimate (2.7) for Π follows from (2.11) and (2.14).

Next let us turn to the case of Π'. Let $\pi_1 = \otimes_p \pi_{1,p}$ (resp. $\pi_2 = \otimes_p \pi_{2,p}$) be a cuspidal representation of $GL_2(\mathbb{Q}_\mathbb{A})$ (resp. $GL_3(\mathbb{Q}_\mathbb{A})$). We may just consider those p such that at least one of $\pi_{1,p}$ and $\pi_{2,p}$ is not tempered. By the same construction as before (2.10), the Satake parameters of $\pi_{1,p}$ and $\pi_{2,p}$ are as follows:

$$\begin{aligned}&\pi_{1,p} : \mathrm{diag}\big[u_1p^a,\, u_1p^{-a}\big], && \text{where } \ 0 \leqslant a \leqslant \tfrac{7}{64},\\ &\pi_{2,p} : \mathrm{diag}\big[u_2p^b,\, u_3,\, u_2p^{-b}\big], && \text{where } \ 0 \leqslant b \leqslant \tfrac{1}{2} - \tfrac{1}{10},\end{aligned}$$

where u_1, u_2 u_3 are complex numbers of absolute value 1. Here we used the parabolic subgroups of type (2) for $\pi_{1,p}$, and of type (3) or $(2,1)$ for $\pi_{2,p}$. Thus the Satake parameters of $\Pi'_p = \pi_{1,p} \boxtimes \pi_{2,p}$ are:

$$\mathrm{diag}\big[u_1u_2p^{a+b},\, u_1u_2p^{b-a},\, u_1u_3p^a,\, u_1u_3p^{-a},\, u_1u_2p^{-(b-a)},\, u_1u_2p^{-(a+b)}\big].$$

If Π' is cuspidal, following the bound (2.3) proved in [10], we get

$$0 < a + b \leqslant \tfrac{1}{2} - \tfrac{1}{37}. \tag{2.15}$$

If Π' is not cuspidal, then it is irreducibly induced from unitary cuspidal representations of smaller GL_m's, and (2.15) holds with an even smaller bound. Then

$$\begin{aligned}&|a_{\Pi'}(p^\nu)|\\ &= \big|(u_1u_2)^\nu(p^{(a+b)\nu} + p^{(a-b)\nu} + p^{(b-a)\nu} + p^{-(a+b)\nu}) + (u_1u_3)^\nu(p^{a\nu} + p^{-a\nu})\big|.\end{aligned} \tag{2.16}$$

From (2.16) we can see that

$$|a_{\Pi'}(p^\nu)| \leqslant 6p^{(a+b)\nu}, \qquad |a_{\Pi'}(p)| \geqslant p^{a+b} - p^a. \tag{2.17}$$

Thus in view of (2.15), (2.17) and the fact that $a \leqslant \frac{7}{64}$, (*) we can deduce

$$\begin{aligned}|a_{\Pi'}(p^\nu)|^2 &\ll (|a_{\Pi'}(p)| + p^a)^{2\nu} \\ &\ll_\nu |a_{\Pi'}(p)|^{2\nu} + p^{2a\nu} \\ &\ll_\nu |a_{\Pi'}(p)|^2 p^{(1-2/37)(\nu-1)} + p^{(1/2-9/32)\nu}.\end{aligned} \tag{2.18}$$

Applying Lemma 2.2 with $\pi' = \pi'' = \Pi'$, $\delta' = \frac{2}{37}$ and $\delta'' = \frac{9}{32}$, we now conclude that

$$\sum_{p^\nu \leqslant x} (\log p)|a_{\Pi'}(p^\nu)|^2 \ll x^{1-1/38+\varepsilon}.$$

This completes the proof. ■

The proof of Theorems 1 and 2. Let Π'' be either Π or Π'. We can write

$$\begin{aligned}\sum_{p^\nu > x,\, \nu \geqslant 2} \frac{(\log p)^2 |a_{\Pi''}(p^\nu)|^2}{p^\nu} &= \sum_{j\geqslant 0} \sum_{2^j x < p^\nu \leqslant 2^{j+1}x,\, \nu\geqslant 2} \frac{(\log p)^2 |a_{\Pi''}(p^\nu)|^2}{p^\nu} \\ &\leqslant \sum_{j\geqslant 0} \frac{\log(2^{j+1}x)}{2^j x} \sum_{2^j x < p^\nu \leqslant 2^{j+1}x,\, \nu\geqslant 2} (\log p)|a_{\Pi''}(p^\nu)|^2.\end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned}\sum_{p^\nu > x,\, \nu \geqslant 2} \frac{(\log p)^2 |a_{\Pi''}(p^\nu)|^2}{p^\nu} &\ll \sum_{j\geqslant 0} \frac{\log(2^{j+1}x)}{2^j x} (2^{j+1}x)^{1-1/38+\varepsilon} \\ &\ll \sum_{j\geqslant 0} \frac{\log(2^{j+1}x)}{(2^{j+1}x)^{1/38-\varepsilon}} \\ &\ll x^{-1/38+2\varepsilon}.\end{aligned}$$

This implies the required result. ■

(*) Note that instead of using the bound $0 \leqslant a \leqslant 7/64$, it suffices to use a bound with $7/64$ being replaced by $1/4 - \delta$ for any $\delta > 0$.

3. Proof of Theorem 3

Theorem 3 follows immediately from (1.6) and the following lemma.

