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THE FLOATING BODY IN REAL SPACE FORMS

Florian Besau* & Elisabeth M. Werner†

Abstract

We carry out a systematic investigation on floating bodies in real space forms. A new unifying approach not only allows us to treat the important classical case of Euclidean space as well as the recent extension to the Euclidean unit sphere, but also the new extension of floating bodies to hyperbolic space.

Our main result establishes a relation between the derivative of the volume of the floating body and a certain surface area measure, which we called the floating area. In the Euclidean setting the floating area coincides with the well known affine surface area, a powerful tool in the affine geometry of convex bodies.

1. Introduction

Two important closely related notions in affine convex geometry are the floating body and the affine surface area of a convex body. The floating body of a convex body is obtained by cutting off caps of volume less or equal to a fixed positive constant. Taking the right-derivative of the volume of the floating body gives rise to the affine surface area. This was established for all convex bodies in all dimensions by Schütt and Werner in [**47**]. More results on floating bodies can be found in e.g., [**5**, **29**].

The affine surface area was introduced by Blaschke in 1923 [**8**]. Due to its important properties, which make it an effective and powerful tool, it is omnipresent in geometry. The affine surface area and its generalizations in the rapidly developing L_p and Orlicz Brunn–Minkowski theory are the focus of intensive investigations (see e.g. [**36**, **52**, **53**, **14**, **50**]).

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A first characterization of affine surface area was achieved by Ludwig and Reitzner [33] and had a profound impact on valuation theory of convex bodies. They started a line of research (see e.g. [34, 32]) leading up to the very recent characterization of all centro-affine valuations by Haberl and Parapatits [22].

There is a natural inequality associated with affine surface area, the affine isoperimetric inequality, which states that among all convex bodies, with fixed volume, affine surface area is maximized for ellipsoids. This inequality has sparked interest into affine isoperimetric inequalities with a multitude of results (see e.g. [36, 55, 56, 52]).

There are numerous other applications for affine surface area, such as, the approximation theory of convex bodies by polytopes [19, 20, 30, 10, 9, 44, 48, 49, 42], affine curvature flows [2, 3, 24, 26, 25], information theory [12, 11, 13, 4, 54, 39, 40] and partial differential equations [37].

In this paper we introduce the floating bodies for spaces of constant curvature, i.e., real space forms. Our considerations lead to a new surface area measure for convex bodies, which we call the floating area. This floating area is intrinsic to the constant curvature space and not only coincides with affine surface area in the flat case, but also has similar properties in the general case. Namely, the floating area is a valuation and upper semi-continuous.

We lay the foundation for further investigations of floating bodies and the floating area of convex bodies in more general spaces. The authors believe that both notions are of interest in its own right and will, in particular, be useful for applications, such as, isoperimetric inequalities and approximation theory of convex bodies in spaces of constant curvature.

1.1. Statement of principal results. A real space form is a simply connected complete Riemannian manifold with constant sectional curvature λ. For $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}, n \geq 2$, we denote by $\mathrm{Sp}^n(\lambda)$ the real space form of dimension n and curvature λ. This includes the special cases of the sphere $\mathbb{S}^n = \mathrm{Sp}^n(1)$, hyperbolic space $\mathbb{H}^n = \mathrm{Sp}^n(-1)$ and Euclidean space $\mathbb{R}^n = \mathrm{Sp}^n(0)$. A compact (geodesically) convex set K is called a convex body. The set of convex bodies in $\mathbb{R}^n$ with non-empty interior is denoted by $\mathcal{K}_0(\mathbb{R}^n)$, or $\mathcal{K}_0(A)$ if we consider convex bodies contained in an open subset $A \subset \mathbb{R}^n$. The set of convex bodies in a space form with non-empty interior is denoted by $\mathcal{K}_0(\mathrm{Sp}^n(\lambda))$, $\mathcal{K}_0(\mathbb{S}^n)$ or $\mathcal{K}_0(\mathbb{H}^n)$. For further details we refer to Section 3.

A hyperplane in a real space form $\mathrm{Sp}^n(\lambda)$ is a totally geodesic hypersurface. It is isometric to $\mathrm{Sp}^{n-1}(\lambda)$. Hyperplanes split the space into two open and connected parts which are half-spaces. We denote by H^+ and H^- the closed half-spaces bounded by the hyperplane H. The standard volume measure on $\mathrm{Sp}^n(\lambda)$ is vol_n^λ.

