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OMEGA THEOREMS FOR A CLASS OF L-FUNCTIONS
(A note on the Rankin-Selberg zeta-function)

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Abstract: In this paper we study the Omega theorems for a class of general L-functions satisfying certain conditions and as an important application, we obtain the Omega theorems for the Rankin-Selberg zeta-functions $Z(s_0)$ attached to holomorphic cusp forms of fixed weight for the full modular group when $\frac{1}{2} \leqslant \sigma_0 < 1$.
Keywords: Rankin-Selberg zeta-function, Omega Theorems, Zero-density estimates.

1. Introduction

Omega theorems for the Riemann zeta-function and L-functions of degree 2 have been extensively studied for which we refer to [1], [2], [5], [6] and [13]. Some of these results can collectively be seen in [4] and [14].

The aim of this note is to prove Ω theorems for a class of L-functions satisfying certain conditions and as an application, we obtain Ω theorems for the Rankin-Selberg zeta-functions which are of degree 4. We follow the arguments of Ramachandra and Sankaranarayanan (see [6]).

Let $\mathcal{C}$ be the class of general L-functions $F(s)$ satisfying the following conditions.

(i). $F(s)$ is absolutely convergent in the half-plane $\sigma > 1$ and continuable analytically to the region $\sigma \geqslant 0$ as a meromorphic function possibly with a simple pole at $s = 1$ having the residue κ_1 and there $F(s)$ is of finite order (i.e $|(s-1)F(s)| \ll (|t|+2)^A$ in $\sigma \geqslant 0$). It has an Euler-product representation and a functional equation of the Riemann zeta type. Thus all the non-trivial complex zeros of $F(s)$ lie in the vertical strip $0 \leqslant \sigma \leqslant 1$.

(ii). $\log F(s)$ can be written in the form

$$\log F(s) = \sum_{p} \sum_{m \geqslant 1} \frac{b\,(p^m)}{p^{ms}} \tag{1.1}$$

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with the series in (1.1) being absolutely convergent in $\sigma > 1$ (where the sum runs over all primes p) and the coefficients $b(n)$'s satisfy the estimates:

$$b(n) \ll n^{\epsilon}, \tag{1.2}$$

$b(p)$'s are real and the asymptotic relation

$$\sum_{p \leqslant x} b(p) = \kappa \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right), \tag{1.3}$$

holds where κ is any positive constant. We also assume that

$$\sum_{p \leqslant u} \left|b\left(p^2\right)\right| \ll u(\log u). \tag{1.4}$$

(iii). Let

$$N_F(\mu, T) = \#\left\{\rho = \beta + i\gamma : F(\rho) = 0,\ \beta \geqslant \mu > 0,\ |\gamma| \leqslant T\right\}. \tag{1.5}$$

We make the following zero-density hypothesis.

Hypothesis. *For fixed μ satisfying $1 > \mu > \frac{1}{2}$ and for $T \geqslant T_0$ (with T_0 sufficiently large), there exists a $\delta > 0$ such that $N_F(\mu, T) \ll T^{1-\delta}$ where the implied constant depends on μ and δ.*

Throughout the paper, we assume that $x \geqslant x_0$ and $T \geqslant T_0$ (where x_0 and T_0 are sufficiently large), and the parameter α satisfies the inequality $0 < \alpha \leqslant \frac{1}{100} \log\log x$. The alphabets $A, B, C \cdots$ (with or without suffixes denote positive constants) and ϵ, δ denote small positive constants. Now, We prove

Theorem. *Let $F(s) \in \mathcal{C}$ and thus the conditions (i), (ii) and (iii) hold for $F(s)$ by our assumption. Let $\frac{1}{2} < \mu_1 \leqslant \sigma_0 < 1$, $0 \leqslant \theta < 2\pi$, $\epsilon > 0$. Let y be the positive solution of the equation $e^y = 2y + 1$. Let l be an integer $\geqslant 6$, $C_2 = \frac{2y}{(2y+1)^2}$, $0 < C_1 < C_2$. Then, for $T \geqslant T_0$, we have*

$$\Re\left(e^{-i\theta} \log F\left(\sigma_0 + it_0\right)\right) \geqslant \kappa\ \left(1 - \sigma_0\right)^{-1} C_0 C_1 \left(\log t_0\right)^{1-\sigma_0} \left(\log\log t_0\right)^{-\sigma_0}$$

for at least one t_0 satisfying $T^{\epsilon} \leqslant t_0 \leqslant T$ where $C_0 = \cos\left(\frac{2\pi}{l}\right)\left(\frac{\delta}{\log l}\right)^{1-\sigma_0}$. Here $\delta = 1$ if we assume Riemann hypothesis for $F(s)$. Otherwise, $\delta = \delta\left(\mu_1\right)$.

