

VANISHING POHOZAEV CONSTANT AND REMOVABILITY OF SINGULARITIES

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Abstract

Conformal invariance of two-dimensional variational problems is a condition known to enable a blow-up analysis of solutions and to deduce the removability of singularities. In this paper, we identify another condition that is not only sufficient, but also necessary for such a removability of singularities. This is the validity of the Pohozaev identity. In situations where such an identity fails to hold, we introduce a new quantity, called the *Pohozaev constant*, which on one hand measures the extent to which the Pohozaev identity fails and, on the other hand, provides a characterization of the singular behavior of a solution at an isolated singularity. We apply this to the blow-up analysis for super-Liouville type equations on Riemann surfaces with conical singularities, because in the presence of such singularities, conformal invariance no longer holds and a local singularity is in general non-removable unless the Pohozaev constant is vanishing.

1. Introduction

Many variational problems of profound interest in geometry and physics are borderline cases of the Palais–Smale condition, and standard theory does not apply to deduce the existence and to control the behavior of solutions. One needs additional ingredients and tools. For two-dimensional problems, like harmonic maps from Riemann surfaces (or in physics, the nonlinear sigma model), minimal and prescribed mean curvature surfaces in Riemannian manifolds, pseudoholomorphic curves, Liouville type problems as occurring, for instance, in prescribing the Gauss curvature of a surface, Ginzburg–Landau and Toda type problems, and as inspired by quantum field theory and super string theory, Dirac-harmonic maps and super-Liouville equations, etc., it turned out that conformal invariance is a key property that enables a successful analysis. The fundamental technical aspect of all such problems is the

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existence of bubbles, that is, the concentration of solutions at isolated points. Since the fundamental work of Sacks–Uhlenbeck [29] and Wente [34], we know that even when such a bubble splits off, the remaining solution is smooth, that is, can be extended through the point where the bubble singularity had been developing. This is called blow-up analysis, and it depends on a precise characterization of the bubble type solutions. In other words, conformal invariance is a sufficient condition for such a blow-up analysis. In technical terms, conformal invariance produces a holomorphic quadratic differential. For harmonic map type problems, it is well known that finiteness of the energy functional in question implies that that differential is in L^1 . This then yields important estimates. For (super-)Liouville equations, the energy functional and the holomorphic quadratic differential are defined in a different way. Finiteness of the energy is not sufficient to get the L^1 bound of that differential and, hence, this is an extra assumption leading to the removability of local singularities (Prop 2.6, [19]).

It turns out, however, that some important problems in the class mentioned no longer satisfy conformal invariance. An example that we shall investigate in this paper are (super-)Liouville equations on surfaces with conical singularities. Another example, which we shall treat in a subsequent paper, is the super-Toda system. Also, some inhomogeneous lower order terms in a problem can destroy conformal invariance.

Thus, in order to both understand the scope of the blow-up analysis in general and to handle some concrete two-dimensional geometric variational problems, we have searched for a condition that is not only sufficient, but also necessary for the blow-up analysis. The condition that we have identified is the Pohozaev identity. This condition is already known to play a crucial role in geometric analysis (see, for instance, [30] or [3] for a version of the Pohozaev identity for non-linear eigenvalue equations of the Dirac operator on Riemannian spin manifolds with boundary), but what is new here is that we can show that this identity by itself suffices for the blow-up analysis. In fact, there are situations where this identity fails to hold. In order to handle these more complicated cases, we introduce a new quantity that is associated to a solution, called the *Pohozaev constant*. By definition, this quantity measures the extent to which the Pohozaev identity fails. In other words, that identity holds iff the Pohozaev constant vanishes. On the other hand, it turns out that this quantity also provides a characterization of the singular behavior of a solution at an isolated singularity. As already mentioned, we demonstrate the scope of this strategy at a rather difficult and subtle example, the (super-)Liouville equation on surfaces with conical singularities. We hope that the general scheme will become clear from our treatment of this particular example.

Thus, in order to get more concrete, we now introduce that example. The classical Liouville functional for a real-valued function u on a smooth Riemann surface M with conformal metric g is

$$E(u) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u - e^{2u} \right\} dv,$$

where K_g is the Gaussian curvature of M . The Euler–Lagrange equation for $E(u)$ is the Liouville equation

$$-\Delta_g u = 2e^{2u} - K_g.$$

Liouville [26] studied this equation in the plane, that is, for $K_g = 0$. The Liouville equation comes up in many problems of complex analysis and differential geometry of Riemann surfaces, for instance, the prescribing curvature problem. The interplay between the geometric and analytic aspects makes the Liouville equation mathematically very interesting.

It also occurs naturally in string theory as discovered by Polyakov [27], from the gauge anomaly in quantizing the string action. There then also is a natural supersymmetric version of the Liouville functional and equation, coupling the bosonic scalar field to a fermionic spinor field. It turns out, however, that we also obtain a very interesting mathematical structure if we consider ordinary instead of fermionic (Grassmann valued) spinor fields. Therefore, in [17], we have introduced the **super-Liouville functional**, a conformally invariant functional that couples a real-valued function and a spinor ψ on a closed smooth Riemannian surface M with conformal metric g and a spin structure,

$$E(u, \psi) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u + \langle (\not{D} + e^u)\psi, \psi \rangle - e^{2u} \right\} dv.$$

The Euler–Lagrange system for $E(u, \psi)$ is

$$\begin{cases} -\Delta_g u &= 2e^{2u} - e^u \langle \psi, \psi \rangle - K_g, \\ \not{D}_g \psi &= -e^u \psi, \end{cases} \quad \text{in } M.$$

The analysis of classical Liouville type equations was developed in [5, 24, 25, 7, 16], and the corresponding analysis for super-Liouville equations in [17, 19, 20, 21, 22]. In particular, the complete blow-up theory for sequences of solutions was established, including the energy identity for the spinor part, the blow-up value at blow-up points and the profile for a sequence of solutions at the blow-up points. For results by physicists about super-Liouville equations, we refer to [28, 1, 15].

In this paper, as an application and a test of our general scheme, we shall study super-Liouville equations on surfaces with conical singularities and establish the geometric and analytic properties for this system. For this purpose, let us first recall the definition of surfaces with conical singularities, following [32]. A conformal metric g on a Riemannian

surface M without boundary has a conical singularity of order α (a real number with $\alpha > -1$) at a point $p \in M$ if in some neighborhood of p

$$g = e^{2u}|z - z(p)|^{2\alpha}|dz|^2,$$

where z is a coordinate of M defined in this neighborhood and u is smooth away from p and continuous at p . The point p is then said to be a conical singularity of *angle* $\theta = 2\pi(1 + \alpha)$. For example, a (somewhat idealized) American football has two singularities of equal angle, while a teardrop has only one singularity. Both these examples correspond to the case $-1 < \alpha < 0$; in case $\alpha > 0$, the angle is larger than 2π , leading to a different geometric picture. Such singularities also appear in orbifolds and branched coverings. They can also describe the ends of complete Riemann surfaces with finite total curvature. If (M, g) has conical singularities of order $\alpha_1, \alpha_2, \dots, \alpha_m$ at q_1, q_2, \dots, q_m , then g is said to represent the divisor $\mathcal{A} = \sum_{j=1}^m \alpha_j q_j$. Importantly, the presence of such conical singularities destroys conformal invariance, because the conical points are different from the regular ones.

Let (M, \mathcal{A}, g) be a compact Riemann surface (without boundary) with conical singularities of divisor \mathcal{A} and with a spin structure. Associated to g , one can define the gradient ∇ and the Laplacian operator Δ in the usual way. We consider the **super-Liouville functional** on M , a conformally invariant functional that couples a real-valued function u and a spinor ψ on M

$$E(u, \psi) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u + \langle (\not{D} + e^u)\psi, \psi \rangle - e^{2u} \right\} dv_g.$$

The Euler–Lagrange system for $E(u, \psi)$ is

$$(1) \quad \begin{cases} -\Delta_g u &= 2e^{2u} - e^u \langle \psi, \psi \rangle - K_g, \\ \not{D}_g \psi &= -e^u \psi, \end{cases} \quad \text{in } M \setminus \{q_1, q_2, \dots, q_m\}.$$

When ψ vanishes, we obtain the classical Liouville equation, or the prescribing curvature equation on M with conical singularities (see [32, 12]). In [10, 11, 6, 31, 7, 9], the blow-up theory of the following Liouville type equations with singular data was systematically studied:

$$-\Delta_g u = \lambda \frac{K e^u}{\int_M K e^u dg} - 4\pi(\sum_{j=1}^m \alpha_j \delta_{q_j} - f),$$

where (M, g) is a smooth surface and the singular data appear in equation. In this paper, we aim to provide an analytic foundation for the system (1).

The local super-Liouville type system (which is deduced in Section 3) we shall study is the following:

$$(2) \quad \begin{cases} -\Delta u(x) &= 2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^\alpha e^{u(x)}|\Psi|^2, \\ \not{D}\Psi &= -V(x)|x|^\alpha e^{u(x)}\Psi, \end{cases} \quad \text{in } B_r(0).$$

Here $\alpha \geq 0$, $V(x)$ is a $C^{1,\beta}$ function satisfying $0 < a \leq V(x) \leq b$ and $B_r = B_r(0)$ is a disc in \mathbb{R}^2 . We also assume that (u, Ψ) satisfy the following energy condition:

$$(3) \quad \int_{B_r(0)} |x|^{2\alpha} e^{2u} + |\Psi|^4 dx < +\infty.$$

Our first result is the following Brezis–Merle type concentration compactness:

Theorem 1.1. *Let (u_n, Ψ_n) be a sequence of solutions satisfying*

$$(4) \quad \begin{cases} -\Delta u_n(x) &= 2V^2(x)|x|^{2\alpha_n} e^{2u_n(x)} - V(x)|x|^{\alpha_n} e^{u_n(x)} |\Psi_n|^2, \\ \not{D}\Psi_n &= -V(x)|x|^{\alpha_n} e^{u_n(x)} \Psi_n, \end{cases} \quad \text{in } B_r,$$

with the energy condition

$$(5) \quad \int_{B_r} |x|^{2\alpha_n} e^{2u_n} dx < C, \text{ and } \int_{B_r} |\Psi_n|^4 dx < C,$$

for some constant $C > 0$. Assume that

- i) $\alpha_n \in \mathbb{R}^+$, $\alpha_n \rightarrow \alpha$ with $\alpha \geq 0$,
- ii) $V \in C^{1,\beta}(B_r)$, $0 < a \leq V(x) \leq b < +\infty$.

Define

$$\begin{aligned} \Sigma_1 &= \{x \in B_r, \text{ there is a sequence } y_n \rightarrow x \text{ such that } u_n(y_n) \rightarrow +\infty\}, \\ \Sigma_2 &= \{x \in B_r, \text{ there is a sequence } y_n \rightarrow x \text{ such that } |\Psi_n(y_n)| \rightarrow +\infty\}. \end{aligned}$$

Then, we have $\Sigma_2 \subset \Sigma_1$. Moreover, (u_n, Ψ_n) admits a subsequence, still denoted by (u_n, Ψ_n) , satisfying

- a) Ψ_n is bounded in $L_{loc}^\infty(B_r \setminus \Sigma_2)$.
- b) For u_n , one of the following alternatives holds:
 - i) u_n is bounded in $L_{loc}^\infty(B_r)$.
 - ii) $u_n \rightarrow -\infty$ uniformly on compact subsets of B_r .
 - iii) Σ_1 is finite, nonempty and either

$$u_n \text{ is bounded in } L_{loc}^\infty(B_r \setminus \Sigma_1),$$

or

$$u_n \rightarrow -\infty \text{ uniformly on compact subsets of } B_r \setminus \Sigma_1.$$

The proof of this concentration result does not yet need the Pohozaev identity. But we shall then proceed to the subtler aspects of the blow-up analysis, and for that, the Pohozaev identity will play a crucial role. We shall first show that global singularities can be removed, that is, an entire solution on the plane can be conformally extended to the sphere. In the subsequent analysis, we shall show that in the blow-up process, no energy will be lost, neither in the Liouville part u_n nor in the spinor part Ψ_n . The technically longest part of our scheme (see Section 6)

consists in exploring the blow-up behavior of (4) and (5) at each blow-up point, to show that the energy identity holds for the spinor parts Ψ_n .

Theorem 1.2. *Notations and assumptions as in Theorem 1.1. Then there are finitely many bubbling solutions of (2) and (3) on \mathbb{R}^2 with $\alpha \geq 0$ and $V \equiv \text{const}$: $(u^{i,k}, \Psi^{i,k})$, $i = 1, 2, \dots, l$; $k = 1, 2, \dots, L_i$, all of which can be conformally extended to S^2 , such that, after selection of a subsequence, Ψ_n converges in C_{loc}^2 to some Ψ on $B_r(0) \setminus \Sigma_1$ and the following energy identity holds:*

$$\lim_{n \rightarrow \infty} \int_{B_r(0)} |\Psi_n|^4 dv = \int_{B_r(0)} |\Psi|^4 dv + \sum_{i=1}^l \sum_{k=1}^{L_i} \int_{S^2} |\Psi^{i,k}|^4 dv.$$

The essential step in the proof of Theorem 1.2 is the removability of a local singularity for solutions of (2) and (3) defined on a punctured disc (see Section 4).

In order to see the scope of our result, we point out that, in general, a local singularity of (u, Ψ) is not removable. For example, when $\alpha = 0$, if we set

$$(6) \quad u(x) = \log \frac{(2 + 2\beta)|x|^\beta}{1 + 2|x|^{2+2\beta}},$$

then u is a solution of

$$-\Delta u = 2e^{2u}, \quad \text{in } \mathbb{R}^2 \setminus \{0\},$$

where $\beta > -1$. Therefore, $(u, 0)$ is a solution of (2) with $\alpha = 0$ and with finite energy in $\mathbb{R}^2 \setminus \{0\}$. It is clear that $x = 0$ is a local singularity which is not removable when $\beta \neq 0$.

So, one needs to find some sufficient condition to remove the local singularity. In [19], the authors considered the following simpler case of $\alpha = 0$ and $V(x) \equiv 1$:

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle \psi, \psi \rangle, \\ \not{D}\psi &= -e^u \psi, \end{cases} \quad \text{in } B_{r_0} \setminus \{0\}.$$

In this case, they defined the following quadratic differential

$$T(z)dz^2 = \{(\partial_z u)^2 - \partial_z^2 u + \frac{1}{4}\langle \psi, dz \cdot \partial_{\bar{z}} \psi \rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z \psi, \psi \rangle\} dz^2,$$

and showed that it is holomorphic in $B_{r_0} \setminus \{0\}$. Then one observes that $\int_{B_r(0)} |T(z)| dz = +\infty$ for $(u, 0)$ in the above example (6). So, in [19], the authors proposed the assumption that $\int_{B_r(0)} |T(z)| dz \leq C$ and showed that this is a sufficient condition for the removability of a local singularity. However, in the more general case considered in this paper, namely, when $\alpha > 0$ or the coefficient function $V(x)$ is nonconstant, then we do not have such a holomorphic quadratic differential and the argument in [19] does not work. Therefore, we need to develop a new method.

To describe our new method, as it applies to the super-Liouville system, let (u, Ψ) be a solution of (2) and (3) defined on a punctured disc. We define a quantity $C(u, \Psi) \in \mathbb{R}$, called *the Pohozaev constant* associated to (u, Ψ) (see Definition 4.1). We shall show that there is a constant $\gamma < 2\pi(1 + \alpha)$ such that

$$u(x) = -\frac{\gamma}{2\pi} \log|x| + h, \quad \text{near } 0,$$

where h is bounded near 0. Moreover, we show that $C(u, \Psi)$ and γ satisfy the following relation:

$$C(u, \Psi) = \frac{\gamma^2}{4\pi}.$$

In particular, we can prove that the local singularity for (u, Ψ) is removable if and only if the associated Pohozaev constant $C(u, \Psi) = 0$, which is equivalent to the fact that the Pohozaev type identity for (u, Ψ) holds (see Theorem 4.2).

Looking back to the example (6) illustrated above, it is easy to see that the Pohozaev constant $C(u, 0) = \pi\beta^2 \neq 0$ when $\beta \neq 0$.

Moreover, applying our new method to the removability of a local singularity, we shall see in Section 7 that the energy identity for the spinor will enable us to derive

Theorem 1.3. *Notations and assumptions as in Theorem 1.1. Assume that the blow-up set $\Sigma_1 \neq \emptyset$. Then*

$$u_n \rightarrow -\infty \quad \text{uniformly on compact subsets of } B_r(0) \setminus \Sigma_1.$$

Furthermore,

$$2V(x)|x|^{2\alpha_n} e^{2u_n} - V(x)|x|^{\alpha_n} e^{u_n} |\Psi_n|^2 \rightharpoonup \sum_{x_i \in \Sigma_1} \beta_i \delta_i,$$

in the sense of distributions, and $\beta_i \geq 4\pi$ for $x_i \in \Sigma_1 \cap B_r(0) \setminus \{0\}$ and $\beta_i \geq 4\pi(1 + \alpha)$ for $x_i \in \Sigma_1 \cap \{0\}$.

To investigate further the blow-up behavior of a sequence of solutions of (4) and (5), let us define the blow-up value at a blow-up point $p \in \Sigma_1$ as follows:

$$(7) \quad m(p) = \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{B_\rho(p)} (2V^2(x)|x|^{2\alpha_n} e^{2u_n} - V(x)|x|^{\alpha_n} e^{u_n} |\Psi_n|^2) dx.$$

In Section 8, we shall then obtain

Theorem 1.4. *Notations and assumptions as in Theorem 1.1. Assume that the blow-up set $\Sigma_1 \neq \emptyset$. Let $p \in \Sigma_1$ and assume that p is the only blow-up point in $\bar{B}_{\rho_0}(p)$ for some small $\rho_0 > 0$. If*

$$(8) \quad \max_{\partial B_{\rho_0}(p)} u_n - \min_{\partial B_{\rho_0}(p)} u_n \leq C,$$

then the blow-up value $m(p) = 4\pi$ when $p \neq 0$ and $m(p) = 4\pi(1 + \alpha)$ when $p = 0$.