Definition 1.1 (λ-Floating Body)**.** Let $\lambda \in \mathbb{R}$ and $K \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$. For $\delta > 0$ the λ-*floating body* $\mathcal{F}_\delta^\lambda K$ is defined by

$$\mathcal{F}_\delta^\lambda K = \bigcap \left\{ H^- : \mathrm{vol}_n^\lambda \left(K \cap H^+\right) \leq \delta^{\frac{n+1}{2}} \right\}.$$

The main theorem of this article is the following:

Theorem 1.2. *Let $n \geq 2$. If $K \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$, then the right-derivative of $\mathrm{vol}_n^\lambda(\mathcal{F}_\delta^\lambda K)$ at $\delta = 0$ exists. More precisely, we have*

$$\lim_{\delta \to 0^+} \frac{\mathrm{vol}_n^\lambda(K) - \mathrm{vol}_n^\lambda(\mathcal{F}_\delta^\lambda K)}{\delta} = c_n \Omega^\lambda(K),$$

where $c_n = \frac{1}{2}\left((n+1)/\kappa_{n-1}\right)^{2/(n+1)}$ and

$$\Omega^\lambda(K) = \int\limits_{\mathrm{bd}\, K} H_{n-1}^\lambda(K, x)^{\frac{1}{n+1}} \, d\mathrm{vol}_{\mathrm{bd}\, K}^\lambda(x).$$

We call $\Omega^\lambda(K)$ the λ-floating area *of K.*

Here $\mathrm{vol}_{\mathrm{bd}\, K}^\lambda$ denotes the natural boundary measure with respect to $\mathrm{Sp}^n(\lambda)$ and $H_{n-1}^\lambda(K, x)$ denotes the (generalized) Gauss–Kronecker curvature on $\mathrm{bd}\, K$, the boundary of K, with respect to $\mathrm{Sp}^n(\lambda)$ (see Section 3 for details). Furthermore, κ_n is the volume of the Euclidean unit Ball $B_e^n(0, 1)$ in $\mathbb{R}^n$, i.e., $\kappa_n = \mathrm{vol}_n^e(B_e^n(0, 1))$.

For $\lambda = 0$, i.e. Euclidean space, Theorem 1.2 was first proved in this form by Schütt and Werner [**47**]. For $\lambda = 1$, the theorem was established only very recently by the authors [**7**]. In this article we now prove the complete form for all $\lambda \in \mathbb{R}$ with a new unifying approach. In Section 2, we recall important notions from Euclidean convex geometry. In particular, we investigate the weighted floating body. In Section 3, we recall basic facts from hyperbolic geometry. We use the projective Euclidean model and relate hyperbolic convex bodies with Euclidean convex bodies. It is well-known that real space forms admit Euclidean models. We make use of this fact to generalize our results in Subsection 3.2 to real space forms. The Euclidean models and the results on the weighted floating body are the main tool to prove Theorem 1.2 in Section 4. In Section 5, we investigate the floating area and also the surface area measure of Euclidean convex bodies related to it. In particular, we show the following.

Theorem 1.3. *Let $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq 2$. Then the λ-floating area $\Omega^\lambda \colon \mathcal{K}(\mathrm{Sp}^n(\lambda)) \to \mathbb{R}$ is*

(a) *upper semi-continuous,*
(b) *a valuation, that is, for $K, L \in \mathcal{K}(\mathrm{Sp}^n(\lambda))$ such that $K \cup L \in \mathcal{K}(\mathrm{Sp}^n(\lambda))$ we have that*

$$\Omega^\lambda(K) + \Omega^\lambda(L) = \Omega^\lambda(K \cup L) + \Omega^\lambda(K \cap L),$$

(c) *and invariant under isometries of* $\mathrm{Sp}^n(\lambda)$. *For* $\lambda = 0$, Ω^0 *coincides with the affine surface area and is invariant not only under isometries, but all (equi-)affine transformations of* $\mathbb{R}^n$.

All the properties in Theorem 1.3 are well known for the affine surface area, that is, $\lambda = 0$, see e.g. [**45**, **31**, **35**, **29**]. Also, in the spherical case, $\lambda = 1$, we were able to establish similar results [**7**].

Finally, in Subsection 5.2, we briefly consider an isoperimetric inequality for the floating area.

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2. The weighted floating body

In this section, we recall the notion of weighted floating bodies introduced in [**51**]. It will serve as a unifying framework for dealing with Euclidean, spherical and hyperbolic floating bodies. In the following we also recall facts from Euclidean convex geometry. For a general reference we refer to [**21**, **17**, **43**]. The final goal of this section is to establish Lemma 2.9, which is a crucial step in the proof of our main Theorem 1.2 in Section 4.