2. Some Lemmas

Lemma 2.1. *Let $\theta_1, \cdots, \theta_M$ be distinct positive real numbers and suppose that $l \geqslant 6$ is an integer. For any given positive integer R, then there exist at least R integers r'_k such that $1 \leqslant r'_k \leqslant J = l^M R$ and $||r'_k \theta_m|| < \frac{1}{l}$ for $1 \leqslant m \leqslant M$.*

Proof. See for example [6]. ■

Lemma 2.2. *For $\frac{1}{2} \leqslant \sigma_0 < 1$, we have*

$$\begin{aligned} S =: & \sum_{\left|\log\left(\frac{p}{x}\right)\right| \leqslant 2\alpha} p^{-\sigma_0} b(p) \left(2\alpha - \left|\log\left(\frac{p}{x}\right)\right|\right) \\ = & \ \kappa \left(\frac{2\sinh\left(\alpha\left(1-\sigma_0\right)\right)}{\left(1-\sigma_0\right)}\right)^2 \frac{x^{1-\sigma_0}}{\log x} \\ & + O\left(\left(\kappa+1\right)\left(1+\alpha^3\right) \ x^{1-\sigma_0} (\log x)^{-2}\right). \end{aligned} \tag{2.2.1}$$

Proof. Let β_1 be a positive solution of the exponential equation

$$e^y = 2y + 1.$$

Ultimately, we are going to choose α such that $\beta_1 = 2\alpha(1-\sigma_0)$ (a fixed positive constant). We note that $1 < \beta_1 < 2$. Keeping this in mind, we prove this Lemma in the following. We have

$$\begin{aligned} S &= \sum_{e^{-2\alpha} x \leqslant p \leqslant x} \cdots + \sum_{x \leqslant p \leqslant e^{2\alpha} x} \cdots + O\left(\alpha \ x^{-\sigma_0+\epsilon}\right) \\ &= S_1 + S_2 + O\left(\alpha \ x^{-\sigma_0+\epsilon}\right). \ \text{(say)} \end{aligned} \tag{2.2.2}$$

We note that (from the condition (1.3) on $F(s)$)

$$K(u) =: \sum_{p \leqslant u} b(p) = \kappa \ \frac{u}{\log u} + O\left(\frac{u}{(\log u)^2}\right), \tag{2.2.3}$$

Now,

$$\begin{aligned} S_1 &= \int_{xe^{-2\alpha}}^{x} u^{-\sigma_0} \left(2\alpha - \log\left(\frac{x}{u}\right)\right) dK(u) \\ &= \kappa \int_{xe^{-2\alpha}}^{x} u^{-\sigma_0} \left(2\alpha - \log\left(\frac{x}{u}\right)\right) \frac{du}{\log u} \\ &\quad + O\left(\left(\kappa+1\right)\left(1+\alpha+\alpha^2\right) \ x^{1-\sigma_0} (\log x)^{-2}\right) \\ &= \kappa \ (2\alpha - \log x) \left\{\frac{u^{1-\sigma_0}}{(1-\sigma_0)\log u} + \frac{u^{1-\sigma_0}}{(1-\sigma_0)^2 (\log u)^2} \Big|_{xe^{-2\alpha}}^{x}\right\} \\ &\quad + \kappa \ \frac{x^{1-\sigma_0}}{(1-\sigma_0)} \left\{1 - e^{-2\alpha(1-\sigma_0)}\right\} \\ &\quad + O\left(\left(\kappa+1\right)\left(1+\alpha+\alpha^2+\alpha^3\right) \ x^{1-\sigma_0} (\log x)^{-2}\right). \end{aligned} \tag{2.2.4}$$

Similarly, we obtain

$$\begin{aligned}
S_2 = {}& \kappa\ (2\alpha + \log x) \left\{ \frac{u^{1-\sigma_0}}{(1-\sigma_0)\log u} + \frac{u^{1-\sigma_0}}{(1-\sigma_0)^2 (\log u)^2} \Big|_x^{xe^{2\alpha}} \right\} \\
& + \kappa\ \frac{x^{1-\sigma_0}}{(1-\sigma_0)} \left\{ 1 - e^{2\alpha(1-\sigma_0)} \right\} \\
& + O\left((\kappa+1) \left(1 + \alpha + \alpha^2 + \alpha^3\right)\ x^{1-\sigma_0} (\log x)^{-2} \right). \qquad (2.2.5)
\end{aligned}$$