For the global super-Liouville equations, if we let (M, \mathcal{A}, g) be a compact Riemann surface with conical singularities represented by the divisor $\mathcal{A} = \sum_{j=1}^m \alpha_j q_j$, $\alpha_j > 0$ and with a spin structure. Writing $g = e^{2\phi} g_0$, where g_0 is a smooth metric on M , in Section 9, we can deduce from the results for the local super-Liouville equations:

Theorem 1.5. *Let (u_n, ψ_n) be a sequence of solutions of (1) with energy conditions:*

$$\int_M e^{2u_n} dg < C, \quad \int_M |\psi_n|^4 dg < C.$$

Define

$$\Sigma_1 = \{x \in M, \text{ there is a sequence } y_n \rightarrow x \text{ such that } u_n(y_n) \rightarrow +\infty\}.$$

Then there exists $G \in W^{1,q}(M, g_0) \cap C_{loc}^2(M \setminus \Sigma_1)$ with $\int_M G dg_0 = 0$ for $1 < q < 2$ such that

$$u_n + \phi - \frac{1}{|M|} \int_M (u_n + \phi) dg_0 \rightarrow G,$$

in $C_{loc}^2(M \setminus \Sigma_1)$ and weakly in $W^{1,q}(M, g_0)$. Moreover, in $\Sigma_1 = \{p_1, \dots, p_l\}$, then for $R > 0$ small such that $B_R(p_k) \cap \Sigma_1 \cap \{q_1, \dots, q_m\} = \{p_k\}$, $k = 1, 2, \dots, l$, we have

(9)

$$G(x) = \begin{cases} -\frac{1}{2\pi} m(p_k) \log d(x, p_k) + g(x), & \text{if } p_k \neq q_1, \dots, q_m, \\ -(\frac{1}{2\pi} m(p_k) - \alpha_j) \log d(x, p_k) + g(x), & \text{if } p_k = q_j, j = 1, \dots, m, \end{cases}$$

for $x \in B_R(p_k) \setminus \{p_k\}$ with $g \in C^2(B_R(p_k))$, where $d(x, p_k)$ denotes the Riemannian distance between x and p_k with respect to g_0 and

$$m(p_k) = \lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \int_{B_R(p_k)} (2e^{2(u_n + \phi)} - e^{u_n + \phi} |e^{\frac{\phi}{2}} \psi_n|^2 - K_{g_0}) dg_0.$$

It is clear from the above theorem that

$$\max_{\partial B_{\rho_0}(p)} u_n - \min_{\partial B_{\rho_0}(p)} u_n \leq C,$$

if $p \in \Sigma_1$ and p is the only blow-up point in $\bar{B}_{\rho_0}(p)$ for some small $\rho_0 > 0$. Then we get the blow-up value $m(p) = 4\pi$ when p is not a conical singularity of M and $m(p) = 4\pi(1 + \alpha)$ when p is a conical singularity of M with order α .

On the other hand, on the surface (M, \mathcal{A}, g) with the divisor $\mathcal{A} = \sum_{j=1}^m \alpha_j q_j$, $\alpha_j > 0$, by the Gauss–Bonnet formula,

$$\frac{1}{2\pi} \int_M K_g dg = \mathcal{X}(M, \mathcal{A}).$$

Here $\mathcal{X}(M, \mathcal{A})$ is the Euler characteristic of (M, \mathcal{A}) defined by

$$\mathcal{X}(M, \mathcal{A}) = \mathcal{X}(M) + |\mathcal{A}|,$$

where $\mathcal{X}(M) = 2 - 2g_M$ is the topological Euler characteristic of M itself, g_M is the genus of M and $|\mathcal{A}| = \sum_{j=1}^m \alpha_j$ is the degree of \mathcal{A} . Then we deduce that

$$\begin{aligned} \int_M 2e^{2u_n} - e^{u_n} |\psi_n|^2 dg &= \int_M 2e^{2(u_n+\phi)} - e^{u_n+\phi} |e^{\frac{\phi}{2}} \psi_n|^2 dg_0 \\ &= 4\pi(1 - g_M) + 2\pi \sum_{j=1}^m \alpha_j. \end{aligned}$$

Since the possible values of $\lim_{n \rightarrow \infty} \int_M 2e^{2(u_n+\phi)} - e^{u_n+\phi} |e^{\frac{\phi}{2}} \psi_n|^2 dg_0$ are

$$4\pi k_0 + \sum_{j=1}^m 4\pi(1 + \alpha_j)k_j,$$

for some nonnegative integers k_0 and $k_j, j = 1, \dots, m$. Therefore, we have the following:

Theorem 1.6. *Let (M, \mathcal{A}, g) be a surface with divisor*

$$\mathcal{A} = \sum_{j=1}^m \alpha_j q_j, \quad \alpha_j > 0.$$

Then

- (i) *if $4\pi(1 - g_M) + 2\pi \sum_{j=1}^m \alpha_j = 4\pi$, then the blow-up set Σ_1 contains at most one point. In particular, Σ_1 contains at most one point if $g_M = 0$ and $\mathcal{A} = 0$.*
- (ii) *if $4\pi(1 - g_M) + 2\pi \sum_{j=1}^m \alpha_j < 4\pi$, then the blow-up set $\Sigma_1 = \emptyset$.*

Remark 1.7. Our method can also be applied to deal with a sequence of solutions (u_n, Ψ_n) of the following local super-Liouville type equations with two coefficient functions

$$(10) \quad \begin{cases} -\Delta u_n(x) = 2V_n^2(x)|x|^{2\alpha_n} e^{2u_n(x)} - W_n(x)|x|^{\alpha_n} e^{u_n(x)} |\Psi_n|^2, \\ \not\Delta \Psi_n = -W_n(x)|x|^{\alpha_n} e^{u_n(x)} \Psi_n, \end{cases} \quad \text{in } B_r,$$

and satisfying the energy condition

$$(11) \quad \int_{B_r} |x|^{2\alpha_n} e^{2u_n} dx < C, \quad \text{and} \quad \int_{B_r} |\Psi_n|^4 dx < C.$$

for some constant $C > 0$, where

- i) $\alpha_n > -1$ and $\alpha_n \rightarrow \alpha > -1$,
- ii) $V_n, W_n \in C^0(\overline{B_r}), 0 < a \leq V_n(x), W_n(x) \leq b < +\infty,$
 $\|\nabla V_n\|_{L^\infty(\overline{B_r})} + \|\nabla W_n\|_{L^\infty(\overline{B_r})} \leq C.$

By slightly modifying the proofs of some analytical properties in Section 3, Section 4, Section 5 as well as Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, Theorem 1.5, the corresponding blow-up results hold (see more details in Section 10). For similar results for Liouville type equations with singular data and with $-1 < \alpha < 0$, we refer to [9].

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2. Invariance of the global system and special solutions

In this section, we start with the invariance of the global super-Liouville equations under conformal diffeomorphisms that preserve the conical points. Then, we shall provide two special solutions.

Proposition 2.1. *The functional $E(u, \psi)$ is invariant under conformal diffeomorphisms $\varphi : M \rightarrow M$ preserving the divisor, that is, $\varphi^* \mathcal{A} = \mathcal{A}$ and $\varphi^*(ds^2) = \lambda^2 ds^2$, where $\lambda > 0$ is the conformal factor of the conformal map φ . Set*

$$\begin{aligned}\tilde{u} &= u \circ \varphi - \ln \lambda, \\ \tilde{\psi} &= \lambda^{-\frac{1}{2}} \psi \circ \varphi,\end{aligned}$$

Then $E(u, \psi) = E(\tilde{u}, \tilde{\psi})$. In particular, if (u, ψ) is a solution of (1), so is $(\tilde{u}, \tilde{\psi})$.

The proof of Proposition 2.1 is the same as that of the case of $\mathcal{A} = 0$ considered in [17].

As we will see later (Section 6), however, the local super-Liouville type system (2) we shall study is not conformally invariant near the conical singularity. During the blow-up process, after suitable rescaling and translation in the domain, we can obtain bubbling solutions of (2) and (3) on \mathbb{R}^2 with $\alpha \geq 0$ and $V \equiv \text{const}$, at which point we can apply the above invariance of the global system and the singularity removability results in Section 4 and Section 5 to conclude that these bubbling solutions can be conformally extended to S^2 .

Now we present some examples of solutions of the super-Liouville equations (1). Let (M, ds^2) be the mathematical version of an American football, i.e., M is a sphere with two antipodal singularities of equal angle. From [33], (M, ds^2) is conformally equivalent to $\mathbf{C} \cup \infty$ with constant curvature $K = 1$ and conical singularities at $z = 0$ and $z = \infty$ with the same angle α , and with the conformal metric

$$\frac{(2 + 2\alpha)^2 |z|^{2\alpha} dz^2}{(1 + |z|^{2+2\alpha})^2},$$

for α being not an integer. Therefore, if we define a conformal map $\varphi : (M, ds^2) \rightarrow \mathbf{C} \cup \infty$ such that

$$(\varphi^{-1})^*(ds^2) = \frac{(2 + 2\alpha)^2 |z|^{2\alpha} dz^2}{(1 + |z|^{2+2\alpha})^2},$$

then $u = \frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log \det |d\varphi|$ are solutions of

$$-\Delta u + 1 - 2e^{2u} = 0 \quad \text{on } M \setminus \{\varphi^{-1}(0), \varphi^{-1}(\infty)\}.$$

In particular, this yields solutions of the form $(u, 0)$ of (1).

There is another example of a solution of (1). Let us recall that a *Killing spinor* is a spinor ψ satisfying

$$\nabla_X \psi = \lambda X \cdot \psi, \quad \text{for any vector field } X,$$

for some constant λ . On the standard sphere, there are Killing spinors with the Killing constant $\lambda = \frac{1}{2}$, see, for instance, [4]. Such a Killing spinor is an eigenspinor, i.e.,

$$\not{D}\psi = -\psi,$$

with constant $|\psi|^2$. Choosing a Killing spinor ψ with $|\psi|^2 = 1$, $(0, \psi)$ is a solution of (1). If we let π be the stereographic projection from $S^2 \setminus \{\textit{northpole}\}$ to the Euclidean plane \mathbb{R}^2 such that the metric of \mathbb{R}^2 is

$$\frac{4}{(1 + |x|^2)^2} |dx|^2,$$

then any Killing spinor has the form

$$\frac{v + x \cdot v}{\sqrt{1 + |x|^2}},$$

up to a translation or a dilation (see [4]). We put $\tilde{\psi} = \frac{v+x \cdot v}{\sqrt{1+|x|^2}}$. Then $(0, \psi) = (0, (\log \det |d\varphi|)^{-\frac{1}{2}} \tilde{\psi} \circ \varphi)$ is a solution of (1).

3. The local super-Liouville system

In this section, we shall first derive the local version of the super-Liouville equations. Then we shall analyze the regularity of solutions under the small energy condition. Consequently, we can prove Theorem 1.1.

It is well known that (see, e.g., [32]), in a small neighborhood $U(p)$ of a given point $p \in M$, we can define an isothermal coordinate system $x = (x_1, x_2)$ centered at p , such that p corresponds to $x = 0$ and $ds^2 = e^{2\phi} |x|^{2\alpha} (dx_1^2 + dx_2^2)$ in $B_{2r}(0) = \{(x_1^2 + x_2^2) < 2r\}$, where ϕ is smooth away from p and continuous at p . We can choose such a neighborhood small enough so that if p is a conical singular point of ds^2 , then $U(p) \cap \mathcal{A} = \{p\}$ and $\alpha > 0$, while, if p is a smooth point of ds^2 , then $U(p) \cap \mathcal{A} = \emptyset$ and $\alpha = 0$. Consequently, with respect to the isothermal coordinates, (u, ψ) satisfies

$$(12) \quad \begin{cases} -\Delta u(x) = e^{2\phi(x)} |x|^{2\alpha} (2e^{2u(x)} - e^{u(x)} |\psi|^2(x) - K_g), \\ \not{D}(e^{\frac{\phi(x)}{2}} |x|^{\frac{\alpha}{2}} \psi) = -e^{\phi(x)} |x|^\alpha e^{u(x)} (e^{\frac{\phi(x)}{2}} |x|^{\frac{\alpha}{2}} \psi), \end{cases} \quad \text{in } B_r(0).$$

Here $\Delta = \partial_{x_1 x_1}^2 + \partial_{x_2 x_2}^2$ is the usual Laplacian. The Dirac operator \not{D} is the usual one, which can be seen as the (doubled) Cauchy–Riemann operator. That is, let $e_1 = \frac{\partial}{\partial x_1}$ and $e_2 = \frac{\partial}{\partial x_2}$ be the standard orthonormal frame on \mathbb{R}^2 . A spinor field is simply a map $\Psi : \mathbb{R}^2 \rightarrow \Delta_2 = \mathbf{C}^2$, and e_1, e_2 acting on spinor fields can be identified with multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If $\Psi := \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbf{C}^2$ is a spinor field, then the Dirac operator is

$$\not{D}\Psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial g}{\partial x_1} \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_2} \end{pmatrix} = 2 \begin{pmatrix} \frac{\partial g}{\partial \bar{z}} \\ -\frac{\partial f}{\partial z} \end{pmatrix},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right).$$

For more details on Dirac operator and spin geometry, we refer to [23].

We note that the last term in the first equation of (12) is $e^{2\phi}|x|^{2\alpha}K_g$, which satisfies

$$-\Delta\phi = e^{2\phi}|x|^{2\alpha}K_g.$$

Since ϕ is continuous, elliptic regularity implies that $\phi \in W_{loc}^{2,p}$ for all $p < +\infty$ if $\alpha \geq 0$ and if the curvature K_g of M is regular enough. Therefore, by Sobolev embedding, $\phi \in C^{1,\delta}$ if $\alpha \geq 0$. If we denote $V(x) = e^\phi$ and $W(x) = e^{2\phi}|x|^{2\alpha}K_g$, then $0 < a \leq V(x) \leq b$ and $W(x)$ is in $L^p(B_r(0))$ for all $p > 1$ if the curvature K_g of M is regular enough.

Therefore, equations (12) can be rewritten as:

$$\begin{cases} -\Delta u(x) = 2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^\alpha e^{u(x)}|\Psi|^2 - W(x), \\ \not{D}\Psi = -V(x)|x|^\alpha e^{u(x)}\Psi, \end{cases} \quad \text{in } B_r(0).$$

Here $\alpha \geq 0$, $V(x)$ and $W(x)$ satisfy the following conditions:

- i) $0 < a \leq V(x) \leq b$;
- ii) $W(x) \in L^p(B_r(0))$, for all $p > 1$.

Furthermore, let $w(x)$ satisfy

$$\begin{cases} -\Delta w(x) = -W(x), & \text{in } B_r(0), \\ w(x) = 0, & \text{on } \partial B_r(0). \end{cases}$$

It is easy to see that $w(x)$ is $C^{1,\beta}$ in $B_r(0)$ for some $0 < \beta < 1$. Setting $v(x) = u(x) - w(x)$, then (v, Ψ) satisfies

$$\begin{cases} -\Delta v(x) = 2V^2(x)e^{2w(x)}|x|^{2\alpha}e^{2v(x)} - V(x)e^{w(x)}|x|^\alpha e^{v(x)}|\Psi|^2, \\ \not{D}\Psi = -V(x)e^{w(x)}|x|^\alpha e^{v(x)}\Psi, \end{cases} \quad \text{in } B_r(0).$$

Now we come to the local version of the singular super-Liouville-type equations

$$(13) \quad \begin{cases} -\Delta u(x) &= 2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^\alpha e^{u(x)}|\Psi|^2, \\ \not{D}\Psi &= -V(x)|x|^\alpha e^{u(x)}\Psi, \end{cases} \quad \text{in } B_r(0).$$

Here $V(x)$ is a $C^{1,\beta}$ function and satisfies $0 < a \leq V(x) \leq b$. We also assume that (u, Ψ) satisfy the energy condition:

$$(14) \quad \int_{B_r(0)} |x|^{2\alpha}e^{2u} + |\Psi|^4 dx < +\infty.$$

Next we consider the regularity of solutions under the energy condition. We put $B_r := B_r(0)$.

First, we define weak solutions of (13) and (14). We say that (u, Ψ) is a weak solution of (13) and (14), if $u \in W^{1,2}(B_r)$ and $\Psi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma B_r))$ satisfy

$$\begin{aligned} \int_{B_r} \nabla u \nabla \phi dx &= \int_{B_r} (2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) \phi dx, \\ \int_{B_r} \langle \Psi, \not{D}\xi \rangle dx &= - \int_{B_r} V(x)|x|^\alpha e^u \langle \Psi, \xi \rangle dx, \end{aligned}$$

for any $\phi \in C_0^\infty(B_r)$ and any spinor $\xi \in C^\infty \cap W_0^{1,\frac{4}{3}}(\Gamma(\Sigma B_r))$. A weak solution is a classical solution by the following:

Proposition 3.1. *Let (u, Ψ) be a weak solution of (13) and (14). Then $(u, \Psi) \in C^2(B_r) \times C^2(\Gamma(\Sigma B_r))$.*

Note that when $\alpha = 0$ this proposition is proved in [17] (see Proposition 4.1). When $\alpha > 0$, it is clear that we can no longer use the inequality $2 \int u^+ < \int e^{2u} < \infty$ to get the L^1 integral of u^+ . So, we need a trick, which was introduced in [10], to prove this proposition.

Proof of Proposition 3.1. By the standard elliptic method, to prove this proposition, it is sufficient to show that $u^+ \in L^\infty(B_{\frac{r}{4}})$ and $|\Psi| \in L^\infty(B_{\frac{r}{4}})$.

In fact, for the regularity of u , let us set

$$f_1 = 2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^\alpha e^{u(x)}|\Psi|^2.$$

Then we have

$$-\Delta u = f_1.$$

We consider the following Dirichlet problem

$$(15) \quad \begin{cases} -\Delta u_1 &= f_1, & \text{in } B_r, \\ u_1 &= 0, & \text{on } \partial B_r. \end{cases}$$

It is clear that $f_1 \in L^1(B_r)$. In view of Theorem 1 in [5] we have

$$(16) \quad e^{k|u_1|} \in L^1(B_r),$$

for some $k > 1$ and, in particular, $u_1 \in L^p(B_r)$ for some $p > 1$.

Let $u_2 = u - u_1$ so that $\Delta u_2 = 0$ on B_r . The mean value theorem for harmonic functions implies that

$$\|u_2^+\|_{L^\infty(B_{\frac{r}{2}})} \leq C \|u_2^+\|_{L^1(B_r)}.$$

On the other hand, it is clear that for some $t > 0$,

$$\int_{B_r(0)} \frac{1}{|x|^{2t\alpha}} dx \leq C.$$

Hence, we can choose $s = \frac{t}{t+1} \in (0, 1)$ when $\alpha > 0$ and $s = 1$ when $\alpha = 0$ such that

$$2s \int_{B_r} u^+ dx \leq \int_{B_r} e^{2su} dx \leq \left(\int_{B_r} |x|^{2\alpha} e^{2u} dx \right)^s \left(\int_{B_r} |x|^{-2t\alpha} dx \right)^{1-s} < \infty.$$

Then by using $u_2^+ \leq u^+ + |u_1|$ we obtain that $u_2^+ \in L^1(B_r)$ and, consequently,

$$(17) \quad \|u_2^+\|_{L^\infty(B_{\frac{r}{2}})} < \infty.$$

Next we rewrite f_1 as

$$f_1 = 2V^2(x)|x|^{2\alpha} e^{2u_2(x)} e^{2u_1(x)} - V(x)|x|^\alpha e^{u_2(x)} e^{u_1(x)} |\Psi|^2.$$

From (16) and (17) we have $f_1 \in L^{1+\varepsilon}(B_{\frac{r}{2}})$ for some $\varepsilon > 0$. Hence, the standard elliptic estimates imply that

$$\|u^+\|_{L^\infty(B_{\frac{r}{4}})} \leq C \|u^+\|_{L^1(B_r)} + C \|f_1\|_{L^{1+\varepsilon}(B_{\frac{r}{2}})} < \infty.$$

Since $u^+ \in L^\infty(B_{\frac{r}{4}})$, then the right hand of the equation

$$\not{D}\Psi = -V(x)|x|^\alpha e^u \Psi$$

is in $L^4(\Gamma(\Sigma B_{\frac{r}{4}}))$. Hence, we have $\Psi \in C^0(\Gamma(\Sigma B_{\frac{r}{4}}))$ and especially $|\Psi| \in L^\infty(B_{\frac{r}{4}})$. q.e.d.