Lemma 3.1. *Let π be a unitary automorphic cuspidal representation for $GL_m(\mathbb{Q}_{\mathbb{A}})$.*

(i) *For each $m \in \{1, \ldots, 4\}$, there is a constant $\delta_m > 0$ such that*

$$\sum_{p^\nu \leqslant x,\, \nu \geqslant 2} (\log p)|a_\pi(p^\nu)|^2 \ll x^{1-\delta_m}.$$

(ii) *If $m \geqslant 5$, under Hypothesis H we have*

$$\sum_{p^\nu \leqslant x,\, \nu \geqslant 2} (\log p)|a_\pi(p^\nu)|^2 \ll x/\log x.$$

Proof. In view of (2.1) of Lemma 2.1 with a suitable choice of ν_0, it suffices to show, for fixed $\nu \geqslant 2$, that (i)

$$\sum_{p^\nu \leqslant x} (\log p)|a_\pi(p^\nu)|^2 \ll_\nu x^{1-\delta_m}, \tag{3.1}$$

if $m \leqslant 4$, and (ii)

$$\sum_{p^\nu \leqslant x} (\log p)|a_\pi(p^\nu)|^2 \ll_\nu x/\log x \tag{3.2}$$

if $m \geqslant 5$ under Hypothesis H.

First we prove (3.2):

$$\begin{aligned}\sum_{p^\nu \leqslant x} (\log p)|a_\pi(p^\nu)|^2 &= \sum_{p^\nu \leqslant x^{1/2}} (\log p)|a_\pi(p^\nu)|^2 + \sum_{x^{1/2} < p^\nu \leqslant x} (\log p)|a_\pi(p^\nu)|^2 \\ &\leqslant x^{1/2} \sum_{p^\nu \leqslant x^{1/2}} \frac{(\log p)^2 |a_\pi(p^\nu)|^2}{p^\nu} \\ &\quad + \frac{2x}{\log x} \sum_{x^{1/2} < p^\nu \leqslant x} \frac{(\log p)^2 |a_\pi(p^\nu)|^2}{p^\nu},\end{aligned}$$

which is $\ll x/\log x$ under Hypothesis H.

Next we prove (3.1) for $m = 4$, since other cases are easier. As before it suffices to consider the sum on the left side of (3.1) taken over $p \neq 2, 3$ with π_p being not tempered. Then for such a p, $\Pi_p \cong \wedge^2 \pi_p$. There are then three possibilities.

If $p \in S_1$ as in (2.10), using Π_p we get $0 < 2a \leqslant \frac{1}{2} - \frac{1}{37}$ as in (2.11). Then

$$\begin{aligned}|a_\pi(p^\nu)|^2 &= |(u_1^\nu + u_2^\nu)(p^{a\nu} + p^{-a\nu})|^2 \\ &\leqslant 16 p^{(1/2 - 1/37)\nu}.\end{aligned}$$

From this, we deduce that

$$\sum_{p^\nu\leqslant x,\, p\in S_1}(\log p)|a_\pi(p^\nu)|^2 \ll \sum_{p^\nu\leqslant x,\, p\in S_1}(\log p)p^{(1/2-1/37)\nu} \ll x^{1-1/37}. \tag{3.3}$$

If $p\in S_2$, we have

$$\begin{aligned}|a_\pi(p^\nu)| &= |u_1^\nu(p^{a\nu}+p^{-a\nu})+u_2^\nu+u_3^\nu| \leqslant p^{a\nu}+3,\\ |a_\pi(p)| &= |u_1(p^a+p^{-a})+u_2+u_3| \geqslant p^a-2\end{aligned}$$

with $0<a\leqslant 1/2-1/17$. Then

$$\begin{aligned}|a_\pi(p^\nu)|^2 &\leqslant \{(|a_\pi(p)|+2)^\nu+3\}^2\\ &\ll_\nu |a_\pi(p)|^{2\nu}+1\\ &\ll_\nu |a_\pi(p)|^2p^{(1-2/17)(\nu-1)}+1.\end{aligned} \tag{3.4}$$

Similarly if $p\in S_3$, then

$$\begin{aligned}\big|a_\pi(p^\nu)\big| &= \big|u_1^\nu\big(p^{a_1\nu}+p^{-a_1\nu}\big)+u_2^\nu\big(p^{a_2\nu}+p^{-a_2\nu}\big)\big| \leqslant 2p^{a_1\nu}+2,\\ \big|a_\pi(p)\big| &= \big|u_1\big(p^{a_1}+p^{-a_1}\big)+u_2\big(p^{a_2}+p^{-a_2}\big)\big| \geqslant p^{a_1}-2p^{a_2}.\end{aligned}$$

From this, (2.3) with $m=4$ and the last inequality of (2.11), we deduce that

$$\begin{aligned}|a_\pi(p^\nu)|^2 &\leqslant \{2(|a_\pi(p)|+2p^{a_2})^\nu+2\}^2\\ &\ll_\nu |a_\pi(p)|^{2\nu}+p^{2a_2\nu}\\ &\ll_\nu |a_\pi(p)|^2p^{(1-2/17)(\nu-1)}+p^{(1/2-1/37)\nu}.\end{aligned} \tag{3.5}$$

As before, we can apply Lemma 2.2 with the choice of parameters

$$(\pi',\pi'',\delta',\delta'')=\begin{cases}(\pi,\pi,\frac{2}{17},\frac{1}{2}) & \text{if } \mathcal{P}=S_2\\ (\pi,\pi,\frac{2}{17},\frac{1}{37}) & \text{if } \mathcal{P}=S_3\end{cases}$$

to write

$$\sum_{p^\nu\leqslant x,\, p\in S_j}(\log p)|a_\pi(p^\nu)|^2 \ll_\nu x^{1-1/37} \qquad (j=2,\,3). \tag{3.6}$$

Now the required result follows from (3.3) and (3.6). ■

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