We note that $\frac{1}{(1-y)} = 1 + y + O\left(y^2\right)$ and $\frac{1}{(1+y)} = 1 - y + O\left(y^2\right)$ for y sufficiently small. Hence from (2.2.4) and (2.2.5), we get

$$\begin{aligned}
S = {}& \frac{\kappa x^{1-\sigma_0}}{(1-\sigma_0)^2 \log x} \left\{ e^{2\alpha(1-\sigma_0)} + e^{-2\alpha(1-\sigma_0)} - 2 \right\} \\
& + O\left((\kappa+1) \left(1 + \alpha + \alpha^2 + \alpha^3\right)\ x^{1-\sigma_0} (\log x)^{-2} \right) \\
= {}& \kappa\ \left(\frac{2\sinh\left(\alpha\left(1-\sigma_0\right)\right)}{(1-\sigma_0)} \right)^2 \frac{x^{1-\sigma_0}}{\log x} \\
& + O\left((\kappa+1) \left(1 + \alpha + \alpha^2 + \alpha^3\right)\ \frac{x^{1-\sigma_0}}{(\log x)^2} \right). \qquad (2.2.6)
\end{aligned}$$

This proves the lemma. ∎

Lemma 2.3. *Let* $0 \leqslant \theta < 2\pi,\ \alpha > 0$ *and* $\mu \leqslant \sigma_0 < 1$ *be constants and let* $s = \sigma + it,\ s_0 = \sigma_0 + it_0$. *Then for all* x *with* $10 \leqslant x \ll (\log t_0)(\log\log t_0)$, *we have*

$$\begin{aligned}
I_1 =: {}& \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \left(e^{-i\theta} \log F(s+s_0) \right) \left(\frac{e^{\alpha s} - e^{-\alpha s}}{s} \right)^2 \left(2 + x^s e^{i\theta} + x^{-s} e^{-i\theta} \right) ds \\
= {}& \sum_{\left|\log\left(\frac{p}{x}\right)\right| \leqslant 2\alpha} p^{-s_0} b(p) \left(2\alpha - \left| \log\left(\frac{p}{x}\right) \right| \right) + O\left((1+\alpha)\ (\log x)^2 \right). \qquad (2.3.1)
\end{aligned}$$

Proof. For $\Re(s+s_0) > 1$, we have

$$\begin{aligned}
\log F(s+s_0) &= \sum_p \frac{b(p)}{p^{s+s_0}} + \sum_{m \geqslant 2,\ p} \frac{b\left(p^m\right)}{p^{m(s+s_0)}} \\
&=: S_3 + S_4. \qquad (2.3.2)
\end{aligned}$$

We observe that if $\alpha > 0,\ x > 0$ and $c > 0$, then we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{e^{\alpha s} - e^{-\alpha s}}{s} \right)^2 x^s ds = \begin{cases} 2\alpha - |\log x| & \text{if } |\log x| \leqslant 2\alpha, \\ 0 & \text{if } |\log x| > 2\alpha. \end{cases} \qquad (2.3.3)$$

Therefore, we have

$$
\begin{aligned}
I_2 =: & \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(e^{-i\theta}S_3\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2\left(x^s e^{i\theta}\right)ds \\
= & \sum_{\left|\log\left(\frac{p}{x}\right)\right|\leqslant 2\alpha} p^{-s_0}b(p)\left(2\alpha-\left|\log\left(\frac{p}{x}\right)\right|\right), \qquad (2.3.4)
\end{aligned}
$$

$$
\begin{aligned}
|I_3| =: & \left|\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(e^{-i\theta}S_3\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2\left(2+x^{-s}e^{-i\theta}\right)ds\right| \\
\leqslant & 2\left|\sum_{|\log p|\leqslant 2\alpha} p^{-s_0}b(p)\left(2\alpha-|\log p|\right)\right| + \left|\sum_{|\log(px)|\leqslant 2\alpha} p^{-s_0}b(p)\left(2\alpha-|\log(px)|\right)\right| \\
\ll & \alpha\left(\sum_{p\leqslant e^{2\alpha}} 1\right) \ll e^{5\alpha} \ll (\log x)^2. \qquad (2.3.5)
\end{aligned}
$$