Next we discuss the blow-up behavior of a sequence of solutions (u_n, Ψ_n) satisfying (4) and (5). First, we study the small energy regularity, i.e., when the energy $\int_{B_r} |x|^{2\alpha_n} e^{2u_n} dx$ is small enough, u_n will be uniformly bounded from above. Our Lemma is:

Lemma 3.2. *Let $0 < \varepsilon_0 < \pi$ be a constant. For any sequence of solutions (u_n, Ψ_n) to (4) with*

$$\int_{B_r} |x|^{2\alpha_n} e^{2u_n} dx < \varepsilon_0, \quad \int_{B_r} |\Psi_n|^4 dx < C,$$

for some fixed constant $C > 0$, we have that $\|u_n^+\|_{L^\infty(B_{\frac{r}{4}})}$ is uniformly bounded.

Proof. We are in the same situation as in Proposition 3.1. When $\alpha_n > 0$, we can no longer use the inequality $2 \int u_n^+ < \int e^{2u_n}$ to get the uniform bound of the L^1 -integral of u_n^+ . But notice that there exists a uniform constant $t > 0$ such that for all n

$$\int_{B_r} \frac{1}{|x|^{2t\alpha_n}} dx \leq C,$$

since $\alpha_n \rightarrow \alpha$ and $\alpha \geq 0$. Consequently, we obtain $s = \frac{t}{t+1} \in (0, 1)$

$$2s \int_{B_r} u_n^+ dx \leq \int_{B_r} e^{2su_n} dx \leq \left(\int_{B_r} |x|^{2\alpha_n} e^{2u_n} dx \right)^s \left(\int_{B_r} |x|^{-2t\alpha_n} dx \right)^{1-s} < C.$$

Then by a similar argument as in the proof of Lemma 4.4 in [17] we can prove this Lemma. q.e.d.

When the energy $\int_{B_r} |x|^{2\alpha_n} e^{2u_n} dx$ is large, the blow-up phenomenon may occur as in the case of a smooth domain.

Proof of Theorem 1.1. By using Lemma 3.2 and applying a similar argument as in the proof of Theorem 5.1 in [17], we can easily prove this theorem. q.e.d.

Remark 3.3. Let $v_n = u_n + \alpha_n \log |x|$, then (v_n, Ψ_n) satisfies

$$\begin{cases} -\Delta v_n(x) = 2V^2(x)e^{2v_n(x)} - V(x)e^{v_n(x)}|\Psi_n|^2 - 2\pi\alpha_n\delta_{p=0}, \\ \not{D}\Psi_n = -V(x)e^{v_n(x)}\Psi_n, \end{cases} \quad \text{in } B_r,$$

with the energy condition

$$\int_{B_r} e^{2v_n} dx < C, \quad \text{and} \quad \int_{B_r} |\Psi_n|^4 dx < C.$$

Then the two blow-up sets of u_n and v_n are the same, by using similar arguments as in [10].

4. The Pohozaev identity and removability of local singularities

This section is the heart of our paper. We shall show that a local singularity is removable if and only if the Pohozaev identity is satisfied. To express this result in compact form, we start by defining a constant that is associated to equations (13) with the constraint (14).

Definition 4.1. Let $(u, \Psi) \in C^2(B_r \setminus \{0\}) \times C^2(\Gamma(\Sigma(B_r \setminus \{0\})))$ be a solution of (13) and (14). For $0 < R < r$, we define the *Pohozaev constant* with respect to equations (13) with the constraint (14)

$$\begin{aligned} C(u, \Psi) &:= R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma \\ &\quad - (1 + \alpha) \int_{B_R(0)} (2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) dx \end{aligned}$$

$$\begin{aligned}
& +R \int_{\partial B_R(0)} V^2(x)|x|^{2\alpha} e^{2u} d\sigma \\
& - \frac{1}{2} \int_{\partial B_R(0)} \left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \right\rangle + \left\langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma \\
& - \int_{B_R(0)} (|x|^{2\alpha} e^{2u} x \cdot \nabla(V^2(x)) - |x|^\alpha e^u |\Psi|^2 x \cdot \nabla V(x)) dx,
\end{aligned}$$

where ν is the outward normal vector of $\partial B_R(0)$.

It is clear that $C(u, \Psi)$ is independent of R for $0 < R < r$.

Thus, the vanishing of the Pohozaev constant $C(u, \Psi)$ is equivalent to the *Pohozaev identity*

$$\begin{aligned}
& R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma \\
& = (1 + \alpha) \int_{B_R(0)} (2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) dx \\
& - R \int_{\partial B_R(0)} V^2(x)|x|^{2\alpha} e^{2u} d\sigma \\
& + \frac{1}{2} \int_{\partial B_R(0)} \left(\left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \right\rangle + \left\langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle \right) d\sigma \\
(18) \quad & + \int_{B_R(0)} (|x|^{2\alpha} e^{2u} x \cdot \nabla(V^2(x)) - |x|^\alpha e^u |\Psi|^2 x \cdot \nabla V(x)) dx,
\end{aligned}$$

for a solution $(u, \Psi) \in C^2(B_r) \times C^2(\Gamma(\Sigma B_r))$ of (13) and (14).

We can now formulate the main result of this section. This result says that a local singularity is removable iff the Pohozaev identity (18) holds, that is, iff the Pohozaev constant vanishes.

Theorem 4.2. (*Removability of a local singularity*) *Let $(u, \Psi) \in C^2(B_r \setminus \{0\}) \times C^2(\Gamma(\Sigma(B_r \setminus \{0\})))$ be a solution of (13) and (14). Then there is a constant $\gamma < 2\pi(1 + \alpha)$ such that*

$$u(x) = -\frac{\gamma}{2\pi} \log|x| + h, \quad \text{near } 0,$$

where h is bounded near 0. The Pohozaev constant $C(u, \Psi)$ and γ satisfy:

$$C(u, \Psi) = \frac{\gamma^2}{4\pi}.$$

In particular, $(u, \Psi) \in C^2(B_r) \times C^2(\Gamma(\Sigma B_r))$, i.e., the local singularity of (u, Ψ) is removable, iff $C(u, \Psi) = 0$.

In the remainder of this section, we shall prove the two directions of Theorem 4.2. We shall first show that for smooth solutions, the Pohozaev identity (18) holds.

Proposition 4.3. *Let $(u, \Psi) \in C^2(B_r) \times C^2(\Gamma(\Sigma B_r))$ be a solution of (13) and (14). Then, for any $0 < R < r$, the Pohozaev type identity (18) holds.*

The case where $\alpha = 0$ and $V \equiv 1$ has already been treated in [19].

Proof. For $x \in \mathbb{R}^2$, we put $x = x_1 e_1 + x_2 e_2$. We multiply all terms in (13) by $x \cdot \nabla u$ and integrate over $B_R(0)$. We obtain

$$\int_{B_R(0)} \Delta u x \cdot \nabla u dx = R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma,$$

and

$$\begin{aligned} & \int_{B_R(0)} 2V^2(x) |x|^{2\alpha} e^{2u} x \cdot \nabla u dx \\ = & R \int_{\partial B_R(0)} V^2(x) |x|^{2\alpha} e^{2u} d\sigma - (2 + 2\alpha) \int_{B_R(0)} V^2(x) |x|^{2\alpha} e^{2u} dx \\ & - \int_{B_R(0)} x \cdot \nabla(V^2(x)) |x|^{2\alpha} e^{2u} dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{B_R(0)} V(x) |x|^\alpha e^u |\Psi|^2 x \cdot \nabla u dx \\ = & R \int_{\partial B_R(0)} V(x) |x|^\alpha e^u |\Psi|^2 d\sigma - \int_{B_R(0)} |x|^\alpha e^u x \cdot \nabla(V(x) |\Psi|^2) dx \\ & - (2 + \alpha) \int_{B_R(0)} V(x) |x|^\alpha e^u |\Psi|^2 dx. \end{aligned}$$

Therefore, we get

$$\begin{aligned} & R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma \\ = & (2 + 2\alpha) \int_{B_R(0)} V^2(x) |x|^{2\alpha} e^{2u} dx - (2 + \alpha) \int_{B_R(0)} V(x) |x|^\alpha e^u |\Psi|^2 dx \\ & - R \int_{\partial B_R(0)} V^2(x) |x|^{2\alpha} e^{2u} d\sigma + R \int_{\partial B_R(0)} V(x) |x|^\alpha e^u |\Psi|^2 d\sigma \\ (19) \quad & + \int_{B_R(0)} |x|^{2\alpha} e^{2u} x \cdot \nabla(V^2(x)) - |x|^\alpha e^u x \cdot \nabla(V(x) |\Psi|^2) dx. \end{aligned}$$

On the other hand, by the Schrödinger–Lichnerowicz formula $\mathcal{D}^2 = -\Delta$ on \mathbb{R}^2 , we have

$$(20) \quad \Delta \Psi = \sum_{\alpha=1}^2 \nabla_{e_\alpha} (V(x) |x|^\alpha e^u) e_\alpha \cdot \Psi - V^2(x) |x|^{2\alpha} e^{2u} \Psi.$$

Here \cdot is the Clifford multiplication and $\{e_1, e_2\}$ is the local orthonormal basis on \mathbb{R}^2 . Using the Clifford multiplication relation

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}, \text{ for } 1 \leq i, j \leq 2,$$

and

$$\langle \Psi, \varphi \rangle = \langle e_i \cdot \Psi, e_i \cdot \varphi \rangle,$$

for any spinors $\Psi, \varphi \in \Gamma(\Sigma M)$, we know that

$$(21) \quad \langle \Psi, e_i \cdot \Psi \rangle + \langle e_i \cdot \Psi, \Psi \rangle = 0,$$

for any $i = 1, 2$. Then we multiply (20) by $x \cdot \Psi$ and integrate over $B_R(0)$ to obtain

$$\begin{aligned} & \int_{B_R(0)} \langle \Delta \Psi, x \cdot \Psi \rangle dx \\ &= \int_{B_R(0)} \sum_{\alpha, \beta=1}^2 \langle \nabla_{e_\alpha} (V(x)|x|^\alpha e^u) e_\alpha \cdot \Psi, e_\beta \cdot \Psi \rangle x_\beta \\ & \quad - V^2(x)|x|^{2\alpha} e^{2u} \langle \Psi, x \cdot \Psi \rangle dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{B_R(0)} \langle x \cdot \Psi, \Delta \Psi \rangle dx \\ &= \int_{B_R(0)} \sum_{\alpha, \beta=1}^2 \langle e_\beta \cdot \Psi, \nabla_{e_\alpha} (V(x)|x|^\alpha e^u) e_\alpha \cdot \Psi \rangle x_\beta \\ & \quad - V^2(x)|x|^{2\alpha} e^{2u} \langle x \cdot \Psi, \Psi \rangle dx. \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} & \int_{B_R(0)} \langle \Delta \Psi, x \cdot \Psi \rangle dx \\ &= \int_{\partial B_R(0)} \left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \right\rangle d\sigma - \int_{B_R(0)} V(x)|x|^\alpha e^u |\Psi|^2 dx \\ & \quad - \int_{B_R(0)} \langle \nabla \Psi, x \cdot \nabla \Psi \rangle dx, \end{aligned}$$

and, similarly, we have

$$\begin{aligned} & \int_{B_R(0)} \langle x \cdot \Psi, \Delta \Psi \rangle dx \\ &= \int_{\partial B_R(0)} \left\langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma - \int_{B_R(0)} V(x)|x|^\alpha e^u |\Psi|^2 dx \\ & \quad - \int_{B_R(0)} \langle x \cdot \nabla \Psi, \nabla \Psi \rangle dx. \end{aligned}$$

Furthermore, we also have

$$\begin{aligned}
& \int_{B_R(0)} \sum_{\alpha, \beta=1}^2 \langle \nabla_{e_\alpha}(V(x)|x|^\alpha e^u) e_\alpha \cdot \Psi, e_\beta \cdot \Psi \rangle x_\beta dx \\
& + \int_{B_R(0)} \sum_{\alpha, \beta=1}^2 \langle e_\beta \cdot \Psi, \nabla_{e_\alpha}(V(x)|x|^\alpha e^u) e_\alpha \cdot \Psi \rangle x_\beta dx \\
& = 2 \int_{B_R(0)} \sum_{\alpha=1}^2 \langle \nabla_{e_\alpha}(V(x)|x|^\alpha e^u) e_\alpha \cdot \Psi, e_\alpha \cdot \Psi \rangle x_\alpha dx \\
& = 2 \int_{B_R(0)} x \cdot \nabla(V(x)|x|^\alpha e^u) |\Psi|^2 dx \\
& = -2 \int_{B_R(0)} V(x)|x|^\alpha e^u x \cdot \nabla(|\Psi|^2) dx - 4 \int_{B_R(0)} V(x)|x|^\alpha e^u |\Psi|^2 dx \\
& \quad + 2R \int_{\partial B_R(0)} V(x)|x|^\alpha e^u |\Psi|^2 dx.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& R \int_{\partial B_R(0)} V(x)|x|^\alpha e^u |\Psi|^2 d\sigma - \int_{B_R(0)} V(x)|x|^\alpha e^u x \cdot \nabla(|\Psi|^2) dx \\
& = \frac{1}{2} \int_{\partial B_R(0)} \left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \right\rangle d\sigma + \frac{1}{2} \int_{\partial B_R(0)} \left\langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma \\
(22) \quad & + \int_{B_R(0)} V(x)|x|^\alpha e^u |\Psi|^2 dx.
\end{aligned}$$

Combining (19) and (22), we obtain our Pohozaev identity (18). q.e.d.

Proposition 4.3 also shows that $C(u, \Psi) = 0$ if (u, ψ) is classical solution of (13) with the condition (14) in B_r . For the converse, let us start with a lemma.

Lemma 4.4. *There exists $0 < \varepsilon_0 < \pi$ such that if (v, ϕ) is a solution of*

$$\begin{cases} -\Delta v & = 2h^2(x)|x|^{2\alpha} e^{2v} - h(x)|x|^\alpha e^v \langle \phi, \phi \rangle, \\ \mathcal{D}\phi & = -h(x)|x|^\alpha e^v \phi, \end{cases} \quad x \in B_{r_0} \setminus \{0\},$$

where $h(x)$ is a $C^{1,\beta}$ function satisfying $0 < a \leq h(x) \leq b$ in B_{r_0} and it satisfies

$$\int_{B_{r_0}} |x|^{2\alpha} e^{2v} dx < \varepsilon_0, \quad \int_{B_{r_0}} |\phi|^4 dx < C,$$

then for any $x \in B_{\frac{r_0}{2}}$ we have

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla \phi(x)||x|^{\frac{3}{2}} \leq C \left(\int_{B_{2|x}} |\phi|^4 dx \right)^{\frac{1}{4}}.$$

Furthermore, if we assume that $e^{2v} = O(\frac{1}{|x|^{2+2\alpha-\varepsilon}})$, then, for any $x \in B_{\frac{r_0}{2}}$, we have

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla\phi(x)||x|^{\frac{3}{2}} \leq C|x|^{\frac{1}{4C}} \left(\int_{B_{r_0}} |\phi|^4 dx \right)^{\frac{1}{4}},$$

for some positive constant C . Here ε is any sufficiently small positive number.

Proof. Set $w(x) = v(x) + \alpha \ln|x|$. Then (w, ϕ) satisfies

$$\begin{cases} -\Delta w &= 2h^2(x)e^{2w} - h(x)e^w \langle \phi, \phi \rangle, \\ \not{D}\phi &= -h(x)e^v \phi, \end{cases} \quad x \in B_{r_0} \setminus \{0\},$$

with the energy conditions

$$\int_{B_{r_0}} e^{2w} dx \leq \varepsilon_0, \quad \int_{B_{r_0}} |\phi|^4(x) dx \leq C.$$

Since $h(x)$ is a $C^{1,\beta}$ function satisfying $0 < a \leq h(x) \leq b$ in B_{r_0} , we can obtain the conclusion of this lemma by applying similar arguments as in the proof of Lemma 6.2 in [17]. q.e.d.

We shall now show the removability of a local singularity when the Pohozaev constant vanishes, thereby completing the proof of Theorem 4.2.

Proposition 4.5. (*Removability of a local singularity*) *Let $(u, \Psi) \in C^2(B_r \setminus \{0\}) \times C^2(\Gamma(\Sigma(B_r \setminus \{0\})))$ be a solution of (13) and (14). Then there is a constant $\gamma < 2\pi(1 + \alpha)$ such that*

$$u(x) = -\frac{\gamma}{2\pi} \log|x| + h, \quad \text{near } 0,$$

where h is bounded near 0. Moreover, the Pohozaev constant $C(u, \Psi)$ and γ are related by

$$C(u, \Psi) = \frac{\gamma^2}{4\pi}.$$

In particular, if $C(u, \Psi) = 0$, then $(u, \Psi) \in C^2(B_r) \times C^2(\Gamma(\Sigma B_r))$, i.e., the local singularity of (u, Ψ) is removable.

Proof. Since $\int_{B_1} |x|^{2\alpha} e^{2u} dx = \int_{B_r} |x|^{2\alpha} e^{2\tilde{u}} dx$ under the following scaling transformation

$$\begin{aligned} \tilde{u}(x) &= u(rx) - (1 + \alpha) \ln r, \\ \tilde{\Psi}(x) &= r^{-\frac{1}{2}} \Psi(rx), \end{aligned}$$

we assume for convenience that $\int_{B_r} |x|^{2\alpha} e^{2u} dx < \varepsilon_0$, where ε_0 is as in Lemma 4.4. By standard potential analysis, it follows that there is a constant γ such that

$$\lim_{|x| \rightarrow 0} \frac{u}{-\log|x|} = \frac{\gamma}{2\pi}.$$

By $\int_{B_r} |x|^{2\alpha} e^{2u} + |\Psi|^4 dx < C$ we obtain that $\gamma \leq 2\pi(1 + \alpha)$. Furthermore, by using Lemma 4.4 and by a similar argument as in the proof of Proposition 2.6 of [19], we can improve this to the strict inequality $\gamma < 2\pi(1 + \alpha)$.