We denote the Euclidean volume by vol_n^e. If a σ-finite Borel measure μ is absolutely continuous to another σ-finite Borel measure ν on an open set $D \subseteq \mathbb{R}^n$, then we write $\mu \ll_D \nu$. The measure μ is equivalent to ν on D, $\mu \sim_D \nu$, if and only if $\mu \ll_D \nu$ and $\nu \ll_D \mu$. Evidently, by the Radon–Nikodym Theorem, for a σ-finite Borel measure μ we have that $\mu \sim_D \mathrm{vol}_n^e$ if and only if there is Borel function $f_\mu \colon D \to \mathbb{R}$ such that $d\mu(x) = f_\mu(x)dx$ and $\mathrm{vol}_n^e(\{f_\mu = 0\}) = 0$. For a convex body $K \in \mathcal{K}_0(\mathbb{R}^n)$ we consider σ-finite measures μ such that $\mu \sim_{\mathrm{int}\, K} \mathrm{vol}_n^e$, where $\mathrm{int}\, K$ denotes the interior of K. Thus, without loss of generality, we may assume μ to be a σ-finite Borel measure on $\mathbb{R}^n$ with support K and for any measurable set A we have

$$\mu(A) = \int_{A \cap \mathrm{int}\, K} f_\mu(x)\, d\mathrm{vol}_n^e(x).$$

Definition 2.1 (Weighted Floating Body [**51**]). Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ and let μ be a finite non-negative Borel measure on $\mathrm{int}\, K$ such that $\mu \sim_{\mathrm{int}\, K} \mathrm{vol}_n^e$. For $\delta > 0$, we define the *weighted floating body* $\mathcal{F}_\delta^\mu\, K$, by

$$\mathcal{F}_\delta^\mu\, K = \bigcap \left\{ H^- : \mu(H^+ \cap K) \leq \delta^{\frac{n+1}{2}} \right\},$$

where $H^\pm$ are the closed half-spaces bounded by the hyperplane H.

We will see that the weighted floating body is non-empty, if δ is small enough. Since it is an intersection of closed half-spaces, it is a convex body contained in K.

Example 2.2. For $\mu = \mathrm{vol}_n^e$ we retrieve the Euclidean floating body, denoted by $\mathcal{F}_\delta^e K$. In the literature different normalizations appear. For instance, in [**47**] the convex floating body is defined as

$$K_t = \bigcap \{H^- : \mathrm{vol}_n^e(H^+ \cap K) \le t\},$$

which is equivalent to our notion since

$$\mathcal{F}_\delta^0 K = K_{\delta(n+1)/2}.$$

We denote by $\cdot$ the Euclidean scalar product and by $\|.\|$ the Euclidean norm in $\mathbb{R}^n$. A convex body is uniquely determined by its *support function* h_K defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

The geometric interpretation of the support function is the following: For a fixed point $x \in \mathbb{R}^n$ and a normal direction $v \in \mathbb{S}^{n-1}$, we denote the hyperplane parallel to the hyperplane through x with normal v at distance $\alpha \in \mathbb{R}$ by $H_{x,v,\alpha}$, i.e.,

$$H_{x,v,\alpha} = \{y \in \mathbb{R}^n : y \cdot v = \alpha + x \cdot v\} = H_{0,v,\alpha+x\cdot v}.$$

For a given direction $v \in \mathbb{S}^{n-1}$, the support function $h_K(v)$ measures the distance of a supporting hyperplane in direction v to the origin. That is, $H_{0,v,h_K(v)}$ is a supporting hyperplane of K in direction v and K is given by

$$K = \bigcap_{v \in \mathbb{S}^{n-1}} H^-_{0,v,h_K(v)}, \tag{2.1}$$

where $H^-_{x,v,\alpha} = \{y \in \mathbb{R}^n : y \cdot v \le \alpha + x \cdot v\}$.

A closed Euclidean ball of radius r and center $x \in \mathbb{R}^n$ is denoted by $B_e^n(x,r)$. For $K \in \mathcal{K}(\mathbb{R}^n)$ the set of points of distance r from K is $B_e^n(K,r)$. For $K, L \in \mathcal{K}(\mathbb{R}^n)$, the *Hausdorff distance* δ^e is defined by

$$\delta^e(K,L) = \inf\left\{r \ge 0 : K \subseteq B_e^n(L,r) \text{ and } L \subseteq B_e^n(K,r)\right\}.$$

Equivalently, we have that

$$\delta^e(K,L) = \sup_{v \in \mathbb{S}^{n-1}} \|h_K(v) - h_L(v)\|.$$

Given a continuous function $f \colon \mathbb{S}^{n-1} \to \mathbb{R}$ the *Wulff shape* $[f]$ (also called Aleksandrov body, see [**16**, Sec. 6]) of f is, unless it is the empty set, the convex body defined by

$$[f] = \bigcap_{v \in \mathbb{S}^{n-1}} H^-_{0,v,f(v)}. \tag{2.2}$$

For a positive continuous function f the Wulff shape is a convex body containing the origin in its interior. For a convex body K we have $K = [h_K]$, i.e., the Wulff shape associated with h_K is K itself. The concept of Wulff shapes has many applications, see e.g. [**43**, Sec. 7.5] for a short exposition.