Similarly, we estimate

$$
\begin{aligned}
|I_4| =: & \left|\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(e^{-i\theta}S_4\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2\left(2+x^s e^{i\theta}+x^{-s}e^{-i\theta}\right)ds\right| \\
= & |S_5+S_6+S_7| \quad \text{say}, \qquad (2.3.6)
\end{aligned}
$$

where

$$
S_5 =: 2e^{-i\theta}\sum_{\substack{|\log(p^m)|\leqslant 2\alpha \\ m\geqslant 2}} b\left(p^m\right)p^{-m\sigma_0}\left(2\alpha-\left|\log\left(p^m\right)\right|\right),
$$

$$
S_6 =: \sum_{\substack{\left|\log\left(\frac{p^m}{x}\right)\right|\leqslant 2\alpha \\ m\geqslant 2}} b\left(p^m\right)p^{-m\sigma_0}\left(2\alpha-\left|\log\left(\frac{p^m}{x}\right)\right|\right),
$$

and

$$
S_7 =: e^{-2i\theta}\sum_{\substack{|\log(p^m x)|\leqslant 2\alpha \\ m\geqslant 2}} b\left(p^m\right)p^{-m\sigma_0}\left(2\alpha-\left|\log\left(p^m x\right)\right|\right).
$$

Using the condition (1.2) (and since $0<\alpha\leqslant\frac{1}{100}\log\log x$ and $\sigma_0>\frac{1}{2}$), we obtain

$$
\begin{aligned}
S_5 \ll & \alpha\, e^{2\alpha}\sum_{m\geqslant 2,\ p^m\leqslant e^{2\alpha}} p^{-m\sigma_0} \\
\ll & \alpha\, e^{2\alpha}\sum_{p\leqslant e^{2\alpha}}\frac{1}{p^{\sigma_0}\left(p^{\sigma_0}-1\right)} \\
\ll & \alpha\, e^{2\alpha}\, e^{2\alpha} \\
\ll & e^{5\alpha} \\
\ll & (\log x)^2 \qquad (2.3.7)
\end{aligned}
$$

and similarly

$$S_7 \ll e^{5\alpha} \ll (\log x)^2. \tag{2.3.8}$$

Let us write

$$w(u) =: \sum_{p \leqslant u} \left| b\left(p^2\right) \right| \ll u \log u \quad (\text{by} \quad (1.4)).$$

From the Riemann-Steiltjes integration and using the average estimate condition (1.4), we note that

$$\begin{aligned}
S_8 =: & \sum_{p^2 \leqslant e^{2\alpha} x =: y} \frac{\left| b\left(p^2\right) \right|}{p^{2\sigma_0}} \\
= & \int_2^{y^{1/2}} \frac{1}{u^{2\sigma_0}} \, d\left(w(u)\right) \\
\ll & \left| u^{-2\sigma_0} \; w(u) \Big|_2^{y^{1/2}} \right| + 2\sigma_0 \int_2^{y^{1/2}} \frac{|w(u)|}{u^{2\sigma_0+1}} \, du \\
\ll & \; y^{\frac{1}{2}-\sigma_0} \log y + (\log y)^2 \\
\ll & \; (\log y)^2 \\
\ll & \; (\alpha + \log x)^2.
\end{aligned} \tag{2.3.9}$$

We also notice that (with $e^{2\alpha} x =: y$)

$$\begin{aligned}
S_9 =: & \sum_{\substack{p^m \leqslant y, \\ m \geqslant 3}} \frac{\left| b\left(p^m\right) \right|}{p^{m\sigma_0}} \\
\ll & \sum_{p \leqslant y^{1/3}} \frac{1}{p^{2(\sigma_0 - \epsilon)} \left(p^{\sigma_0 - \epsilon} - 1\right)} \\
\ll & \; 1
\end{aligned} \tag{2.3.10}$$

and hence from (2.3.9) and (2.3.10), we get

$$S_6 \ll \alpha \; (\alpha + \log x)^2 \ll \alpha \; (\log x)^2. \tag{2.3.11}$$

This proves the lemma. ■

We note (see for example page 56, Lemma α of [14]) if $f(s)$ is regular and

$$\left| \frac{f(s)}{f\left(s_0\right)} \right| < e^M \quad (M > 1)$$

in the circle $|s - s_0| \leqslant r$, then

$$\left| \frac{f'(s)}{f(s)} - \sum_{\rho} \frac{1}{s - \rho} \right| < \frac{AM}{r} \quad \left(|s - s_0| \leqslant \frac{r}{4}\right) \tag{2.1}$$

where ρ runs through the zeros of $f(s)$ such that $|\rho - s_0| \leqslant \frac{r}{2}$.