Define $v(x)$ by

$$v(x) = -\frac{1}{2\pi} \int_{B_r} \log|x-y|(2V^2(y)|y|^{2\alpha}e^{2u} - V(y)|y|^\alpha e^u |\Psi|^2) dy,$$

and set $w = u - v$. It is clear that $-\Delta v = 2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2$ in B_r and $\Delta w = 0$ in $B_r \setminus \{0\}$. One can check that

$$\lim_{|x| \rightarrow 0} \frac{v(x)}{-\log|x|} = 0,$$

which implies that

$$\lim_{|x| \rightarrow 0} \frac{w(x)}{-\log|x|} = \lim_{|x| \rightarrow 0} \frac{u - v}{-\log|x|} = \frac{\gamma}{2\pi}.$$

Since w is harmonic in $B_1 \setminus \{0\}$ we have

$$w = -\frac{\gamma}{2\pi} \log|x| + w_0,$$

with a smooth harmonic function w_0 in B_r . Therefore, we have

$$u = -\frac{\gamma}{2\pi} \log|x| + v + w_0 \quad \text{near } 0.$$

Next we will compute the Pohozaev constant for (u, Ψ) . For this purpose, we want to estimate the decay of (v, Ψ) near the zero. Since

$$-\Delta v = 2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2,$$

and the two terms on the right hand $f_1(x) := 2V^2(x)|x|^{2\alpha}e^{2u(x)}$ and $f_2(x) := -V(x)|x|^\alpha e^{u(x)} |\Psi|^2(x)$ are L^1 integrable, we can obtain $e^{|v(x)|} \in L^p(B_r)$ for any $p \geq 1$. Since

$$f_1(x) = |x|^{-\frac{\gamma}{\pi} + 2\alpha} (2V^2(x)e^{2w_0(x) + 2v(x)}),$$

and

$$f_2(x) = -|x|^{-\frac{\gamma}{2\pi} + \alpha - 1} (V(x)e^{w_0(x) + v(x)} |x| |\Psi|^2(x)),$$

we set $s_1 = \frac{\gamma}{\pi} - 2\alpha$ and $s_2 = \frac{\gamma}{2\pi} - \alpha + 1$. Then $\max\{s_1, s_2\} < 2$. Since $|\Psi| \leq C|x|^{-\frac{1}{2}}$ near 0 and $w_0(x)$ is smooth in B_r , we have by Hölder's inequality that $f_1 \in L^t(B_r)$ for any $t \in (1, \frac{2}{s_1})$ if $s_1 > 0$, and $f_1 \in L^t(B_r)$ for any $t > 1$ if $s_1 \leq 0$. For f_2 , we also have $f_2 \in L^t(B_r)$ for any $t \in (1, \frac{2}{s_2})$ if $s_2 > 0$, and $f_2 \in L^t(B_r)$ for any $t > 1$ if $s_2 \leq 0$. Altogether, there exists some $t > 1$ such that $f \in L^t(B_r)$. In turn, we get that $v(x)$ is in $L^\infty(B_r)$. On the other hand, since $v(x)$ is in $L^\infty(B_r)$, it follows from Lemma 4.4 that there exists a small $\delta_0 > 0$ such that

$$|\Psi| \leq C|x|^{\delta_0 - \frac{1}{2}}, \quad \text{near } 0,$$

and

$$|\nabla\Psi| \leq C|x|^{\delta_0-\frac{3}{2}}, \quad \text{near } 0.$$

Next we estimate $\nabla v(x)$. If $s_1 < 0$ and $s_2 < 0$, then $v(x)$ is in $C^1(B_r)$. If $s_1 > 0$ or $s_2 > 0$, $\nabla v(x)$ will have a decay when $|x| \rightarrow 0$. Without loss of generality, we assume that $s_1 > 0$ and $s_1 > 0$. Denote

$$v_1(x) = -\frac{1}{2\pi} \int_{B_r} (\log|x-y|)(2V^2(y)|y|^{2\alpha}e^{2u(y)})dy,$$

and

$$v_2(x) = \frac{1}{2\pi} \int_{B_r} (\log|x-y|)(V(y)|y|^\alpha e^{u(y)}|\Psi|^2(y))dy.$$

Note that

$$\begin{aligned} |\nabla v_1(x)| &\leq \frac{1}{2\pi} \int_{B_r} \frac{1}{|x-y|} |f_1(y)| dy \\ &= \frac{1}{2\pi} \int_{\{|x-y| \geq \frac{|x|}{2}\} \cap B_r} \frac{1}{|x-y|} |f_1(y)| dy \\ &\quad + \frac{1}{2\pi} \int_{\{|x-y| \leq \frac{|x|}{2}\} \cap B_r} \frac{1}{|x-y|} |f_1(y)| dy \\ &= I_1 + I_2. \end{aligned}$$

Fix $t \in (1, \frac{2}{s_1})$ and choose $0 < \tau_1 < 1$ such that $\frac{\tau_1 t}{t-1} < 2$. Hence, we have $0 < \tau_1 < 2 - s_1$. Then by Hölder's inequality we obtain

$$\begin{aligned} I_1 &\leq \left(\int_{\{|x-y| \geq \frac{|x|}{2}\} \cap B_r} \frac{1}{|x-y|^{\frac{\tau_1 t}{t-1}}} dy \right)^{\frac{t-1}{t}} \\ &\quad \times \left(\int_{\{|x-y| \geq \frac{|x|}{2}\} \cap B_r} \frac{1}{|x-y|^{(1-\tau_1)t}} |f_1|^t dy \right)^{\frac{1}{t}} \\ &\leq \frac{C}{|x|^{1-\tau_1}}. \end{aligned}$$

For I_2 , since $y \in \{|x-y| \leq \frac{|x|}{2}\}$ implies that $|y| \geq \frac{|x|}{2}$, we can get that

$$\begin{aligned} I_2 &\leq C \int_{\{|x-y| \leq \frac{|x|}{2}\} \cap B_r} \frac{1}{|x-y||y|^{s_1}} dy \\ &\leq C|x|^{1-s_1}. \end{aligned}$$

Hence, we have

$$|\nabla v_1(x)| \leq C\left(\frac{1}{|x|^{1-\tau_1}} + |x|^{1-s_1}\right),$$

for suitable $\tau_1 \in (0, 2 - s_1)$. Similarly, we also can get that

$$|\nabla v_2(x)| \leq C\left(\frac{1}{|x|^{1-\tau_2}} + |x|^{1-s_2}\right),$$

for suitable $\tau_2 \in (0, 2 - s_2)$. In conclusion, we have

$$|\nabla v(x)| \leq C\left(\frac{1}{|x|^{1-\tau}} + |x|^{1-s}\right),$$

for suitable $\tau = \min\{\tau_1, \tau_2\}$ and $s = \max\{s_1, s_2\}$.

Now, we can compute the Pohozaev constant $C(u, \Psi)$. Since

$$\nabla u = -\frac{\gamma}{2\pi} \frac{x}{|x|^2} + \nabla(w_0 + v(x)),$$

we have for any $0 < R < r$

$$\begin{aligned} & R \int_{\partial B_R} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma \\ = & R \int_{\partial B_R} \left(\frac{x}{|x|} \cdot \nabla(w_0 + v) - \frac{\gamma}{2\pi} \frac{1}{|x|} \right)^2 d\sigma \\ & - R \int_{\partial B_R} \frac{1}{2} \left(\frac{\gamma^2}{4\pi^2} \frac{1}{|x|^2} - 2 \frac{\gamma}{2\pi} \frac{x}{|x|^2} \cdot \nabla(w_0 + v) + |\nabla(w_0 + v)|^2 \right) d\sigma \\ = & \frac{1}{4\pi} \gamma^2 - \frac{\gamma}{2\pi} R \int_{\partial B_R} \frac{x}{|x|^2} \cdot \nabla(w_0 + v) d\sigma \\ & - \int_{\partial B_R} \frac{R}{2} |\nabla(w_0 + v)|^2 - R \left(\frac{x}{|x|} \cdot \nabla(w_0 + v) \right)^2 d\sigma \\ = & \frac{1}{4\pi} \gamma^2 + o_R(1), \end{aligned}$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow 0$. We also have

$$(1 + \alpha) \int_{B_R} 2V^2(x) |x|^{2\alpha} e^{2u} - V(x) |x|^\alpha e^u |\Psi|^2 dx = o_R(1),$$

and

$$R \int_{\partial B_R} V^2(x) |x|^{2\alpha} e^{2u} d\sigma = o_R(1),$$

and

$$\int_{B_R} (|x|^{2\alpha} e^{2u} x \cdot \nabla(V^2(x)) - |x|^\alpha e^u |\Psi|^2 x \cdot \nabla V(x)) dx = o_R(1),$$

and

$$\int_{\partial B_R} \left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \nabla \Psi \right\rangle d\sigma + \int_{\partial B_R} \left\langle x \cdot \nabla \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma = o_R(1).$$

Putting all together and letting $R \rightarrow 0$, we get

$$C(u, \Psi) = \lim_{R \rightarrow 0} C(u, \Psi, R) = \frac{\gamma^2}{4\pi}.$$

Since $C(u, \Psi) = 0$ for (u, Ψ) , therefore, we get $\gamma = 0$. Then from the proof of Proposition 3.1 we have $(u, \Psi) \in C^2(B_r) \times C^2(\Gamma(\Sigma B_r))$, i.e., the local singularity of (u, Ψ) is removable. q.e.d.

5. Bubble energy

In this section, we shall analyze some properties of a “bubble”, i.e., an entire solution of (13) with finite energy and with constant coefficient function, which can be obtained after a suitable rescaling at a blow-up point. We shall obtain the asymptotic behavior of an entire solution with finite energy and show the global singularity removability. The latter means that an entire solution on \mathbb{R}^2 can be conformally extended to \mathbb{S}^2 .

Without loss of generality, we assume that $V(x) \equiv 1$ and, hence, the considered equations are

$$(23) \quad \begin{cases} -\Delta u &= 2|x|^{2\alpha}e^{2u} - |x|^\alpha e^u |\Psi|^2, \\ \not{D}\Psi &= -|x|^\alpha e^u \Psi, \end{cases} \quad \text{in } \mathbb{R}^2,$$

with $\alpha \geq 0$. The energy condition is

$$(24) \quad I(u, \Psi) = \int_{\mathbb{R}^2} (|x|^{2\alpha}e^{2u} + |\Psi|^4)dx < \infty.$$

First, let $(u, \Psi) \in H_{loc}^{1,2}(\mathbb{R}^2) \times W_{loc}^{1,\frac{4}{3}}(\Gamma(\Sigma\mathbb{R}^2))$ be a weak solution of (23) and (24), then applying similar arguments as in the proof of Proposition 3.1, we get $u^+ \in L^\infty(\mathbb{R}^2)$ and, hence, $(u, \Psi) \in C^2(\mathbb{R}^2) \times C^2(\Gamma(\Sigma\mathbb{R}^2))$.

Next, we denote by (v, Φ) the Kelvin transformation of (u, Ψ) , i.e.,

$$\begin{aligned} v(x) &= u\left(\frac{x}{|x|^2}\right) - (2 + 2\alpha) \ln |x|, \\ \Phi(x) &= |x|^{-1} \Psi\left(\frac{x}{|x|^2}\right). \end{aligned}$$

Then (v, Φ) satisfies

$$(25) \quad \begin{cases} -\Delta v &= 2|x|^{2\alpha}e^{2v} - |x|^\alpha e^v |\Phi|^2, \\ \not{D}\Phi &= -|x|^\alpha e^v \Phi, \end{cases} \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

Now we define the energy of the entire solution, i.e., the bubble energy, by

$$d = \int_{\mathbb{R}^2} 2|x|^{2\alpha}e^{2u} - |x|^\alpha e^u |\Psi|^2 dx,$$

and define a constant spinor $\xi_0 = \int_{\mathbb{R}^2} |x|^\alpha e^u \Psi dx$. It will turn out that the constant spinor ξ_0 is well defined. Then we have

Proposition 5.1. *Let (u, Ψ) be a solution of (23) and (24). Then u satisfies*

$$(26) \quad u(x) = -\frac{d}{2\pi} \ln |x| + C + O(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty,$$

$$(27) \quad \Psi(x) = -\frac{1}{2\pi} \frac{x}{|x|^2} \cdot \xi_0 + o(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty,$$

where \cdot is the Clifford multiplication, $C \in \mathbb{R}$ is some constant, and $d = 4\pi(1 + \alpha)$.

Proof. The proof of this proposition is standard, see [17, 13, 18] and the references therein. The essential facts used in this case are the Pohozaev identity (Proposition 4.3) and the decay estimate for the spinor of (25) (see Lemma 4.4). For readers' convenience, we sketch the proof here.

First, let us define

$$w(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\ln|x - y| - \ln(|y| + 1))(2|y|^{2\alpha}e^{2u} - |y|^\alpha e^u |\Psi|^2) dy.$$

Since

$$\int_{\mathbb{R}^2} (2|x|^{2\alpha}e^{2u} - |x|^\alpha e^u |\Psi|^2) dx < C,$$

it follows from the standard potential argument that

$$\frac{u(x)}{\ln|x|} \rightarrow -\frac{d}{2\pi} \quad \text{as } |x| \rightarrow +\infty.$$

Since $\int_{\mathbb{R}^2} |x|^{2\alpha}e^{2u} dx < +\infty$, the above result implies

$$d \geq 2\pi(1 + \alpha).$$

Furthermore, similarly, as in the case of the usual Liouville or super-Liouville equation [17], we can show that $d > 2\pi(1 + \alpha)$.

Secondly, from $d > 2\pi(1 + \alpha)$, we can improve the estimate for e^{2u} to

$$(28) \quad e^{2u} \leq C|x|^{-2-2\alpha-\varepsilon} \quad \text{for } |x| \text{ near } \infty.$$

Therefore, from Lemma 4.4 and the Kelvin transformation, we obtain the following asymptotic estimates of the spinor $\Psi(x)$:

$$(29) \quad |\Psi(x)| \leq C|x|^{-\frac{1}{2}-\delta_0} \quad \text{for } |x| \text{ near } \infty,$$

and

$$(30) \quad |\nabla\Psi(x)| \leq C|x|^{-\frac{3}{2}-\delta_0} \quad \text{for } |x| \text{ near } \infty,$$

for some positive number δ_0 .

Then, from (28), (29) and (30) and by some standard potential analysis in [13] and [14], we can obtain firstly

$$-\frac{d}{2\pi} \ln|x| - C \leq u(x) \leq -\frac{d}{2\pi} \ln|x| + C,$$

and, furthermore, we can get

$$u(x) = -\frac{d}{2\pi} \ln|x| + C + O(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty,$$

for some constant $C > 0$. Thus, we get the proof of (26).

Next, we want to show that $d = 4\pi(1 + \alpha)$. For sufficiently large $R > 0$, the Pohozaev identity for the solution (u, Ψ) gives

$$\begin{aligned}
 & R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma \\
 &= (1 + \alpha) \int_{B_R(0)} (2|x|^{2\alpha} e^{2u} - |x|^\alpha e^u |\Psi|^2) dx \\
 (31) \quad & -R \int_{\partial B_R(0)} |x|^{2\alpha} e^{2u} d\sigma + \frac{1}{2} \int_{\partial B_R(0)} \left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \right\rangle + \left\langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma,
 \end{aligned}$$

where ν is the outward normal vector to $\partial B_R(0)$. By (26), (29) and (30) we have

$$\lim_{R \rightarrow +\infty} R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma = \frac{1}{4\pi} d^2,$$

and

$$\lim_{R \rightarrow +\infty} R \int_{\partial B_R(0)} |x|^{2\alpha} e^{2u} d\sigma = 0,$$

and

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R} \left| \frac{\partial \Psi}{\partial \nu} \right| |x \cdot \Psi| d\sigma = 0.$$

Let $R \rightarrow \infty$ in (31), we get that

$$\frac{1}{4\pi} d^2 = (1 + \alpha)d.$$

It follows that $d = 4\pi(1 + \alpha)$.

Finally, we show (27). Noting that $d = 4\pi(1 + \alpha)$, we have

$$(32) \quad e^{2u} \leq C|x|^{-(4+4\alpha)} \quad \text{for } |x| \text{ near } \infty.$$

This implies that the constant spinor ξ_0 is well defined. By using the Green function of the Dirac operator in \mathbb{R}^2 ,

$$G(x, y) = \frac{1}{2\pi} \frac{x - y}{|x - y|^2},$$

see [2], if we set

$$\xi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \cdot |x|^\alpha e^u \Psi dy,$$

then we have $\not{D}\xi = -|x|^\alpha e^u \Psi$.

Since

$$\begin{aligned}
 |x \cdot \xi(x) - \frac{1}{2\pi} \xi_0| &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \left(\frac{x \cdot (x - y)}{|x - y|^2} + 1 \right) \cdot |y|^\alpha e^u \Psi(y) dy \right| \\
 &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \frac{(x - y) \cdot y}{|x - y|^2} \cdot |y|^\alpha e^u \Psi(y) dy \right| \\
 (33) \quad &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|y|}{|x - y|} |y|^\alpha e^u |\Psi| dy,
 \end{aligned}$$

and by (32),

$$(34) \quad |x|^\alpha |\Psi| e^u \leq C|x|^{-2-\alpha-\varepsilon} \quad \text{for } |x| \text{ near } \infty,$$

for some positive constants C and ε , we can follow the derivation of gradient estimates in [14] to get

$$(35) \quad |x \cdot \xi(x) - \frac{1}{2\pi} \xi_0| \leq C|x|^{-\varepsilon} \quad \text{for } |x| \text{ near } \infty.$$

Set $\eta(x) = \Psi(x) - \xi(x)$. Then $\not{D}\eta(x) = 0$. It follows from (29) and (35) that $\eta(x) = 0$, i.e., $\Psi(x) = \xi(x)$. Furthermore,

$$\begin{aligned} |\Psi(x) + \frac{1}{2\pi} \frac{x}{|x|^2} \cdot \xi_0| &= \left| \frac{x}{|x|^2} \cdot (x \cdot \Psi(x) - \frac{1}{2\pi} \xi_0) \right| \\ &\leq \frac{1}{|x|} |x \cdot \Psi(x) - \frac{1}{2\pi} \xi_0| \\ &\leq C|x|^{-1-\varepsilon}, \end{aligned}$$

for $|x|$ near ∞ . This proves (27). q.e.d.

Finally, we show that an entire solution can be conformally extended to \mathbb{S}^2 .

Theorem 5.2. (*Removability of a global singularity*) *Let (u, Ψ) be a $C^2(\mathbb{R}^2) \times C^2(\Gamma(\Sigma\mathbb{R}^2))$ solution of (23) and (24). Then (u, Ψ) extends conformally to a solution on \mathbb{S}^2 .*

Proof. Let (v, Φ) be the Kelvin transformation of (u, Ψ) . Then (v, Φ) satisfies (25) on $\mathbb{R}^2 \setminus \{0\}$. To prove this theorem, it is sufficient to show that (v, Φ) is smooth on \mathbb{R}^2 . Applying Proposition 5.1, we have

$$(36) \quad v(x) = \left(\frac{d}{2\pi} - (2 + 2\alpha) \right) \ln |x| + O(1) \quad \text{for } |x| \text{ near } 0.$$

Since $d = 4\pi(1 + \alpha)$, we get that v is bounded near 0. By recalling that Φ is also bounded near 0, standard elliptic theory implies that $(v, \Phi) \in C^2(\mathbb{R}^2) \times C^2(\Gamma(\Sigma\mathbb{R}^2))$. q.e.d.

6. Energy identity for spinors

In this section, which is the technically most demanding one, we shall show an energy identity for the spinors. Firstly, analogously to the case of super-Liouville equations on closed Riemann surfaces (see Lemma 3.4, [19]), we shall derive the following local estimate for the spinor part on an annulus:

Lemma 6.1. *Let (u, Ψ) be a solution of (13) and (14) on the annulus $A_{r_1, r_2} = \{x \in \mathbb{R}^2 | r_1 \leq |x| \leq r_2\}$, where $0 < r_1 < 2r_1 < \frac{r_2}{2} < r_2 < 1$.*

Then we have

$$\begin{aligned}
 & \left(\int_{A_{2r_1, \frac{r_2}{2}}} |\nabla \Psi|^{\frac{4}{3}} \right)^{\frac{3}{4}} + \left(\int_{A_{2r_1, \frac{r_2}{2}}} |\Psi|^4 \right)^{\frac{1}{4}} \\
 (37) \quad & \leq \Lambda \left(\int_{A_{r_1, r_2}} |x|^{2\alpha} e^{2u} \right)^{\frac{1}{2}} \left(\int_{A_{r_1, r_2}} |\Psi|^4 \right)^{\frac{1}{4}} + C \left(\int_{A_{r_1, 2r_1}} |\Psi|^4 \right)^{\frac{1}{4}} \\
 & \quad + C \left(\int_{A_{\frac{r_2}{2}, r_2}} |\Psi|^4 \right)^{\frac{1}{4}},
 \end{aligned}$$

for a positive constant Λ and some universal positive constant C .