The weighted floating body is a Wulff shape.

Proposition 2.3. *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$, μ be a finite non-negative Borel measure on* $\operatorname{int} K$ *such that $\mu \sim_{\operatorname{int} K} \operatorname{vol}_n^e$ and $\delta \in \left(0, \mu(K)^{\frac{2}{n+1}}\right)$. For $v \in \mathbb{S}^{n-1}$, there exists a unique $s_\delta(v) \in \mathbb{R}$ determined by*

$$\mu\left(K \cap H^+_{0,v,h_K(v)-s_\delta(v)}\right) = \delta^{\frac{n+1}{2}}.$$

In particular, $s_\delta(v) = s(\delta, v)$ is continuous on $\left(0, \mu(K)^{\frac{2}{n+1}}\right) \times \mathbb{S}^{n-1}$ and strictly increasing in δ. Moreover, the weighted floating body $\mathcal{F}^\mu_\delta K$ exists if and only if the Wulff shape $[h_K - s_\delta]$ exists and in this case we have that

$$\mathcal{F}^\mu_\delta K = [h_K - s_\delta]. \tag{2.3}$$

Proof. We consider $G\colon \mathbb{S}^{n-1} \times \mathbb{R} \to [0, \mu(K)]$ defined by

$$G(v, \Delta) = \mu\left(K \cap H^+_{0,v,h_K(v)-\Delta}\right).$$

Since μ is a non-negative Borel measure equivalent to vol_n, we can find a Borel function $f_\mu\colon \mathbb{R}^n \to [0,\infty)$ such that $f_\mu > 0$ almost everywhere on $\operatorname{int} K$ and $f_\mu = 0$ else. We can, therefore, write

$$G(v,\Delta) = \int\limits_{H^+_{0,v,h_K(v)-\Delta}} f_\mu(x)\, dx = \int\limits_{h_K(v)-\Delta}^{h_K(v)} \int\limits_{v^\perp} f_\mu(w + tv)\, dw\, dt.$$

For the second equality we used Fubini's theorem and the substitution $x = w + tv$, where $w \in v^\perp = \{y \in \mathbb{R}^n : y \cdot v = 0\}$ and $t \in \mathbb{R}$ are uniquely determined by x. Thus, G is strictly increasing in Δ for $\Delta \in (0, h_K(v) + h_K(-v))$ from 0 to $\mu(K)$. To see that G is continuous, first note that $K(v,\Delta) := K \cap H^+_{0,v,h_K(v)-\Delta}$ depends continuously on $(v,\Delta) \in \mathbb{S}^{n-1} \times \mathbb{R}$ with respect to the Hausdorff distance δ^e. This follows, since h_K is continuous and the map $(v,\lambda) \to H^+_{0,v,\lambda} \cap K$ is continuous in $v \in \mathbb{S}^{n-1}$ and $\lambda \in \mathbb{R}$. Now, since vol_n^e is continuous on $\mathcal{K}_0(\mathbb{R}^n)$, see [**43**, Thm. 1.8.20], and since $\mu \sim_{\operatorname{int} K} \operatorname{vol}_n^e$, we conclude that μ is continuous on $\mathcal{K}_0(\mathbb{R}^n) \cap K$ and, therefore, $\mu(K(v,\Delta)) = G(v,\Delta)$ is continuous in v.

Hence, for $\delta \in \left(0, \mu(K)^{\frac{2}{n+1}}\right)$ there is a unique $s_\delta(v) \in (0, h_K(v) + h_K(-v))$ such that

$$\delta^{\frac{n+1}{2}} = G(v, s_\delta(v)),$$

which is strictly increasing in δ and continuous.