Therefore, we get

$$\frac{F'}{F}(s) = \sum_{|t-\gamma|\leqslant 1} (s-\rho)^{-1} + O(\log t). \tag{2.2}$$

Here $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $F(s)$. Integrating (2.2) from s to $2+it$ and assuming that t is not the ordinate of any zero of $F(s)$, we obtain

$$\log F(s) - \log F(2+it) = \sum_{|t-\gamma|\leqslant 1} \{\log(s-\rho) - \log(2+it-\rho)\} + O(\log t). \tag{2.3}$$

Proceeding as in Theorem 9.6 B of [14] we get

$$\log F(s) = \sum_{|t-\gamma|\leqslant 1} \log(s-\rho) + O(\log t). \tag{2.4}$$

Let t_0 be sufficiently large and $\tau = (\log t_0)^2$. If the region $\{\sigma > 0, |\sigma| \leqslant 2\tau\}$ is zero-free for $F(s+s_0)$ for $|t| \leqslant 2\tau - \sigma$, then in $0 < \sigma \leqslant 1$, we have the estimate,

$$\log F(s+s_0) = O\left((\log t_0)\left(\log\left(\frac{2}{\sigma}\right)\right)\right). \tag{2.5}$$

This can be seen easily as follows. From (2.4), we already have,

$$\log F(s+s_0) = \sum_{|t+t_0-\gamma|\leqslant 1} \log(s+s_0-\rho) + O(\log t_0). \tag{2.6}$$

We only need to estimate the first sum appearing in the right hand side of (2.6). Since, $|\Im \log(s+s_0)| \leqslant \pi$, we have

$$|\log(s+s_0-\rho)| \leqslant |\log|s+s_0-\rho|| + \pi. \tag{2.7}$$

We observe if $1 \leqslant |s+s_0-\rho| < 2$, then each term in the sum in (2.6) is in absolute value $\leqslant \log 2$ and the number of terms in the sum is $O(\log t_0)$.

When $0 < |s+s_0-\rho| < 1$, we observe that

$$|s+s_0-\rho|^2 = (\sigma+\sigma_0-\beta)^2 + (t+t_0-\gamma)^2, \tag{2.8}$$

and the rectangle $\{0 < \sigma \leqslant 1, |t| \leqslant 2\tau - \sigma\}$ is zero-free for $F(s+s_0)$. If ρ lies on the left border of this region, i.e on the line $\Re s\ (=: \beta) = \sigma_0$, then $|s+s_0-\rho|^2 \geqslant \sigma^2$ and in this case, we have $|\log\sigma| = \left|\log\left(\frac{1}{\sigma}\right)\right|$. As before, the number of terms in the sum (2.6) is $O(\log t_0)$ and we are through.

If ρ lies inside the rectangular region, then again we obtain the same estimate since $|s+s_0-\rho| \geqslant |t+t_0-\gamma| \geqslant \sigma$. Thus we arrive at the estimate (2.5).

Lemma 2.4. *Let θ, α, σ_0 and t_0 be as in lemma 2.3. The contribution of the tail portion $|t| \geqslant (\log t_0)^2$ to the integral in lemma 2.3 is $O\left((\log x)^2\right)$. Also the contribution from the integrals over $[i\tau, 1+i\tau]$ and $[-i\tau, 1-i\tau]$ are $O\left((\log x)^2\right)$.*

Proof. The proof follows from the estimate

$$\log F(s+s_0) \ll (\log t_0)\left(\log\left(\frac{2}{\sigma}\right)\right). \qquad \blacksquare$$

Lemma 2.5. *With $\tau = (\log t_0)^2$, we have*

$$\begin{aligned} I_5 =: \Re\left\{\frac{1}{2\pi i}\int_{-i\tau}^{i\tau}\left(e^{-i\theta}\log F(s+s_0)\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2\left(2+x^s e^{i\theta}+x^{-s}e^{-i\theta}\right)ds\right\} \\ = \sum_{\left|\log\left(\frac{p}{x}\right)\right|\leqslant 2\alpha} b(p)p^{-\sigma_0}\cos(t_0\log p)\left(2\alpha - \left|\log\left(\frac{p}{x}\right)\right|\right) + O\left((1+\alpha)\,(\log x)^2\right). \end{aligned}$$

Proof. Note that the coefficients $b(p)$'s are real numbers (by our assumption). Now, the proof follows from the above lemmas. $\blacksquare$

Lemma 2.6. *We have*

$$\begin{aligned} Q_1 =: \left(\max_{|t|\leqslant\tau,\sigma=0}\left(\Re e^{-i\theta}\log F(s+s_0)\right)\right)\times \\ \times\left(\frac{1}{2\pi i}\int_{|t|\leqslant\tau,\ \sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2\left(2+x^s e^{i\theta}+x^{-s}e^{-i\theta}\right)ds\right) \\ \geqslant \sum_{\left|\log\left(\frac{p}{x}\right)\right|\leqslant 2\alpha} b(p)p^{-\sigma_0}\cos(t_0\log p)\left(2\alpha - \left|\log\left(\frac{p}{x}\right)\right|\right) + O\left((1+\alpha)\,(\log x)^2\right). \end{aligned}$$