Proof. In view of the second equation in (13), one can apply the L^p estimates for the Dirac operator \not{D} and use similar arguments as in the proof of Lemma 3.4 of [19] to prove the lemma. q.e.d.

Then, we can show the energy identity for the spinors – Theorem 1.2.

Proof of Theorem 1.2. We shall follow closely the arguments for the case of super-Liouville equations on closed Riemann surfaces [19]. One crucial step here is to use the local singularity removability to get a contradiction.

We assume that D_{δ_i} be a small ball which is centered at a blow-up point $x_i \in \Sigma_1$ such that $D_{2\delta_i} \cap D_{2\delta_j} = \emptyset$ for $i \neq j, i, j = 1, 2, \dots, l$, and on $B_r(0) \setminus \bigcup_{i=1}^l D_{\delta_i}$, Ψ_n converges strongly to some limit Ψ in L^4 and $\int_{B_r(0)} |\Psi|^4 < \infty$. Then, it suffices to prove that for each fixed blow-up point $x_i \in \Sigma_1$, there are solutions (u^k, ξ^k) of (13) and (14) on S^2 with $\alpha \geq 0$ and V being a constant function, $k = 1, 2, \dots, K$ such that

$$\lim_{\delta_i \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_{\delta_i}} |\Psi_n|^4 dx = \sum_{k=1}^K \int_{S^2} |\xi^k|^4 dx.$$

Without loss of generality, we assume that there is only one bubble at each blow-up point p (the general case of multiple bubbles at p can be handled by induction). Furthermore, we may assume that $p = 0$. The case of $p \neq 0$ can be handled in an analogous way and, in fact, this case is simpler, as $|x|^{2\alpha_n}$ is a smooth function near $p \neq 0$. Then what we need to prove is that there exists a bubble (u, ξ) such that

$$(38) \quad \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\delta} |\Psi_n|^4 dx = \int_{S^2} |\xi|^4 dx,$$

where D_δ is a disc of radius $\delta > 0$ centered at the blow-up point $p = 0$.

We rescale each (u_n, Ψ_n) near the blow-up point p . Choose $x_n \in \overline{D_\delta}$ such that $u_n(x_n) = \max_{\overline{D_\delta}} u_n(x)$. Then we have $x_n \rightarrow p = 0$ and $u_n(x_n) \rightarrow +\infty$. Let $\lambda_n = e^{\frac{-u_n(x_n)}{\alpha_n + 1}} \rightarrow 0$ and define $t_n = \max\{\lambda_n, |x_n|\} \rightarrow 0$. Now there are two cases: (i) $\frac{t_n}{\lambda_n} = O(1)$ as $n \rightarrow +\infty$ and (ii) $\frac{t_n}{\lambda_n} \rightarrow +\infty$ as $n \rightarrow +\infty$.

Case I: $\frac{t_n}{\lambda_n} = O(1)$ as $n \rightarrow +\infty$.

In this case, we define

$$\begin{cases} \tilde{u}_n(x) &= u_n(t_n x) + (\alpha_n + 1) \ln t_n, \\ \tilde{\Psi}_n(x) &= t_n^{\frac{1}{2}} \Psi_n(t_n x), \end{cases}$$

for any $x \in \bar{D}_{\frac{\delta}{2t_n}}$. Then $(\tilde{u}_n(x), \tilde{\Psi}_n(x))$ satisfies

$$\begin{cases} -\Delta \tilde{u}_n(x) &= 2V^2(t_n x) |x|^{2\alpha_n} e^{2\tilde{u}_n(x)} \\ &\quad - V(t_n x) |x|^{\alpha_n} e^{\tilde{u}_n(x)} |\tilde{\Psi}_n(x)|^2, \\ \not{D} \tilde{\Psi}_n(x) &= -V(t_n x) |x|^{\alpha_n} e^{\tilde{u}_n(x)} \tilde{\Psi}_n(x), \end{cases} \quad \text{in } \bar{D}_{\frac{\delta}{2t_n}},$$

with energy conditions

$$\int_{D_{\frac{\delta}{2t_n}}} \left(|x|^{2\alpha_n} e^{2\tilde{u}_n(x)} + |\tilde{\Psi}_n(x)|^4 \right) dx < C.$$

Notice that

$$\begin{aligned} 0 \leq \max_{\bar{D}_{\frac{\delta}{2t_n}}} \tilde{u}_n(x) = \tilde{u}_n\left(\frac{x_n}{t_n}\right) &= u_n(x_n) + (\alpha_n + 1) \ln t_n \\ &= -(\alpha_n + 1) \ln \lambda_n + (\alpha_n + 1) \ln t_n \leq C. \end{aligned}$$

Moreover, since the maximum point of $\tilde{u}_n(x)$, i.e., $\frac{x_n}{t_n}$, is bounded, namely $|\frac{x_n}{t_n}| \leq 1$. So by taking a subsequence, we can assume that $\frac{x_n}{t_n} \rightarrow x_0 \in \mathbb{R}^2$ with $|x_0| \leq 1$. Therefore, it follows from Theorem 1.1 that, by passing to a subsequence, $(\tilde{u}_n, \tilde{\Psi}_n)$ converges in $C_{loc}^2(\mathbb{R}^2) \times C_{loc}^2(\Gamma(\Sigma\mathbb{R}^2))$ to some $(\tilde{u}, \tilde{\Psi})$ satisfying

$$(39) \quad \begin{cases} -\Delta \tilde{u} &= 2V^2(0) |x|^{2\alpha} e^{2\tilde{u}} - V(0) |x|^{\alpha} e^{\tilde{u}} |\tilde{\Psi}|^2, \\ \not{D} \tilde{\Psi} &= -V(0) |x|^{\alpha} e^{\tilde{u}} \tilde{\Psi}, \end{cases} \quad \text{in } \mathbb{R}^2,$$

with the energy condition $\int_{\mathbb{R}^2} (|x|^{2\alpha} e^{2\tilde{u}} + |\tilde{\Psi}|^4) dx < \infty$. By Proposition 5.1, there holds

$$\int_{\mathbb{R}^2} (2V^2(0) |x|^{2\alpha} e^{2\tilde{u}} - V(0) |x|^{\alpha} e^{\tilde{u}} |\tilde{\Psi}|^2) dx = 4\pi(1 + \alpha),$$

and by the removability of a global singularity (Theorem 5.2), we get a bubbling solution of (13) and (14) on S^2 .

Case II: $\frac{t_n}{\lambda_n} \rightarrow +\infty$ as $n \rightarrow +\infty$.

In this case, necessarily $t_n = |x_n|$ and, hence, $\frac{|x_n|}{\lambda_n} \rightarrow +\infty$ as $n \rightarrow +\infty$. Set $\tau_n = \frac{e^{-u_n(x_n)}}{|x_n|^{\alpha_n}} = \lambda_n \left(\frac{\lambda_n}{|x_n|}\right)^{\alpha_n}$. Then $\tau_n \rightarrow 0$ and $\frac{|x_n|}{\tau_n} \rightarrow +\infty$, as $n \rightarrow +\infty$. Now define

$$\begin{cases} \tilde{u}_n(x) &= u_n(x_n + \tau_n x) - u_n(x_n), \\ \tilde{\Psi}_n(x) &= \tau_n^{\frac{1}{2}} \Psi_n(x_n + \tau_n x), \end{cases}$$

for any $x \in \overline{D}_{\frac{t_n \delta}{2\tau_n}}(x_n)$. Then $(\tilde{u}_n(x), \tilde{\Psi}_n(x))$ satisfies

$$\begin{cases} -\Delta \tilde{u}_n(x) &= 2V^2(x_n + \tau_n x) \left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{2\alpha_n} e^{2\tilde{u}_n(x)} \\ &\quad - V(x_n + \tau_n x) \left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{\alpha_n} e^{\tilde{u}_n(x)} |\tilde{\Psi}_n(x)|^2, \\ \not\Delta \tilde{\Psi}_n(x) &= -V(x_n + \tau_n x) \left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{\alpha_n} e^{\tilde{u}_n(x)} \tilde{\Psi}_n(x), \end{cases}$$

in $D_{\frac{t_n \delta}{2\tau_n}}(x_n)$ and with energy conditions

$$\int_{D_{\frac{t_n \delta}{2\tau_n}}} \left(\left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{2\alpha_n} e^{2\tilde{u}_n(x)} + |\tilde{\Psi}_n(x)|^4 \right) dx < C.$$

It is clear that $\tilde{u}_n(x) \leq \max_{\overline{D}_{\frac{t_n \delta}{2\tau_n}}(x_n)} \tilde{u}_n(x) = \tilde{u}_n(0) = 0$, and $\left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{2\alpha_n} \rightarrow 1$ uniformly in $C_{loc}^0(\mathbb{R}^2)$. Then from Theorem 1.1, by passing to a subsequence, $(\tilde{u}_n, \tilde{\Psi}_n)$ converges in $C_{loc}^2(\mathbb{R}^2) \times C_{loc}^2(\Gamma(\Sigma\mathbb{R}^2))$ to some $(\tilde{u}, \tilde{\Psi})$ satisfying

$$\begin{cases} -\Delta \tilde{u} &= 2V^2(0)e^{2\tilde{u}} - V(0)e^{\tilde{u}}|\tilde{\Psi}|^2, \quad \text{in } \mathbb{R}^2, \\ \not\Delta \tilde{\Psi} &= -V(0)e^{\tilde{u}}\tilde{\Psi}, \end{cases}$$

with the energy condition $\int_{\mathbb{R}^2} (e^{2\tilde{u}} + |\tilde{\Psi}|^4) dx < \infty$. By the removability of a global singularity (see Proposition 6.3 and Theorem 6.4 in [17]), there holds

$$\int_{\mathbb{R}^2} (2V^2(0)e^{2\tilde{u}} - V(0)e^{\tilde{u}}|\tilde{\Psi}|^2) dx = 4\pi,$$

and we get a bubbling solution of (13) and (14) on S^2 .

In order to prove (38) we need to estimate the energy of Ψ_n in the neck domain. We shall proceed separately for Case I and for Case II.

For **Case I**, the neck domain is

$$A_{\delta, R, n} = \{x \in \mathbb{R}^2 | t_n R \leq |x| \leq \delta\}.$$

Then to prove (38), it suffices to prove the following

$$(40) \quad \lim_{\delta \rightarrow 0} \lim_{R \rightarrow +\infty} \lim_{n \rightarrow \infty} \int_{A_{\delta, R, n}} |\Psi_n|^4 dx = 0.$$

Next we shall show two claims.

Claim I.1: For any $\epsilon > 0$, there is an $N > 1$ such that for any $n \geq N$, we have

$$\int_{D_r \setminus D_{e^{-1}r}} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) dx < \epsilon, \quad \forall r \in [et_n R, \delta].$$

To show this claim, we firstly note the following two facts:

Fact I.1: For any $\epsilon > 0$ and any $T > 0$, there exists some $N(T) > 0$ such that for any $n \geq N(T)$, we have

$$(41) \quad \int_{D_\delta \setminus D_{\delta e^{-T}}} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) dx < \epsilon.$$

Actually, since (u_n, Ψ_n) has no blow-up point in $\overline{D_{2\delta}} \setminus \{p\}$, we know that Ψ_n converges strongly to Ψ in $L^4_{loc}(\overline{D_{2\delta}} \setminus \{p\})$, and u_n will either be uniformly bounded on any compact subset of $\overline{D_{2\delta}} \setminus \{p\}$ or uniformly tend to $-\infty$ on any compact subset of $\overline{D_{2\delta}} \setminus \{p\}$.

If u_n uniformly tends to $-\infty$ on any compact subset of $\overline{D_{2\delta}} \setminus \{p\}$, it is clear that, for any given $T > 0$, there is an $N(T) > 0$ big enough such that when $n \geq N(T)$, we have

$$\int_{D_\delta \setminus D_{\delta e^{-T}}} |x|^{2\alpha_n} e^{2u_n} dx < \frac{\epsilon}{2}.$$

Moreover, since Ψ_n converges to Ψ in $L^4_{loc}(\overline{D_{2\delta}} \setminus \{p\})$ and, hence,

$$\int_{D_\delta \setminus D_{\delta e^{-T}}} |\Psi_n|^4 \rightarrow \int_{D_\delta \setminus D_{\delta e^{-T}}} |\Psi|^4.$$

For any given $\epsilon > 0$ small, we can choose $\delta > 0$ small enough such that $\int_{D_\delta} |\Psi|^4 < \frac{\epsilon}{4}$, then for any given $T > 0$, there is an $N(T) > 0$ big enough such that when $n \geq N(T)$

$$\int_{D_\delta \setminus D_{\delta e^{-T}}} |\Psi_n|^4 < \frac{\epsilon}{2}.$$

Consequently, we get (41).

If (u_n, Ψ_n) is uniformly bounded on any compact subset of $\overline{D_{2\delta}} \setminus \{p\}$, then (u_n, Ψ_n) converges to a limit solution (u, Ψ) with bounded energy $\int_{D_{2\delta}} (|x|^{2\alpha} e^{2u} + |\Psi|^4) < \infty$ strongly on any compact subset of $\overline{D_{2\delta}} \setminus \{p\}$ and, hence,

$$\int_{D_\delta \setminus D_{\delta e^{-T}}} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) \rightarrow \int_{D_\delta \setminus D_{\delta e^{-T}}} (|x|^{2\alpha} e^{2u} + |\Psi|^4).$$

Therefore, we can choose $\delta > 0$ small enough such that, for any given $\epsilon > 0$ and any given $T > 0$, there exists an $N(T) > 0$ big enough so that, when $n \geq N(T)$, (41) holds.

Fact I.2: For any small $\epsilon > 0$, and $T > 0$, we may choose an $N(T) > 0$ such that when $n \geq N(T)$

$$\begin{aligned} & \int_{D_{t_n R e^T} \setminus D_{t_n R}} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) \\ &= \int_{D_{R e^T} \setminus D_R} (|x|^{2\alpha_n} e^{2\tilde{u}_n} + |\tilde{\Psi}_n|^4) \end{aligned}$$

$$\begin{aligned} &\rightarrow \int_{D_{Re^T} \setminus D_R} (|x|^{2\alpha} e^{2\tilde{u}} + |\tilde{\Psi}|^4) \\ &< \epsilon, \end{aligned}$$

if R is big enough.

Now we can deal with **Claim I.1**. We argue by contradiction by using the above two facts. Suppose that there exists $\epsilon_0 > 0$ and a sequence $r_n \in [et_n R, \delta]$ such that

$$\int_{D_{r_n} \setminus D_{e^{-1}r_n}} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) \geq \epsilon_0.$$

Then, by the above two facts, we know that $\frac{\delta}{r_n} \rightarrow +\infty$ and $\frac{t_n R}{r_n} \rightarrow 0$, in particular, $r_n \rightarrow 0$ and $\frac{t_n}{r_n} \rightarrow 0$ as $n \rightarrow +\infty$.

Scaling again, we set

$$(42) \quad \begin{cases} v_n(x) &= u_n(r_n x) + (\alpha_n + 1) \ln r_n, \\ \varphi_n(x) &= r_n^{\frac{1}{2}} \Psi_n(r_n x). \end{cases}$$

It is clear that

$$(43) \quad \int_{(D_1 \setminus D_{e^{-1}})} (|x|^{2\alpha_n} e^{2v_n} + |\varphi_n|^4) \geq \epsilon_0,$$

and (v_n, φ_n) satisfies

$$\begin{cases} -\Delta v_n(x) &= 2V^2(r_n x) |x|^{2\alpha_n} e^{2v_n(x)} - V(r_n x) |x|^{\alpha_n} e^{v_n(x)} |\varphi_n(x)|^2, \\ \not\partial \varphi_n(x) &= -V(r_n x) |x|^{\alpha_n} e^{v_n(x)} \varphi_n(x), \end{cases}$$

in $D_{\frac{\delta}{r_n}} \setminus D_{\frac{t_n R}{r_n}}$. By Theorem 1.1, there are three possible cases:

(1). There exists some $R > 0$, some point $q \in D_R \setminus D_{\frac{1}{R}}$ and energy concentration occurs near q , namely along some subsequence

$$\lim_{n \rightarrow \infty} \int_{D_r(q)} (|x|^{2\alpha_n} e^{2v_n} + |\varphi_n|^4) \geq \epsilon_0 > 0,$$

for any small $r > 0$. In such a case, we still obtain a second bubble on S^2 by the rescaling argument. Thus, we get a contradiction to the assumption that there is only one bubble at the blow-up point p .

(2). For any $R > 0$, there is no blow-up point in $D_R \setminus D_{\frac{1}{R}}$ and v_n tends to $-\infty$ uniformly in $\overline{D_R} \setminus \overline{D_{\frac{1}{R}}}$. Then, there is a solution φ satisfying

$$\not\partial \varphi = 0, \text{ in } \mathbb{R}^2 \setminus \{0\},$$

with bounded energy $\|\varphi\|_{L^4(\mathbb{R}^2)} < \infty$, such that

$$\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\|_{L^4(D_R \setminus D_{\frac{1}{R}})} = 0, \quad \text{for any } R > 0.$$

By the same arguments as in the case of super-Liouville equations [19], we know that φ can be conformally extended to a harmonic spinor on S^2 , which has to be identically 0. This will contradict (43).

(3). For any $R > 0$, there is no blow-up point in $D_R \setminus D_{\frac{1}{R}}$ and (v_n, φ_n) is uniformly bounded in $\overline{D_R \setminus D_{\frac{1}{R}}}$. Then, there is a solution (v, φ) satisfying

$$(44) \quad \begin{cases} -\Delta v &= 2V^2(0)|x|^{2\alpha}e^{2v} - V(0)|x|^\alpha e^v |\varphi|^2, & \text{in } \mathbb{R}^2 \setminus \{0\}, \\ \not\partial \varphi &= -V(0)|x|^\alpha e^v \varphi, & \text{in } \mathbb{R}^2 \setminus \{0\}, \end{cases}$$

with finite energy $\int_{\mathbb{R}^2} (|x|^{2\alpha}e^{2v} + |\varphi|^4)dx < \infty$, such that

$$\lim_{n \rightarrow \infty} \left(\|v_n - v\|_{C^2(D_R \setminus D_{\frac{1}{R}})} + \|\varphi_n - \varphi\|_{C^2(D_R \setminus D_{\frac{1}{R}})} \right) = 0,$$

for any $R > 0$.

In this case, we shall show that the local singularities at 0 and at ∞ of (v, φ) are removable. Firstly, since (u_n, Ψ_n) satisfies (4) and (5) in $D_{2\delta}$, the following Pohozaev identity holds for any $\rho > 0$ with $r_n\rho < 2\delta$,

$$\begin{aligned} & r_n\rho \int_{\partial D_{r_n\rho}} \left| \frac{\partial u_n}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u_n|^2 d\sigma \\ &= (1 + \alpha_n) \int_{D_{r_n\rho}} (2V^2(x)|x|^{2\alpha_n}e^{2u_n} - V(x)|x|^{\alpha_n}e^{u_n}|\Psi_n|^2)dx \\ & \quad - r_n\rho \int_{\partial D_{r_n\rho}} V^2(x)|x|^{2\alpha_n}e^{2u_n} d\sigma \\ & \quad + \frac{1}{2} \int_{\partial D_{r_n\rho}} \left\langle \frac{\partial \Psi_n}{\partial \nu}, x \cdot \Psi_n \right\rangle + \left\langle x \cdot \Psi_n, \frac{\partial \Psi_n}{\partial \nu} \right\rangle d\sigma \\ & \quad + \int_{D_{r_n\rho}} (|x|^{2\alpha_n}e^{2u_n}x \cdot \nabla(V^2(x)) - |x|^{\alpha_n}e^{u_n}|\Psi_n|^2x \cdot \nabla V(x))dx. \end{aligned}$$

It follows that the associated Pohozaev constant of $(v_n(x), \varphi_n(x))$ (see (42)) satisfies

$$\begin{aligned} & C(v_n, \varphi_n) \\ &= C(v_n, \varphi_n, \rho) \\ &= \rho \int_{\partial D_\rho} \left| \frac{\partial v_n}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla v_n|^2 d\sigma \\ & \quad - (1 + \alpha_n) \int_{D_\rho} (2V^2(r_nx)|x|^{2\alpha_n}e^{2v_n} - V(r_nx)|x|^{\alpha_n}e^{v_n}|\varphi_n|^2)dx \\ & \quad + \rho \int_{\partial D_\rho} V^2(r_nx)|x|^{2\alpha_n}e^{2v_n} d\sigma \\ & \quad - \frac{1}{2} \int_{\partial D_\rho} \left\langle \frac{\partial \varphi_n}{\partial \nu}, x \cdot \varphi_n \right\rangle + \left\langle x \cdot \varphi_n, \frac{\partial \varphi_n}{\partial \nu} \right\rangle d\sigma \\ & \quad - \int_{D_\rho} (|x|^{2\alpha_n}e^{2v_n}x \cdot \nabla(V^2(r_nx)) - |x|^{\alpha_n}e^{v_n}|\varphi_n|^2x \cdot \nabla(V(r_nx))) dx \\ &= 0. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} & \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\rho} (|x|^{2\alpha_n} e^{2v_n} x \cdot \nabla(V^2(r_n x)) \\ & \quad - |x|^{\alpha_n} e^{v_n} |\varphi_n|^2 x \cdot \nabla(V(r_n x))) dx = 0. \end{aligned}$$

Since (v_n, φ_n) converges to (v, φ) in $C_{loc}^2(\mathbb{R}^2 \setminus \{0\}) \times C_{loc}^2(\Gamma(\Sigma\mathbb{R}^2 \setminus \{0\}))$, we have

$$\begin{aligned} 0 &= \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} C(v_n, \varphi_n, \rho) \\ &= \lim_{\rho \rightarrow 0} C(v, \varphi, \rho) \\ &\quad - (1 + \alpha) \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\delta} (2V^2(r_n x) |x|^{2\alpha_n} e^{2v_n} - V(r_n x) |x|^{\alpha_n} e^{v_n} |\varphi_n|^2) dx \\ &= C(v, \varphi) - (1 + \alpha)\beta. \end{aligned}$$

Here

$$\beta = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\delta} (2V^2(r_n x) |x|^{2\alpha_n} e^{2v_n} - V(r_n x) |x|^{\alpha_n} e^{v_n} |\varphi_n|^2) dx,$$

and $C(v, \varphi) = C(v, \varphi, \rho)$ is the Pohozaev constant with respect to equation (44), i.e.,

$$\begin{aligned} & C(v, \varphi) \\ &= C(v, \varphi, \rho) \\ &= \rho \int_{\partial D_\rho} \left| \frac{\partial v}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla v|^2 d\sigma \\ &\quad - (1 + \alpha) \int_{D_\rho} (2V^2(0) |x|^{2\alpha} e^{2v} - V(0) |x|^\alpha e^v |\varphi|^2) dx \\ &\quad + \rho \int_{\partial D_\rho} V^2(0) |x|^{2\alpha} e^{2v} d\sigma - \frac{1}{2} \int_{\partial D_\rho} \left\langle \frac{\partial \varphi}{\partial \nu}, x \cdot \varphi \right\rangle + \langle x \cdot \varphi, \frac{\partial \varphi}{\partial \nu} \rangle d\sigma. \end{aligned}$$

On the other hand, since (v_n, φ_n) converges to (v, φ) in $C_{loc}^2(\mathbb{R}^2 \setminus \{0\}) \times C_{loc}^2(\Gamma(\Sigma\mathbb{R}^2 \setminus \{0\}))$, we have

$$\begin{aligned} & 2V^2(r_n x) |x|^{2\alpha_n} e^{2v_n} - V(r_n x) |x|^{\alpha_n} e^{v_n} |\varphi_n|^2 \\ & \rightarrow \nu = 2V^2(0) |x|^{2\alpha} e^{2v} - V(0) |x|^\alpha e^v |\varphi|^2 + \beta \delta_{p=0} \end{aligned}$$

weakly in the sense of measures in B_R for any small $R > 0$. Using Green's representation formula for (v_n, φ_n) in B_R , we derive that

$$v(x) = -\frac{\beta}{2\pi} \log |x| + w(x) + h(x),$$

with

$$w(x) = -\frac{1}{2\pi} \int_{B_R} (\log |x-y|) (2V^2(0) |y|^{2\alpha} e^{2v(y)} - V(0) |y|^\alpha e^{v(y)} |\varphi|^2(y)) dy,$$

and

$$h(x) = \frac{1}{2\pi} \int_{\partial B_R} (\log|x-y|) \frac{\partial v(y)}{\partial \nu} dy - \frac{1}{2\pi} \int_{\partial B_R} \frac{(x-y) \cdot \nu}{|x-y|^2} v(y) dy.$$

It is clear that $h(x)$ is a regular term and $h(x) \in C^1(B_R)$ and that $w(x)$ satisfies

$$-\Delta(w(x) + h(x)) = 2V^2(0)|x|^{2\alpha} e^{2v(x)} - V(0)|x|^\alpha e^{v(x)} |\varphi|^2(x), \quad \text{in } B_R.$$

Therefore, applying similar arguments as in the proof of Proposition 4.5, we know that $w(x)$ is bounded in B_R , and, furthermore, we obtain

$$C(v, \varphi) = \frac{\beta^2}{4\pi}.$$

Thus, there holds

$$\frac{\beta^2}{4\pi} = (1 + \alpha)\beta.$$

Since $\int_{B_R} |x|^{2\alpha} e^{2v} dx < \infty$, we have $\beta \leq 2\pi(1 + \alpha)$. Therefore, we conclude that $\beta = 0$ and, hence, $C(v, \varphi) = 0$. Then, by Proposition 4.5, the singularity at 0 can be removed. Furthermore, the singularity at ∞ can be removed by applying the removability of a global singularity (see Theorem 5.2). Then we get another bubble on \mathbb{S}^2 . Thus, we get a contradiction and complete the proof of **Claim I.1**.

Claim I.2: We can separate $A_{\delta,R,n}$ into finitely many parts

$$A_{\delta,R,n} = \bigcup_{k=1}^{N_k} A_k,$$

such that on each part

$$(45) \quad \int_{A_k} |x|^{2\alpha_n} e^{2u_n} dx \leq \frac{1}{4\Lambda^2}, \quad k = 1, 2, \dots, N_k.$$

Where $N_k \leq N_0$ with N_0 being a uniform integer for all n large enough, $A_k = D_{r^{k-1}} \setminus D_{r^k}$, $r^0 = \delta$, $r^{N_k} = t_n R$, $r^k < r^{k-1}$ for $k = 1, 2, \dots, N_k$, and Λ is the constant as in Lemma 6.1.

The proof of the above claim is standard, see the case of super-Liouville equations in [19, 21] as well as the cases of other Dirac equations in [35, 36]. Here we omit it.

Now using **Claim I.1** and **Claim I.2**, we can show (40). The arguments are similar to the case of super-Liouville equations in [19, 21]. For the sake of completeness, we provide the details here.

Let $0 < \epsilon < 1$ be small, δ be small enough, and let R and n be large enough. We apply Lemma 6.1 to each part A_l and use (45) to calculate

$$\left(\int_{A_l} |\Psi_n|^4 \right)^{\frac{1}{4}} \leq \Lambda \left(\int_{D_{\epsilon r^{l-1}} \setminus D_{\epsilon^{-1} r^l}} |x|^{2\alpha_n} e^{2u_n} \right)^{\frac{1}{2}} \left(\int_{D_{\epsilon r^{l-1}} \setminus D_{\epsilon^{-1} r^l}} |\Psi_n|^4 \right)^{\frac{1}{4}}$$

$$\begin{aligned}
& + C\left(\int_{D_{er^{l-1}} \setminus D_{r^{l-1}}} |\Psi_n|^4\right)^{\frac{1}{4}} + C\left(\int_{D_{r^l} \setminus D_{e^{-1}r^l}} |\Psi_n|^4\right)^{\frac{1}{4}} \\
\leq & \Lambda\left(\left(\int_{A_l} |x|^{2\alpha_n} e^{2u_n}\right)^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{2}}\right)\left(\left(\int_{A_l} |\Psi_n|^4\right)^{\frac{1}{4}} + \epsilon^{\frac{1}{4}} + \epsilon^{\frac{1}{4}}\right) \\
& + C\epsilon^{\frac{1}{4}} \\
\leq & \Lambda\left(\int_{A_l} |x|^{2\alpha_n} e^{2u_n}\right)^{\frac{1}{2}}\left(\int_{A_l} |\Psi_n|^4\right)^{\frac{1}{4}} + C(\epsilon^{\frac{1}{4}} + \epsilon^{\frac{1}{2}} + \epsilon^{\frac{3}{4}}) \\
\leq & \frac{1}{2}\left(\int_{A_l} |\Psi_n|^4\right)^{\frac{1}{4}} + C\epsilon^{\frac{1}{4}},
\end{aligned}$$

which gives

$$(46) \quad \left(\int_{A_l} |\Psi_n|^4\right)^{\frac{1}{4}} \leq C\epsilon^{\frac{1}{4}}.$$

Then, using Lemma 6.1, (45), (46) and applying similar arguments, we obtain

$$(47) \quad \left(\int_{A_l} |\nabla \Psi_n|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq C\epsilon^{\frac{1}{4}}.$$

Summing up (46) and (47) on A_l , we conclude that

$$(48) \quad \int_{A_{\delta, R, n}} |\Psi_n|^4 + \int_{A_{\delta, R, n}} |\nabla \Psi_n|^{\frac{4}{3}} = \sum_{l=1}^{N_0} \int_{A_l} |\Psi_n|^4 + |\nabla \Psi_n|^{\frac{4}{3}} \leq C\epsilon^{\frac{1}{3}}.$$

This proves (40) and finishes the proof of theorem in this case.

For **Case II**, the neck domain is different from **Case I** and it is

$$A_{S, R, n}(x_n) = \{x \in \mathbb{R}^2 \mid \tau_n R \leq |x - x_n| \leq t_n S\}.$$

In fact, in this case, we can rescale twice to get the bubble. First, since $t_n = |x_n|$, we define the rescaling functions

$$\begin{cases} \bar{u}_n(x) & = u_n(t_n x) + (\alpha_n + 1) \ln t_n, \\ \bar{\Psi}_n(x) & = t_n^{\frac{1}{2}} \Psi_n(t_n x), \end{cases}$$

for any $x \in \bar{D}_{\frac{\delta}{2t_n}}$. Then $(\bar{u}_n(x), \bar{\Psi}_n(x))$ satisfies

$$\begin{cases} -\Delta \bar{u}_n(x) & = 2V^2(t_n x) |x|^{2\alpha_n} e^{2\bar{u}_n(x)} \\ & \quad - V(t_n x) |x|^{\alpha_n} e^{\bar{u}_n(x)} |\bar{\Psi}_n(x)|^2, \quad \text{in } \bar{D}_{\frac{\delta}{2t_n}}, \\ \not{D} \bar{\Psi}_n(x) & = -V(t_n x) |x|^{\alpha_n} e^{\bar{u}_n(x)} \bar{\Psi}_n(x), \end{cases}$$

with energy conditions

$$\int_{D_{\frac{\delta}{2t_n}}} \left(|x|^{2\alpha_n} e^{2\bar{u}_n(x)} + |\bar{\Psi}_n(x)|^4\right) dx < C.$$

Set that $y_n = \frac{x_n}{t_n}$. Noticing that $\bar{u}_n(y_n) = u_n(x_n) + (\alpha_n + 1) \ln t_n = (\alpha_n + 1) \ln t_n - (\alpha_n + 1) \ln \lambda_n \rightarrow +\infty$, we set that $\delta_n = e^{-\bar{u}_n(y_n)}$ and define the rescaling function

$$\begin{cases} \tilde{u}_n(x) &= \bar{u}_n(\delta_n x + y_n) + \ln \delta_n, \\ \tilde{\Psi}_n(x) &= \delta_n^{\frac{1}{2}} \bar{\Psi}_n(\delta_n x + y_n), \end{cases}$$

for any $\delta_n x + y_n \in \bar{D}_{\frac{\delta}{2t_n}}$. We can see that $(\tilde{u}_n, \tilde{\Psi}_n)$ is exactly the same as that defined before. Without loss of generality, we assume that $y_0 = \lim_{n \rightarrow \infty} \frac{x_n}{t_n}$. Notice that

$$\begin{aligned} \int_{D_\delta} |\Psi_n|^4 dx &= \int_{D_{\frac{\delta}{t_n}}} |\bar{\Psi}_n|^4 dx \\ &= \int_{D_{\frac{\delta}{t_n}} \setminus D_{R_1}(y_n)} |\bar{\Psi}_n|^4 dx + \int_{D_{R_1}(y_n) \setminus D_{\delta_n R_2}(y_n)} |\bar{\Psi}_n|^4 dx \\ &\quad + \int_{D_{\delta_n R_2}(y_n)} |\bar{\Psi}_n|^4 dx \\ &= \int_{D_{\frac{\delta}{t_n}} \setminus D_{R_1}(y_n)} |\bar{\Psi}_n|^4 dx \\ &\quad + \int_{D_{t_n R_1}(x_n) \setminus D_{t_n \delta_n R_2}(x_n)} |\Psi_n|^4 dx + \int_{D_{\delta_n R_2}(y_n)} |\bar{\Psi}_n|^4 dx. \end{aligned}$$

Since we have assumed that (u_n, Ψ_n) has only one bubble at the blow-up point $p = 0$, $(\bar{u}_n, \bar{\Psi}_n)$ also has only one bubble at the blow-up point $p = y_0$. Therefore, we have

$$\lim_{R_1 \rightarrow +\infty} \lim_{n \rightarrow \infty} \int_{D_{\frac{\delta}{t_n}} \setminus D_{R_1}(y_n)} |\bar{\Psi}_n|^4 dx = 0,$$

uniformly for any small δ , and since $D_{\delta_n R_2}(y_n)$ is a bubble domain, we know $A_{S,R,n}$ is the neck domain for sufficiently large $S, R > 0$, and it is sufficient to prove

$$(49) \quad \lim_{S \rightarrow +\infty} \lim_{R \rightarrow +\infty} \lim_{n \rightarrow \infty} \int_{A_{S,R,n}(x_n)} |\Psi_n|^4 dv = 0.$$

For this purpose, we shall prove two claims.

Claim II.1: for any $\epsilon > 0$, there is an $N > 1$ such that for any $n \geq N$, we have

$$(50) \quad \int_{D_r(x_n) \setminus D_{e^{-1}r}(x_n)} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) dx < \epsilon, \quad \forall r \in [e\tau_n R, t_n S].$$

To get (50), similarly, to the **Case I**, we firstly note the following two facts:

Fact II.1: For any $\epsilon > 0$ and any $T > 0$, there exists some $N(T) > 0$ such that for any $n \geq N(T)$, we have

$$\int_{D_{t_n S}(x_n) \setminus D_{t_n S e^{-T}}(x_n)} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) dx < \epsilon,$$

if S is large enough.

Fact II.2: For any small $\epsilon > 0$, and $T > 0$, we may choose an $N(T) > 0$ such that when $n \geq N(T)$

$$\begin{aligned} & \int_{D_{\tau_n R e^T}(x_n) \setminus D_{\tau_n R}(x_n)} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) \\ &= \int_{D_{R e^T} \setminus D_R} \left(\left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{2\alpha_n} e^{2\tilde{u}_n} + |\tilde{\Psi}_n|^4 \right) \\ &\rightarrow \int_{D_{R e^T} \setminus D_R} (e^{2\tilde{u}} + |\tilde{\Psi}|^4) \\ &< \epsilon, \end{aligned}$$

if R is large enough.

Now we argue by contradiction to show (50) by using the above two facts. We assume that there exists $\epsilon_0 > 0$ and a sequence $r_n \in [e\tau_n R, t_n S]$ such that

$$\int_{D_{r_n}(x_n) \setminus D_{e^{-1}r_n}(x_n)} (|x|^{2\alpha_n} e^{2u_n} + |\Psi_n|^4) \geq \epsilon_0.$$

Then, by the above two facts, we know that $\frac{t_n S}{r_n} \rightarrow +\infty$ and $\frac{\tau_n R}{r_n} \rightarrow 0$, in particular, $r_n \rightarrow 0$ as $n \rightarrow +\infty$. Note that $|\frac{x_n}{r_n}| = |\frac{t_n}{r_n}| \rightarrow +\infty$ as $n \rightarrow \infty$. We define

$$(51) \quad \begin{cases} v_n(x) &= u_n(r_n x + x_n) + \ln(r_n |x_n|^{\alpha_n}), \\ \varphi_n(x) &= r_n^{\frac{1}{2}} \Psi_n(r_n x + x_n). \end{cases}$$

Then (v_n, φ_n) satisfies

$$\begin{cases} -\Delta v_n(x) &= 2V^2(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{2\alpha_n} e^{2v_n(x)} \\ &\quad - V(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{\alpha_n} e^{v_n(x)} |\varphi_n|^2, \\ \mathcal{D}\varphi_n(x) &= -V(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{\alpha_n} e^{v_n(x)} \varphi_n(x), \end{cases}$$

in $D_{\frac{t_n S}{r_n}} \setminus D_{\frac{\tau_n R}{r_n}}$, and

$$\int_{e^{-1} \leq |x| \leq 1} \left(\left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{2\alpha_n} e^{2v_n} + |\varphi_n|^4 \right) \geq \epsilon_0.$$

By Theorem 1.1, there are three possible cases. However, similarly, to the Case I, we can rule out the first and the second possible cases. If the third case happens, then for any $R > 0$, there is no blow-up point in

$D_R \setminus D_{\frac{1}{R}}$ and (v_n, φ_n) is uniformly bounded in $\overline{D_R \setminus D_{\frac{1}{R}}}$. Then, there is a solution (v, φ) satisfying

$$(52) \quad \begin{cases} -\Delta v &= 2V^2(0)e^{2v} - V(0)e^v|\varphi|^2, & \text{in } \mathbb{R}^2 \setminus \{0\}, \\ \not{D}\varphi &= -V(0)e^v\varphi, & \text{in } \mathbb{R}^2 \setminus \{0\}, \end{cases}$$

with finite energy $\int_{\mathbb{R}^2}(e^{2v} + |\varphi|^4)dx < \infty$, such that

$$\lim_{n \rightarrow \infty} \left(\|v_n - v\|_{C^2(D_R \setminus D_{\frac{1}{R}})} + \|\varphi_n - \varphi\|_{C^2(D_R \setminus D_{\frac{1}{R}})} \right) = 0,$$

for any $R > 0$.

Next we shall use the Pohozaev identity to remove the two singularities to get another bubble.