Proof. It follows from lemma 2.5. $\blacksquare$

Lemma 2.7. *For $\tau = (\log t_0)^2$ and $2\alpha \leqslant |\log x|$, we have*

$$\frac{1}{2\pi i}\int_{|t|\leqslant\tau,\ \sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2\left(2+x^s e^{i\theta}+x^{-s}e^{-i\theta}\right)ds = 4\alpha + O\left(\frac{1}{\tau}\right).$$

Proof. This is lemma 3.11 of [6]. $\blacksquare$

Lemma 2.8. *Let C be a positive constant to be chosen later. Let p be the set of primes satisfying*

$$Ce^{-2\alpha}(\log P\log\log P) \leqslant p \leqslant Ce^{2\alpha}(\log T\log\log T),$$

where we refer to lemma 2.1 and put $T = l^M RP$. Here M will be greater than or equal to the number of primes satisfying the inequalities just stated. we put

$M = \left[\left(Ce^{2\alpha}+\epsilon\right)\log T\right]$ where $\epsilon > 0$ is an arbitrary but fixed constant. Let $x = C(\log t_0)(\log\log t_0)$ where C is a small positive constant and $t_0 = 2\pi l_k$ ($k = 1, 2, \cdots, R$) for any k. Then, for all primes p satisfying $\left|\log\left(\frac{p}{x}\right)\right| \leqslant 2\alpha$, we have

$$\begin{aligned} Q_2 =: &\sum_{\left|\log\left(\frac{p}{x}\right)\right|\leqslant 2\alpha} b(p)p^{-\sigma_0}\cos(t_0\log p)\left(2\alpha - \left|\log\left(\frac{p}{x}\right)\right|\right) \\ &\geqslant \kappa\ \cos\left(\frac{2\pi}{l}\right)C^{1-\sigma_0}\left(\frac{2\sinh\alpha(1-\sigma_0)}{1-\sigma_0}\right)^2\left(\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}\right). \end{aligned}$$

Proof. The proof follows from the Lemma 2.2. ∎

3. Proof of the Theorem

Consider the rectangles $\left\{\sigma_0 \leqslant \sigma < 1, |t_j - t| \leqslant 2(\log t_0)^2\right\}$ $(j = 1, 2, \cdots, R)$. These rectangles are disjoint and the number of such rectangles is R. If $R > DT^{1-\delta} + 2$ where D is the constant coming from the hypothesis, then at least two of these rectangles are zero-free. We select the rectangle for which $t_0 + \tau \leqslant T$ (T to be defined) and fix $P = T^{\epsilon_1}$, $R = T^{1-\delta+\epsilon_2}$ where ϵ_1, ϵ_2 are small positive constants. Then we put, $M = \left[\left(Ce^{2\alpha}+\epsilon\right)\log T\right]$ and $l^M RP = T$. If we choose $C = \frac{\delta}{e^{2\alpha}\log l} - \frac{\epsilon_3}{e^{2\alpha}\log l}$ for a small positive constant ϵ_3, then from the last three lemmas 2.6, 2.7 and 2.8, we get

$$\begin{aligned} Q_3 =: &\max_{|t|\leqslant\tau,\sigma=0}\left(\Re e^{-i\theta}\log F(s+s_0)\right) \\ &\geqslant \frac{\kappa}{4\alpha}\cos\left(\frac{2\pi}{l}\right)(\log l)^{-(1-\sigma_0)}\frac{\delta^{1-\sigma_0}}{e^{2\alpha(1-\sigma_0)}}\left(\frac{2\sinh\alpha(1-\sigma_0)}{1-\sigma_0}\right)^2\left(\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}\right) \\ &= \frac{\kappa}{2}\frac{\cos(\frac{2\pi}{l})\delta^{1-\sigma_0}}{(\log l)^{1-\sigma_0}(1-\sigma_0)}\left(\frac{(1-e^{-\beta_1})}{\sqrt{\beta_1}}\right)^2\left(\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}\right), \end{aligned} \tag{3.1}$$

where $\beta_1 = 2\alpha(1-\sigma_0)$.

By choosing $\beta_1 > 0$ such that $\frac{1-e^{-\beta_1}}{\sqrt{\beta_1}}$ is maximum, we see that the expression in the right hand side of (3.1) becomes

$$\frac{\kappa\cos(\frac{2\pi}{l})\delta^{1-\sigma_0}}{(\log l)^{1-\sigma_0}(1-\sigma_0)}\left(\frac{C_1(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}\right),$$

where C_1 is a positive constant independent of δ, l, and σ_0 and $C_2 = \frac{2y}{(2y+1)^2} > C_1$ with y is the positive solution of the equation $e^y = 2y+1$. This proves the theorem.

4. Some interesting examples

Example 1. The Riemann zeta-function $\zeta(s)$:

In this case, in the Theorem, we can take $\mu_1 = \frac{1}{2}$. Here $\delta = 1$ if we assume the Riemann hypothesis namely *" all the non-trivial complex zeros of $\zeta(s)$ are on the critical line $\Re s = \frac{1}{2}$ "*. Otherwise we have to assume $\frac{1}{2} < \sigma_0 < 1$ and then we can take $\delta = 1 - \frac{3(1-\sigma_0)}{(2-\sigma_0)}$ (due to Ingham's zero-density estimate, see [14]).

Example 2. The Dedekind zeta-function $\zeta_K(s)$ of an algebraic number field K:

Let K be an algebraic number field. The Dedekind zeta-function of K is defined for $\Re s > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{A} \neq 0} (N\mathfrak{A})^{-s} \tag{4.1}$$

where $N\mathfrak{A}$ denotes the norm of the ideal $\mathfrak{A}$ and the sum is extended over all non-zero integral ideals of the ring of integers of the field K. If we write,

$$\log \zeta_K(s) = \sum_{n=1}^{\infty} e_n n^{-s} \quad (\text{for } \ \sigma > 2), \tag{4.2}$$

then, we notice that $e_n \geqslant 0$ for all n. Also from the prime ideal theorem, it is well-known that

$$\sum_{n \leqslant x} e_n \asymp \sum_{p \leqslant x} e_p \asymp \sum_{N\mathfrak{P} \leqslant x} \asymp \frac{x}{\log x}. \tag{4.3}$$

If K is an algebraic number field abelian over K'. Let the degrees of K and K' be n and k respectively. Then,

$$\zeta_K(s) = L_1(s) \cdots L_j(s) \tag{4.4}$$

where $j = n/k$ and $L_i(s)$ are abelian L-functions of K. Therefore we can take any $\mu > 1 - \frac{3}{2k+6}$ in our zero-density hypothesis of the condition (iii).

Let μ' be the smallest real number for which

$$\int_0^T |L_i(\mu' + it)|^2 \, dt \ll T^{1+\epsilon}. \tag{4.5}$$

Then, $\mu' = \frac{1}{2}$ happens when $K' = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{d})$. Then for $\sigma_0 > \frac{1}{2}$, we can take $\mu > \frac{1}{2}$. If $\mu' > \frac{1}{2}$, by standard arguments, we can take any $\mu > \mu'$ in the zero-density hypothesis of the condition (iii). For a detailed discussion of the above cases, we refer to section 5 of [6].

For instance, if the degree n of K exceeds 3, then we observe (see [3])

$$N_{\zeta_K}(\sigma, T) \ll T^{(n+\epsilon)(1-\sigma)} (\log T)^C \tag{4.6}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$. Then, we can take $\delta = 1 - (n+\epsilon)(1-\sigma_0)$ in the zero-density hypothesis of the condition (iii) and $\mu_1 = 1 - \frac{1}{n+\epsilon}$ in the Theorem.

Example 3. Rankin-Selberg zeta-functions:

Let f be a holomorphic cusp form of *fixed even integral weight* k for the full modular group $SL(2,\mathbb{Z})$ which is a normalised eigenfunction of all the Hecke operators. We denote by $Z_{f,f}(s)$ the L-function of the Rankin-Selberg convolution of F with itself. We recall here that

$$Z(s) =: Z_{f,f}(s) = \zeta(2s)\left(\sum_{n=1}^{\infty} \lambda_f^2(n) n^{-s}\right) \tag{4.7}$$

where f has the Fourier series expansion $f(z) = \sum \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$. Here $z \in \mathfrak{H}$ and ζ is the Riemann zeta-function. It has meromorphic continuation to the whole complex plane with a simple pole at $s=1$ and it satisfies the functional equation,

$$\Gamma(s+k-1)\Gamma(s)Z(s) = (2\pi)^{4s-2}\Gamma(k-s)\Gamma(1-s)Z(1-s). \tag{4.8}$$