Firstly, since (u_n, Ψ_n) satisfies (4) and (5) in $D_{2\delta}$, the following Pohozaev identity holds for any $\rho > 0$ with $r_n\rho < t_n$,

$$\begin{aligned} & r_n\rho \int_{\partial D_{r_n\rho}(x_n)} \left| \frac{\partial u_n}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u_n|^2 d\sigma \\ &= \int_{D_{r_n\rho}(x_n)} (2V^2(x)|x|^{2\alpha_n}e^{2u_n} - V(x)|x|^{\alpha_n}e^{u_n}|\Psi_n|^2) dx \\ & \quad - r_n\rho \int_{\partial D_{r_n\rho}(x_n)} V^2(x)|x|^{2\alpha_n}e^{2u_n} d\sigma \\ & \quad + \frac{1}{2} \int_{\partial D_{r_n\rho}(x_n)} \left\langle \frac{\partial \Psi_n}{\partial \nu}, (x - x_n) \cdot \Psi_n \right\rangle + \left\langle (x - x_n) \cdot \Psi_n, \frac{\partial \Psi_n}{\partial \nu} \right\rangle d\sigma \\ & \quad + \int_{D_{r_n\rho}(x_n)} e^{2u_n} (x - x_n) \cdot \nabla (V^2(x)|x|^{2\alpha_n}) dx \\ & \quad - \int_{D_{r_n\rho}(x_n)} e^{u_n} |\Psi_n|^2 (x - x_n) \cdot \nabla (V(x)|x|^{\alpha_n}) dx. \end{aligned}$$

Here we have used the fact that $|x|^{2\alpha_n}$ is smooth in $D_{r_n\rho}(x_n) \subset \mathbb{R}^2 \setminus \{0\}$.

Noticing again that

$$\begin{cases} v_n(x) &= u_n(r_nx + x_n) + \ln(r_n|x_n|^{\alpha_n}), \\ \varphi_n(x) &= r_n^{\frac{1}{2}}\Psi_n(r_nx + x_n). \end{cases}$$

Hence, the Pohozaev constant associated with (v_n, φ_n) (see definition (51)) satisfies

$$\begin{aligned} & C(v_n, \varphi_n) \\ &= C(v_n, \varphi_n, \rho) \\ &= \rho \int_{\partial D_\rho} \left| \frac{\partial v_n}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla v_n|^2 d\sigma \\ & \quad - \int_{D_\rho} 2V^2(r_nx + x_n) \left| \frac{x_n}{|x_n|} \right| + \frac{r_n}{|x_n|} |x|^{2\alpha_n} e^{2v_n} dx \end{aligned}$$

$$\begin{aligned}
& + \int_{D_\rho} V(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{\alpha_n} e^{v_n} |\varphi_n|^2 dx \\
& + \rho \int_{\partial D_\rho} V^2(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{2\alpha_n} e^{2v_n} d\sigma \\
& - \frac{1}{2} \int_{\partial D_\rho} \left\langle \frac{\partial \varphi_n}{\partial \nu}, x \cdot \varphi_n \right\rangle + \left\langle x \cdot \varphi_n, \frac{\partial \varphi_n}{\partial \nu} \right\rangle d\sigma \\
& - \int_{D_\rho} e^{2v_n} x \cdot \nabla \left(V^2(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{2\alpha_n} \right) dx \\
& + \int_{D_\rho} e^{v_n} |\varphi_n|^2 x \cdot \nabla \left(V(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{\alpha_n} \right) dx \\
& = 0.
\end{aligned}$$

Note that (v_n, φ_n) converges to (v, φ) in $C_{loc}^2(\mathbb{R}^2 \setminus \{0\}) \times C_{loc}^2(\Gamma(\Sigma\mathbb{R}^2 \setminus \{0\}))$ and $\left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{\alpha_n}$ is a smooth function in D_δ for $\delta > 0$ small enough. Therefore, we have

$$\begin{aligned}
0 & = \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} C(v_n, \varphi_n, \rho) \\
& = \lim_{\rho \rightarrow 0} C(v, \varphi, \rho) \\
& \quad - \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\delta} \left(2V^2(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{2\alpha_n} e^{2v_n} \right. \\
& \quad \left. - V(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n x}{|x_n|} \right|^{\alpha_n} e^{v_n} |\varphi_n|^2 \right) dx \\
& = C(v, \varphi) - \beta.
\end{aligned}$$

Here

$$\begin{aligned}
\beta & = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\delta} \left(2V^2(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{2\alpha_n} e^{2v_n} \right. \\
& \quad \left. - V(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{\alpha_n} e^{v_n} |\varphi_n|^2 \right) dx,
\end{aligned}$$

and $C(v, \varphi)$ is the Pohozaev constant with respect to (52), i.e.,

$$\begin{aligned}
C(v, \varphi) & = \rho \int_{\partial D_\rho} \left| \frac{\partial v}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla v|^2 d\sigma \\
& \quad - \int_{D_\rho} (2V^2(0)e^{2v} - V(0)e^v |\varphi|^2) dx \\
& \quad + \rho \int_{\partial D_\rho} V^2(0)e^{2v} d\sigma - \frac{1}{2} \int_{\partial D_\rho} \left\langle \frac{\partial \varphi}{\partial \nu}, x \cdot \varphi \right\rangle + \left\langle x \cdot \varphi, \frac{\partial \varphi}{\partial \nu} \right\rangle d\sigma.
\end{aligned}$$

On the other hand, since (v_n, φ_n) converges to (v, φ) in $C_{loc}^2(\mathbb{R}^2 \setminus \{0\}) \times C_{loc}^2(\Gamma(\Sigma\mathbb{R}^2 \setminus \{0\}))$, we have

$$\begin{aligned}
 & 2V^2(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{2\alpha_n} e^{2v_n} \\
 & - V(r_n x + x_n) \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{\alpha_n} e^{v_n} |\varphi_n|^2 \\
 \rightarrow \quad & \nu = 2V^2(0)e^{2v} - V(0)e^v |\varphi|^2 + \beta \delta_{p=0}
 \end{aligned}$$

weakly in the sense of measures in B_R for any sufficient small $R > 0$. Then, applying similar arguments as in **Case I**, we can show that

$$v(x) = -\frac{\beta}{2\pi} \log|x| + w(x) + h(x),$$

with $w(x)$ being a bounded term and $h(x)$ being a regular term and, furthermore, we have

$$C(v, \varphi) = \frac{\beta^2}{4\pi}.$$

Hence, there holds

$$\frac{\beta^2}{4\pi} = \beta.$$

Since $\int_{B_R} e^{2v} dx < \infty$, we have $\beta \leq 2\pi$. Therefore, we deduce that $C(v, \varphi) = 0$, $\beta = 0$ and, hence, the singularities at 0 and ∞ of (52) can be removed. Then we get another bubble on \mathbb{S}^2 . Thus, we get a contradiction and complete the proof of (50).

Next, similarly, to **Case I**, we can prove the following:

Claim II.2: We can separate $A_{\delta, R, n}(x_n)$ into finitely many parts

$$A_{\delta, R, n}(x_n) = \bigcup_{k=1}^{N_k} A_k,$$

such that on each part

$$(53) \quad \int_{A_k} |x|^{2\alpha_n} e^{2u_n} dx \leq \frac{1}{4\Lambda^2}, \quad k = 1, 2, \dots, N_k,$$

where $N_k \leq N_0$ with N_0 being an uniform integer for all n large enough, $A_k = D_{r^{k-1}}(x_n) \setminus D_{r^k}(x_n)$, $r^0 = t_n S$, $r^{N_k} = \tau_n R$, $r^k < r^{k-1}$ for $k = 1, 2, \dots, N_k$, and $\Lambda > 0$ is the constant as in Lemma 6.1.

Then, we can use **Claim II.1** and **Claim II.2** to show (49). This finishes the proof of the theorem in the second case. q.e.d.

7. Blow-up behavior

With the energy identity for spinors in place, we can now rule out the possibility that u_n is uniformly bounded in $L^\infty_{loc}(B_r \setminus \Sigma_1)$ in Theorem 1.1 and, hence, the result can be improved.

Proof of Theorem 1.3. We shall prove this by contradiction. Assume that the conclusion of the theorem is false. Then by Theorem 1.1, u_n is uniformly bounded in L^∞ on any compact subset of $B_r(0) \setminus \Sigma_1$.

Since (u_n, Ψ_n) is a sequence of solutions to (4) with uniformly bounded energy (5), by classical elliptic estimates for both the Laplacian Δ and the Dirac operator \mathcal{D} , we know that (u_n, Ψ_n) converges in C^2 on any compact subset of $B_r(0) \setminus \Sigma_1$ to some limit solution (u, Ψ) of (13) with bounded energy $\int_{B_r(0)} (|x|^{2\alpha} e^{2u} + |\Psi|^4) < +\infty$.

Since the blow-up set Σ_1 is not empty, we can take a point $p \in \Sigma_1$. Choose a small $\delta_0 > 0$ such that p is the only point of Σ_1 in $\overline{B_{2\delta_0}}(p) \subset B_r(0)$. Without loss of generality, we assume that $p = 0$. The case of $p \neq 0$ can be handled in an analogous way.

We shall first show that the limit (u, Ψ) is C^2 at the isolated singularity $p = 0$. In fact, since (u_n, Ψ_n) satisfies the Pohozaev identity on D_ρ for $0 < \rho < \delta_0$, the Pohozaev constant $C(u_n, \Psi_n) = C(u_n, \Psi_n, \rho)$ satisfies

$$\begin{aligned}
0 &= C(u_n, \Psi_n) \\
&= C(u_n, \Psi_n, \rho) \\
&= \rho \int_{\partial D_\rho} \left| \frac{\partial u_n}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u_n|^2 d\sigma \\
&\quad - (1 + \alpha_n) \int_{D_\rho} (2V^2(x) |x|^{2\alpha_n} e^{2u_n} - V(x) |x|^{\alpha_n} e^{u_n} |\Psi_n|^2) dx \\
&\quad + \rho \int_{\partial D_\rho} V^2(x) |x|^{2\alpha_n} e^{2u_n} d\sigma \\
&\quad - \frac{1}{2} \int_{\partial D_\rho} \left\langle \frac{\partial \Psi_n}{\partial \nu}, x \cdot \Psi_n \right\rangle + \left\langle x \cdot \Psi_n, \frac{\partial \Psi_n}{\partial \nu} \right\rangle d\sigma \\
&\quad - \int_{D_\rho} (|x|^{2\alpha_n} e^{2u_n} x \cdot \nabla(V^2(x)) - |x|^{\alpha_n} e^{u_n} |\Psi_n|^2 x \cdot \nabla V(x)) dx.
\end{aligned}$$

Since (u_n, Ψ_n) converges to (u, Ψ) in C^2 on any compact subset of $\overline{B_{2\delta_0}} \setminus \{0\}$, we have

$$\begin{aligned}
0 &= \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} C(u_n, \Psi_n, \rho) \\
&= \lim_{\rho \rightarrow 0} C(u, \Psi, \rho) \\
&\quad - (1 + \alpha) \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\delta} (2V^2(x) |x|^{2\alpha_n} e^{2u_n} - V(x) |x|^{\alpha_n} e^{u_n} |\Psi_n|^2) dx \\
&= C(u, \Psi) - (1 + \alpha)\beta,
\end{aligned}$$

where

$$\begin{aligned}
C(u, \Psi) &= C(u, \Psi, \rho) \\
&= \rho \int_{\partial D_\rho} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma
\end{aligned}$$

$$\begin{aligned}
 & - (1 + \alpha) \int_{D_\rho} (2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) dx \\
 & + \rho \int_{\partial D_\rho} V^2(x)|x|^{2\alpha} e^{2u} d\sigma \\
 & - \frac{1}{2} \int_{\partial D_\rho} \langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \rangle + \langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \rangle d\sigma \\
 & - \int_{D_\rho} (|x|^{2\alpha} e^{2u} x \cdot \nabla(V^2(x)) - |x|^\alpha e^u |\Psi|^2 x \cdot \nabla V(x)) dx,
 \end{aligned}$$

and

$$\beta = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_{D_\delta} (2V^2(x)|x|^{2\alpha_n} e^{2u_n} - V(x)|x|^{\alpha_n} e^{u_n} |\Psi_n|^2) dx.$$

Moreover, we can also assume that

$$\begin{aligned}
 & 2V^2(x)|x|^{2\alpha_n} e^{2u_n} - V(x)|x|^{\alpha_n} e^{u_n} |\Psi_n|^2 \\
 & \rightarrow \nu = 2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2 + \beta \delta_{p=0},
 \end{aligned}$$

in the sense of distributions in B_R for any small $R > 0$. Then, applying similar arguments as in the proof of the local singularity removability in Claim I.1, Theorem 1.2, we can show that $C(u, \Psi) = 0$, $\beta = 0$ and, hence, (u, Ψ) is a C^2 solution of (13) on $B_{2\delta_0}$ with bounded energy

$$\int_{B_{2\delta_0}} (|x|^{2\alpha} e^{2u} + |\Psi|^4) < +\infty.$$

Now we can choose some small $\delta_1 \in (0, \delta_0)$ such that for any $\delta \in (0, \delta_1)$,

$$(54) \quad \int_{B_\delta} (2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) dx < \min\left\{\frac{1+\alpha}{10}, \frac{1}{10}\right\}.$$

Next, as in the proof of Theorem 1.2, we rescale (u_n, Ψ_n) near $p = 0$. Choose $x_n \in B_{\delta_1}$ with $u_n(x_n) = \max_{\bar{B}_{\delta_1}} u_n(x)$. Then we have $x_n \rightarrow p$ and $u_n(x_n) \rightarrow +\infty$. Let $\lambda_n = e^{\frac{-u_n(x_n)}{\alpha_n+1}} \rightarrow 0$ and denote $t_n = \max\{\lambda_n, |x_n|\} \rightarrow 0$. We distinguish the following two cases:

Case I: $\frac{t_n}{\lambda_n} = O(1)$, as $n \rightarrow +\infty$.

In this case, the rescaling functions are

$$\begin{cases} \tilde{u}_n(x) &= u_n(t_n x) + (\alpha_n + 1) \ln t_n, \\ \tilde{\Psi}_n(x) &= t_n^{\frac{1}{2}} \Psi_n(t_n x), \end{cases}$$

for any $x \in \bar{D}_{\frac{\delta_1}{2t_n}}$. And by passing to a subsequence, $(\tilde{u}_n, \tilde{\Psi}_n)$ converges in $C_{loc}^2(\mathbb{R}^2)$ to some $(\tilde{u}, \tilde{\Psi})$ satisfying

$$\begin{cases} -\Delta \tilde{u} &= 2V^2(0)|x|^{2\alpha} e^{2\tilde{u}} - V(0)|x|^\alpha e^{\tilde{u}} |\tilde{\Psi}|^2, \\ \not{D} \tilde{\Psi} &= -V(0)|x|^\alpha e^{\tilde{u}} \tilde{\Psi}, \end{cases} \quad \text{in } \mathbb{R}^2,$$

with

$$(55) \quad \int_{\mathbb{R}^2} (2V^2(0)|x|^{2\alpha}e^{2\tilde{u}} - V(0)|x|^\alpha e^{\tilde{u}}|\tilde{\Psi}|^2)dx = 4\pi(1 + \alpha).$$

Then for $\delta \in (0, \delta_1)$ small enough, $R > 0$ large enough and n large enough, we have

$$\begin{aligned} & \int_{B_\delta} (2V^2(x)|x|^{2\alpha_n}e^{2u_n} - V(x)|x|^{\alpha_n}e^{u_n}|\Psi_n|^2)dx \\ = & \int_{B_{t_n R}} (2V^2(x)|x|^{2\alpha_n}e^{2u_n} - V(x)|x|^{\alpha_n}e^{u_n}|\Psi_n|^2)dx \\ & + \int_{B_\delta \setminus B_{t_n R}} (2V^2(x)|x|^{2\alpha_n}e^{2u_n} - V(x)|x|^{\alpha_n}e^{u_n}|\Psi_n|^2)dx \\ \geq & \int_{B_R} (2V^2(t_n x)|x|^{2\alpha_n}e^{2\tilde{u}_n} - V(t_n x)|x|^{\alpha_n}e^{\tilde{u}_n}|\tilde{\Psi}_n|^2) \\ & - \int_{B_\delta \setminus B_{t_n R}} V(x)|x|^{\alpha_n}e^{u_n}|\Psi_n|^2 \\ (56) \quad \geq & 4\pi(1 + \alpha) - \frac{1 + \alpha}{10}. \end{aligned}$$

Here in the last step, we have used (55) and the fact from Theorem 1.2 that the neck energy of the spinor field Ψ_n is converging to zero. We remark that in the above estimate, if there are multiple bubbles then we need to decompose $B_\delta \setminus B_{t_n R}$ further into bubble domains and neck domains and then apply the no neck energy result in Theorem 1.2 to each of these neck domains.

On the other hand, we fix some $\delta \in (0, \delta_1)$ small such that (56) holds and then let $n \rightarrow \infty$ to conclude that

$$\begin{aligned} 4\pi(1 + \alpha) - \frac{1 + \alpha}{10} & \leq \int_{B_\delta} (2V^2(x)|x|^{2\alpha_n}e^{2u_n} - V(x)|x|^{\alpha_n}e^{u_n}|\Psi_n|^2)dx \\ & = - \int_{B_\delta} \Delta u_n = - \int_{\partial B_\delta} \frac{\partial u_n}{\partial n} \\ & \rightarrow - \int_{\partial B_\delta} \frac{\partial u}{\partial n} = - \int_{B_\delta} \Delta u \\ & = \int_{B_\delta} (2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^\alpha e^u|\Psi|^2)dx < \frac{1 + \alpha}{10}. \end{aligned}$$

Here in the last step, we have used (54). Thus, we get a contradiction and finish the proof of the Theorem in this case.

Case II: $\frac{t_n}{\lambda_n} \rightarrow +\infty$, as $n \rightarrow \infty$.

In this case, we should rescale twice to get the bubble. First, since $t_n = |x_n|$, we define the rescaling functions

$$\begin{cases} \bar{u}_n(x) &= u_n(t_n x) + (\alpha_n + 1) \ln t_n, \\ \bar{\Psi}_n(x) &= t_n^{\frac{1}{2}} \Psi_n(t_n x), \end{cases}$$

for any $x \in \bar{D}_{\frac{\delta}{2t_n}}$. Set $y_n := \frac{x_n}{t_n}$. Noticing that $\bar{u}_n(y_n) \rightarrow +\infty$, we set that $\delta_n = e^{-\bar{u}_n(y_n)}$ and define the rescaling function

$$\begin{cases} \tilde{u}_n(x) &= \bar{u}_n(\delta_n x + y_n) + \ln \delta_n, \\ \tilde{\Psi}_n(x) &= \delta_n^{\frac{1}{2}} \bar{\Psi}_n(\delta_n x + y_n), \end{cases}$$

for any $\delta_n x + y_n \in \bar{D}_{\frac{\delta}{2t_n}}$. Without loss of generality, we assume that $y_0 = \lim_{n \rightarrow \infty} \frac{x_n}{t_n}$. Then by also passing to a subsequence, $(\tilde{u}_n, \tilde{\Psi}_n)$ converges in $C_{loc}^2(\mathbb{R}^2)$ to some $(\tilde{u}, \tilde{\Psi})$ satisfying

$$\begin{cases} -\Delta \tilde{u} &= 2V^2(0)e^{2\tilde{u}} - V(0)e^{\tilde{u}}|\tilde{\Psi}|^2, \\ \not{D}\tilde{\Psi} &= -V(0)e^{\tilde{u}}\tilde{\Psi}, \end{cases} \text{ in } \mathbb{R}^2,$$

with

$$(57) \quad \int_{\mathbb{R}^2} (2V^2(0)e^{2\tilde{u}} - V(0)e^{\tilde{u}}|\tilde{\Psi}|^2) dx = 4\pi.$$

Now fixing $\delta \in (0, \delta_1)$ small enough, $S, R > 0$ large enough and n large enough, by using (57) and the fact that the neck energy of the spinor field Ψ_n is converging to zero, we have

$$\begin{aligned} & \int_{B_\delta} (2V^2(x)|x|^{2\alpha_n} e^{2u_n} - V(x)|x|^{\alpha_n} e^{u_n} |\Psi_n|^2) dx \\ &= \int_{B_{\frac{\delta}{t_n}}} (2V^2(t_n x)|x|^{2\alpha_n} e^{2\bar{u}_n} - V(t_n x)|x|^{\alpha_n} e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx \\ &= \int_{B_{\frac{\delta}{t_n}} \setminus B_S(y_n)} (2V^2(t_n x)|x|^{2\alpha_n} e^{2\bar{u}_n} - V(t_n x)|x|^{\alpha_n} e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx \\ & \quad + \int_{B_S(y_n) \setminus B_{\frac{\tau_n}{t_n} R}(y_n)} (2V^2(t_n x)|x|^{2\alpha_n} e^{2\bar{u}_n} - V(t_n x)|x|^{\alpha_n} e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx \\ & \quad + \int_{B_{\frac{\tau_n}{t_n} R}(y_n)} (2V^2(t_n x)|x|^{2\alpha_n} e^{2\bar{u}_n} - V(t_n x)|x|^{\alpha_n} e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx \\ &\geq \int_{B_R} \left(2V^2(x_n + \tau_n x) \left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{2\alpha_n} e^{2\tilde{u}_n(x)} \right. \\ & \quad \left. - V(x_n + \tau_n x) \left| \frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|} x \right|^{\alpha_n} e^{\tilde{u}_n(x)} |\tilde{\Psi}_n|^2 \right) dx \\ & \quad - \int_{B_{t_n S} \setminus B_{\tau_n R}(x_n)} V(x)|x|^{\alpha_n} e^{u_n} |\Psi_n|^2 \end{aligned}$$

$$\begin{aligned}
& - \int_{B_{\frac{\delta}{t_n}} \setminus B_S(y_n)} V(t_n x) |x|^{\alpha_n} e^{\bar{u}_n} |\bar{\Psi}_n|^2 \\
& \geq 4\pi - \frac{1}{10}.
\end{aligned}$$

Then, applying similar arguments as in **Case I** we get a contradiction. Thus, we finish the proof of the Theorem. q.e.d.

8. Blow-up value

In this section, we shall further investigate the blow-up behavior of a sequence of solutions of (4) and (5). Let $m(p)$ be the blow-up value at a blow-up point $p \in \Sigma_1$ defined as in (7). It is clear from the result in Theorem 1.3 that $m(p) \geq 4\pi$. Now we shall determine the precise value of $m(p)$ under a boundary condition.

Proof of Theorem 1.4. Without loss of generality, we assume $p = 0$. The case of $p \neq 0$ can be handled analogously. It follows from the boundary condition in (8) that $0 \leq u_n - \min_{\partial B_{\rho_0}(p)} u_n \leq C$ on $\partial B_{\rho_0}(p)$. Define w_n as the unique solution of the following Dirichlet problem

$$\begin{cases} -\Delta w_n = 0, & \text{in } B_{\rho_0}(p), \\ w_n = u_n - \min_{\partial B_{\rho_0}} u_n, & \text{on } \partial B_{\rho_0}(p). \end{cases}$$

By the maximum principle, w_n is uniformly bounded in $B_{\rho_0}(p)$ and, consequently, w_n is C^2 in $B_{\rho_0}(p)$. Furthermore, the function $v_n = u_n - \min_{\partial B_{\rho_0}(p)} u_n - w_n$ solves the Dirichlet problem

$$\begin{cases} -\Delta v_n = 2V^2(x) |x|^{2\alpha_n} e^{2u_n} - V(x) |x|^{\alpha_n} e^{u_n} |\Psi_n|^2, & \text{in } B_{\rho_0}(p), \\ v_n = 0, & \text{on } \partial B_{\rho_0}(p), \end{cases}$$

with the energy condition

$$\int_{B_{\rho_0}(p)} (2V^2(x) |x|^{2\alpha_n} e^{2u_n} - V(x) |x|^{\alpha_n} e^{u_n} |\Psi_n|^2) dx \leq C.$$

By Green's representation formula, we have

$$\begin{aligned}
v_n(x) &= \frac{1}{2\pi} \int_{B_{\rho_0}(p)} \log \frac{1}{|x-y|} (2V^2(y) |y|^{2\alpha_n} e^{2u_n} \\
& \quad - V(y) |y|^{\alpha_n} e^{u_n} |\Psi_n|^2) dy + R_n(x),
\end{aligned}$$

where $R_n(x) \in C^1(B_{\rho_0}(p))$ is a regular term. Since $p = 0$ is the only blow-up point in $B_{\rho_0}(p)$, from Theorem 1.3, we know

$$(58) \quad v_n(x) \rightarrow \frac{m(p)}{2\pi} \ln \frac{1}{|x|} + R(x), \text{ in } C_{loc}^1(B_{\rho_0}(p) \setminus \{0\}),$$

for $R(x) \in C^1(B_{\rho_0}(p))$. On the other hand, we observe that (v_n, Ψ_n) satisfies

$$\begin{cases} -\Delta v_n &= 2K_n^2(x)|x|^{2\alpha_n}e^{2v_n} - K_n(x)|x|^{\alpha_n}e^{v_n}|\Psi_n|^2, \\ \not{D}\Psi_n &= -K_n(x)e^{v_n}\Psi_n, \end{cases} \quad \text{in } B_{\rho_0}(p),$$

where $K_n = V(x)e^{\min_{\partial B_{\rho_0}(p)} u_n + w_n}$. Noticing that $p = 0$, the Pohozaev identity of (v_n, Ψ_n) in $B_\rho(p)$ for $0 < \rho < \rho_0$ is

$$\begin{aligned} & \rho \int_{\partial B_\rho(0)} \left| \frac{\partial v_n}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla v_n|^2 d\sigma \\ &= (1 + \alpha_n) \int_{B_\rho(0)} (2K_n^2(x)|x|^{2\alpha_n}e^{2v_n} - K_n(x)|x|^{\alpha_n}e^{v_n}|\Psi_n|^2) dx \\ & \quad - \rho \int_{\partial B_\rho(0)} K_n^2(x)|x|^{2\alpha_n}e^{2v_n} d\sigma \\ & \quad + \frac{1}{2} \int_{\partial B_\rho(0)} \left\langle \frac{\partial \Psi_n}{\partial \nu}, x \cdot \Psi_n \right\rangle + \left\langle x \cdot \Psi_n, \frac{\partial \Psi_n}{\partial \nu} \right\rangle d\sigma \\ (59) \quad & + \int_{B_\rho(0)} (|x|^{2\alpha_n}e^{2v_n}x \cdot \nabla(K_n^2(x)) - |x|^{\alpha_n}e^{v_n}|\Psi_n|^2x \cdot \nabla K_n(x)) dx. \end{aligned}$$

By (58), we have

$$\begin{aligned} & \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \rho \int_{\partial B_\rho(0)} \left| \frac{\partial v_n}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla v_n|^2 d\sigma \\ &= \lim_{\rho \rightarrow 0} \rho \int_{\partial B_\rho(0)} \frac{1}{2} \left| \frac{\partial \left(\frac{m(p)}{2\pi} \ln \frac{1}{|x|} \right)}{\partial \nu} \right|^2 d\sigma \\ &= \frac{1}{4\pi} m^2(p). \end{aligned}$$

Since $u_n \rightarrow -\infty$ uniformly on $\partial B_\rho(0)$, we also have

$$\begin{aligned} & \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \rho \int_{\partial B_\rho(0)} K_n^2(x)|x|^{2\alpha_n}e^{2v_n} d\sigma \\ &= \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \rho \int_{\partial B_\rho(0)} V^2(x)|x|^{2\alpha_n}e^{2u_n} d\sigma \\ &= 0. \end{aligned}$$

Noticing that $\int_{B_{\rho_0}(0)} (|x|^{2\alpha_n}e^{2u_n} + \Psi_n^4) dx \leq C$, we can obtain that

$$\lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{B_\rho(0)} (|x|^{2\alpha_n}e^{2v_n}x \cdot \nabla(K_n^2(x)) - |x|^{\alpha_n}e^{v_n}|\Psi_n|^2x \cdot \nabla K_n(x)) dx = 0.$$

Since $u_n \rightarrow -\infty$ uniformly in $B_{2\rho}(0) \setminus B_{\frac{\rho}{4}}(0)$, and $|\Psi_n|$ is uniformly bounded in $B_{2\rho}(0) \setminus B_{\frac{\rho}{4}}(0)$ for any $\rho > 0$, we know

$$\not{D}\Psi = 0, \quad \text{in } B_{\rho_0} \setminus \{0\}.$$

Since the local singularity of a harmonic spinor with finite energy is removable, we have

$$\not{D}\Psi = 0, \quad \text{in } B_{\rho_0}.$$

It follows that Ψ is smooth in B_{ρ_0} . Therefore, we obtain that

$$\lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\partial B_\rho(0)} |\Psi_n| |x \cdot \nabla \Psi_n| d\sigma = 0.$$

Let $n \rightarrow \infty$ and then $\rho \rightarrow 0$ in (59), we get that

$$\frac{1}{4\pi} m^2(p) = (1 + \alpha)m(p).$$

It follows that $m(p) = 4\pi(1 + \alpha)$. Thus, we finish the proof of Theorem 1.4. q.e.d.

9. The global super-Liouville system on a singular Riemann surface

In this section, we study the blow-up behavior of a sequence of solutions of the global super-Liouville system on a singular Riemann surface and prove Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.5. Since $g = e^{2\phi}g_0$ with g_0 being smooth, then by the well known properties of ϕ (see, e.g., [32] or [8], p. 5639), we know that (u_n, ψ_n) satisfies

$$\begin{cases} -\Delta_{g_0}(u_n + \phi) = 2e^{2(u_n + \phi)} - e^{u_n + \phi} \left\langle e^{\frac{\phi}{2}} \psi_n, e^{\frac{\phi}{2}} \psi_n \right\rangle - K_{g_0} - \sum_{j=1}^m 2\pi\alpha_j \delta_{q_j}, \\ \not{D}_{g_0}(e^{\frac{\phi}{2}} \psi_n) = -e^{u_n + \phi}(e^{\frac{\phi}{2}} \psi_n), \end{cases}$$

in M and with the energy conditions:

$$\int_M e^{2(u_n + \phi)} dg_0 < C, \quad \int_M |e^{\frac{\phi}{2}} \psi_n|^4 dg_0 < C.$$

If we define the blow-up set of $u_n + \phi$ as

$$\Sigma'_1 = \left\{ x \in M, \text{ there is a sequence } y_n \rightarrow x \text{ such that } (u_n + \phi)(y_n) \rightarrow +\infty \right\},$$

then by Remark 3.3, we have $\Sigma_1 = \Sigma'_1$. By the blow-up results of the local system, it follows that one of the following alternatives holds:

- i) u_n is bounded in $L^\infty(M)$.
- ii) $u_n \rightarrow -\infty$ uniformly on M .
- iii) Σ_1 is finite, nonempty and

$$u_n \rightarrow -\infty \text{ uniformly on compact subsets of } M \setminus \Sigma_1.$$

Furthermore,

$$2e^{2(u_n+\phi)} - e^{u_n+\phi}|e^{\frac{\phi}{2}}\psi_n|^2 \rightharpoonup \sum_{p_i \in \Sigma_1} m(p_i)\delta_{p_i},$$

in the sense of distributions.

Now let $p = \frac{q}{q-1} > 2$. We have

$$\begin{aligned} & \|\nabla(u_n + \phi)\|_{L^q(M, g_0)} \\ & \leq \sup \left\{ \left| \int_M \nabla(u_n + \phi)\nabla\varphi dg_0 \right| \mid \varphi \in W^{1,p}(M, g_0), \right. \\ & \quad \left. \int_M \varphi dg_0 = 0, \|\varphi\|_{W^{1,p}(M, g_0)} = 1 \right\}. \end{aligned}$$

By the Sobolev embedding theorem, we get

$$\|\varphi\|_{L^\infty(M, g_0)} \leq C.$$

It is clear that

$$\begin{aligned} & \left| \int_M \nabla(u_n + \phi)\nabla\varphi dg_0 \right| \\ & = \left| \int_M \Delta_{g_0}(u_n + \phi)\varphi dg_0 \right| \\ & \leq \int_M (2e^{2(u_n+\phi)} + e^{u_n+\phi}|e^{\frac{\phi}{2}}\psi_n|^2 + |K_{g_0}|)|\varphi| dg_0 \\ & \quad + \sum_{j=1}^m \left| \int_M 2\pi\alpha_j\delta_{q_j}\varphi dg_0 \right| \leq C. \end{aligned}$$

Therefore, $u_n + \phi - \frac{1}{|M|} \int_M (u_n + \phi) dg_0$ is uniformly bounded in $W^{1,q}(M, g_0)$.

Next, we define the Green function G by

$$\begin{cases} -\Delta_{g_0}G = \sum_{p \in \Sigma_1} m(p)\delta_p - K_{g_0} - \sum_{j=1}^m 2\pi\alpha_j\delta_{q_j}, \\ \int_M G dg_0 = 0. \end{cases}$$

Then G satisfies (9). We have for any $\varphi \in C^\infty(M)$

$$\begin{aligned} & \int_M \nabla(u_n + \phi - G)\nabla\varphi dg_0 = - \int_M \Delta_{g_0}(u_n + \phi - G)\varphi dg_0 \\ & = \int_M (2e^{2(u_n+\phi)} - e^{u_n+\phi} \langle e^{\frac{\phi}{2}}\psi_n, e^{\frac{\phi}{2}}\psi_n \rangle - \sum_{p \in \Sigma_1} m(p)\delta_p)\varphi dg_0 \\ & \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Combining this with the fact that $u_n + \phi - \frac{1}{|M|} \int_M (u_n + \phi) dg_0$ is uniformly bounded in $W^{1,q}(M, g_0)$, we get the conclusion of the lemma. q.e.d.

Proof of Theorem 1.6. The result follows from Theorem 1.5 and the Gauss–Bonnet formula. q.e.d.

10. The local super-Liouville equations with two coefficient functions

In this section, we discuss the following local super-Liouville type equations with two different coefficient functions:

$$(60) \quad \begin{cases} -\Delta u(x) &= 2V^2(x)|x|^{2\alpha}e^{2u(x)} - W(x)|x|^\alpha e^{u(x)}|\Psi|^2, \\ \not{D}\Psi &= -W(x)|x|^\alpha e^{u(x)}\Psi, \end{cases} \quad \text{in } B_r(0),$$

and with the energy condition

$$(61) \quad \int_{B_r(0)} |x|^{2\alpha}e^{2u} + |\Psi|^4 dx < +\infty,$$

where $\alpha > -1$ and $V(x), W(x) \in W^{1,\infty}(\overline{B_r(0)})$ satisfying

$$0 < a \leq V(x), W(x) \leq b < +\infty.$$

In analogy to the case considered in Section 3, we can define the notion of weak solutions $(u, \Psi) \in W^{1,2}(B_r(0)) \times W^{1,\frac{4}{3}}(\Gamma(\Sigma B_r(0)))$ of (60) and (61) and show that any such weak solution (u, Ψ) is *regular* in the sense that $(u, \Psi) \in W^{1,p}(B_r(0)) \times W^{1,q}(\Gamma(\Sigma B_r(0)))$ for some $p > 2$ and some $q > 2$ and (u, Ψ) is $C_{loc}^2 \times C_{loc}^2$ in $B_r(0) \setminus \{0\}$.

Firstly, it is easy to check that the following Pohozaev type identity holds:

$$\begin{aligned} & R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma \\ &= (1 + \alpha) \int_{B_R(0)} (2V^2(x)|x|^{2\alpha}e^{2u} - W(x)|x|^\alpha e^u |\Psi|^2) dx \\ &\quad - R \int_{\partial B_R(0)} V^2(x)|x|^{2\alpha}e^{2u} d\sigma \\ &\quad + \frac{1}{2} \int_{\partial B_R(0)} \left(\left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \right\rangle + \left\langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle \right) d\sigma \\ &\quad + \int_{B_R(0)} (|x|^{2\alpha}e^{2u} x \cdot \nabla(V^2(x)) - |x|^\alpha e^u |\Psi|^2 x \cdot \nabla W(x)) dx, \end{aligned}$$

for any regular solution (u, Ψ) of (60) and (61) on $B_r(0)$ and for any $0 < R < r$.

Secondly, when (u, Ψ) is a regular solution of (60) and (61) in $B_r(0) \setminus \{0\}$, we define the Pohozaev constant associated to (u, Ψ) as follows

$$\begin{aligned} C(u, \Psi) &= R \int_{\partial B_R(0)} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla u|^2 d\sigma \\ &\quad - (1 + \alpha) \int_{B_R(0)} (2V^2(x)|x|^{2\alpha}e^{2u} - W(x)|x|^\alpha e^u |\Psi|^2) dx \end{aligned}$$

$$\begin{aligned}
 &+R \int_{\partial B_R(0)} V^2(x)|x|^{2\alpha} e^{2u} d\sigma \\
 &- \frac{1}{2} \int_{\partial B_R(0)} \left\langle \frac{\partial \Psi}{\partial \nu}, x \cdot \Psi \right\rangle + \left\langle x \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma \\
 &- \int_{B_R(0)} (|x|^{2\alpha} e^{2u} x \cdot \nabla(V^2(x)) - |x|^\alpha e^u |\Psi|^2 x \cdot \nabla W(x)) dx.
 \end{aligned}$$

Then the local singularity removability as in Proposition 4.5 holds.

Thirdly, for a bubble, namely an entire regular solution on \mathbb{R}^2 with bounded energy, we consider the following equation:

$$\begin{cases} -\Delta u &= 2a|x|^{2\alpha} e^{2u} - b|x|^\alpha e^u |\Psi|^2, \\ \not{D}\Psi &= -b|x|^\alpha e^u \Psi, \end{cases} \quad \text{in } \mathbb{R}^2,$$

with $\alpha > -1$ and for two real numbers $a > 0$ and $b > 0$. The energy condition is

$$I(u, \Psi) = \int_{\mathbb{R}^2} (|x|^{2\alpha} e^{2u} + |\Psi|^4) dx < \infty.$$

By using its corresponding Pohozaev type identity, we can prove the same results as in Proposition 5.1 and Theorem 5.2. In particular, we have

$$d = \int_{\mathbb{R}^2} 2a|x|^{2\alpha} e^{2u} - b|x|^\alpha e^u |\Psi|^2 dx = 4\pi(1 + \alpha).$$

Finally, for a sequence of regular solutions (u_n, Ψ_n) to (10) and (11), we define the blow-up value $m(p)$ at a blow-up point p as

$$m(p) = \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{B_\rho(p)} (2V_n^2(x)|x|^{2\alpha_n} e^{2u_n} - W_n(x)|x|^{\alpha_n} e^{u_n} |\Psi_n|^2) dx,$$

and we can show that the blow-up behaviors for (u_n, Ψ_n) as in Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5 hold.

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