These L- functions are of degree 4. From the Shimura's split (see [12] or lemma 3.1 of [9] and see also the related references [7] and [8]), we observe that the Rankin-Selberg zeta-function splits into two factors as

$$Z(s) = \zeta(s)\ D(s), \tag{4.9}$$

where $D(s)$ is the normalised symmetric square L-function attached to the Hecke eigenform f. For $\Re s > 1$, $Z(s)$ has the Euler product,

$$Z(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_p}^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p \overline{\alpha_p}}{p^s}\right)^{-1}, \tag{4.10}$$

where $\lambda_f(p) = \alpha_p + \overline{\alpha_p} \in \mathbb{R}$, $\alpha_p\ \overline{\alpha_p} = 1$ and $|\alpha_p| = 1$. In [10], the first author established certain zero density theorems for these symmetric square L-functions. Therefore, for example from theorem 1.1 of [10], we infer that (for $\frac{1}{2} < \mu < 1$)

$$N_D(\mu, T) \ll T^{\frac{5(1-\mu)}{(3-2\mu)}} (\log T)^A$$

and in turn, this implies that

$$N_Z(\mu, T) \ll T^{\frac{5(1-\mu)}{(3-2\mu)}} (\log T)^A$$

where A is an absolute positive constant. Hence, the zero-density hypothesis in condition (iii) holds when $\frac{2}{3} < \mu < 1$.

By the prime number theorem (related to the weighted coefficients $\lambda_f^2(p)$, see for example [11]), we have

$$\sum_{p \leqslant u} \lambda_f^2(p) \log p = u + O\left(ue^{-c\sqrt{\log u}}\right), \tag{4.11}$$

We also notice that (for $m \geqslant 2$)

$$b(p^m) = \frac{\alpha_p^{2m} + \overline{\alpha_p}^{2m} + 2}{m} \ll 1. \tag{4.12}$$

Therefore, we deduce from the Theorem,

Corollary. *Let* $\frac{1}{2} \leqslant \sigma_0 < 1$, $0 \leqslant \theta < 2\pi$, $\epsilon > 0$. *Let* y *be the positive solution of the equation* $e^y = 2y+1$. *Let* l *be an integer* $\geqslant 6$, $C_2 = \frac{2y}{(2y+1)^2}$, $0 < C_1 < C_2$. *Then, for* $T \geqslant T_0$, *we have*

$$\Re\left(e^{-i\theta}\log Z\left(\sigma_0 + it_0\right)\right) \geqslant (1-\sigma_0)^{-1} C_0 C_1 \left(\log t_0\right)^{1-\sigma_0} \left(\log\log t_0\right)^{-\sigma_0}$$

for at least one t_0 *satisfying* $T^\epsilon \leqslant t_0 \leqslant T$ *where* $C_0 = \cos\left(\frac{2\pi}{l}\right)\left(\frac{\delta}{\log l}\right)^{1-\sigma_0}$. *Here* $\delta = 1$ *if we assume Riemann hypothesis for* $Z(s)$, *otherwise we have to assume* $\frac{2}{3} < \sigma_0 < 1$ *and then we can take* $\delta = 1 - \frac{5(1-\sigma_0)}{(3-2\sigma_0)}$.

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References

[1] R. Balasubramanian, *On the frequency of Titchmarsh phenomenon for* $\zeta(s)$-IV, Hardy-Ramanujan J., **9** (1986), 1–10.
[2] R. Balasubramanian and K. Ramachandra, *On the frequency of Titchmarsh phenomenon for* $\zeta(s)$-III, Proc. Indian Acad. Sci., **86 A** (1977), 341–351.
[3] D.R. Heath-Brown, *On the density of the zeros of the Dedekind zeta-function*, Acta Arith., **33** (1977), 169–181.
[4] A. Ivić, *The Riemann zeta-function*, Wiley, (1985).
[5] H.L. Montgomery, *Extreme values of the Riemann zeta-function*, Comment. Math. Helv., **52** (1977), 511–518.
[6] K. Ramachandra and A. Sankaranarayanan, *Note on a paper by H.L. Montgomery*-I *(Omega theorems for the Riemann zeta-function)*, Publ. Inst. Math. (Beograd) (N.S.), **50 (64)** (1991), 51–59.
[7] R.A. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions*-I, Proc. Cambridge. Phil. Soc., **35** (1939), 351–356.
[8] R.A. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions*-II, Proc. Cambridge. Phil. Soc., **35** (1939), 357–372.
[9] A. Sankaranarayanan, *Fundamental properties of symmetric square L-functions*-I, Illinois J. Math., **46** (2002), 23–43.
[10] A. Sankaranarayanan, *Fundamental properties of symmetric square L-functions*-II, Funct. Approx. Comment. Math., **30** (2002), 89–115.
[11] A. Sankaranarayanan, *On Hecke L-functions associated with cusp forms*-II *(On the sign changes of* $S_f(T)$*)*, Annales Acad. Scientiarum Fenn., **31** (2006), no: 1, 213–238.
[12] G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London. Math. Soc., **31** (1975), 79–98.

[13] A. Selberg, *Contributions to the theory of the Riemann zeta-function* in: Collected Papers, Vol I, Springer, (1989), 214–280.

[14] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd Edition (edited by D.R. Heath-Brown), Clarendon Press (Oxford) (1986).

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