To prove (2.3), we first consider a fixed $v \in \mathbb{S}^{n-1}$. For $t_1 < t_2$, we have that $H^+_{0,v,h_K(v)-t_1} \subseteq H^+_{0,v,h_K(v)-t_2}$. The maximal t such that

$G(v,t) \leq \delta^{\frac{n+1}{2}}$ is $t = s_\delta(v)$. Hence, we have that

$$\bigcap \left\{ H^-_{0,v,h_K(v)-t} : t \in \mathbb{R} \text{ such that } G(v,t) \leq \delta^{\frac{n+1}{2}} \right\} = H^-_{0,v,h_K(v)-s_\delta(v)}.$$

Finally, we conclude that

$$\begin{aligned} \mathcal{F}^\mu_\delta K &= \bigcap \left\{ H^-_{0,v,h_K(v)-t} : v \in \mathbb{S}^{n-1}, t \in \mathbb{R} \text{ such that } G(v,t) \leq \delta^{\frac{n+1}{2}} \right\} \\ &= \bigcap_{v \in \mathbb{S}^{n-1}} H^-_{0,v,h_K(v)-s_\delta(v)} = [h_K - s_\delta]. \end{aligned}$$

q.e.d.

For a convex body $K \in \mathcal{K}_0(\mathbb{R}^n)$ and a boundary point $x \in \operatorname{bd} K$ we define the set of normal vectors $\sigma(K,x)$ of K in x, also called the *spherical image* of K at x (see [**43**, p. 88]), by

$$\sigma(K,x) = \{v \in \mathbb{S}^{n-1} : H_{x,v,0} \text{ is a supporting hyperplane to } K \text{ in } x\}.$$

A boundary point x is called *regular* if $\sigma(K,x)$ is a single point, that is, K has a unique outer unit normal vector N_x at x. Note that for a convex body almost all boundary points are regular (see [**43**, Thm. 2.2.5]). A boundary point x is *exposed* if and only if there is a support hyperplane H such that $K \cap H = \{x\}$.

A subset $S \subseteq \mathbb{S}^{n-1}$ is a *spherical convex body* if and only if the positive hull $\operatorname{pos} S = \{\lambda s : \lambda \geq 0, s \in S\}$ is a closed convex cone in $\mathbb{R}^n$. The spherical image at a boundary point x is a spherical convex body and the closed convex cone generate by it is the *normal cone* $N(K,x) = \operatorname{pos} \sigma(K,x)$. If K has non-empty interior, then $\sigma(K,x)$ is proper for any boundary point, that is, the normal cone does not contain any linear subspace.

The *spherical Hausdorff distance* δ^s is a metric on spherical convex bodies induced by the spherical distance

$$d_s(x,y) = \arccos(x \cdot y) \tag{2.4}$$

on $\mathbb{S}^{n-1}$ in the following way: For a subset $A \subset \mathbb{S}^{n-1}$ we denote by A_ε the ε-neighborhood of A, i.e., $A_\varepsilon = \{a \in \mathbb{S}^{n-1} : d_s(a,A) < \varepsilon\}$. Then, for spherical convex bodies S and T, we have that

$$\delta^s(S,T) = \inf\{\varepsilon \geq 0 : S \subseteq T_\varepsilon \text{ and } S \subseteq T_\varepsilon\}.$$

The spherical Hausdorff distance induces a metric on the closed convex cones with apex at the origin via the positive hull pos. Hence, we say that a sequence of closed convex cones C_i converges to a closed convex cone C if and only if the sequence of spherical convex bodies $C_i \cap \mathbb{S}^{n-1}$ converges to $C \cap \mathbb{S}^{n-1}$ with respect to δ^s.

Fix $z \in \mathbb{S}^{n-1}$ and let $(C_i)_{i=1}^\infty$ be a sequence of closed convex cones contained in the open half-space $\operatorname{int} H^+_{0,z,0}$. Then C_i converges to a closed convex cone C contained in the same open-half space with respect to δ^s if and only if the sections of the convex cones with the affine hyperplane

$H_{0,z,1}$ converge with respect to the Euclidean Hausdorff metric δ^e in $H_{0,z,1}$.

We define the *gnomonic projection* $g_z \colon \operatorname{int} H^+_{0,z,0} \to z^\perp \cong \mathbb{R}^{n-1}$ by

$$g_z(x) = (x \cdot z)^{-1} x - z. \tag{2.5}$$

Then g_z maps closed convex cones contained in the open half-space $\operatorname{int} H^+_{0,z,0}$ to convex bodies in $z^\perp \cong \mathbb{R}^{n-1}$. By the previous statement we find, that the gnomonic projection induces an homeomorphism between the space of closed convex cones in $\operatorname{int} H^+_{0,z,0}$ with respect to the δ^s and the space of convex bodies in $z^\perp \cong \mathbb{R}^{n-1}$ with respect to δ^e. Compare also [**6**, Cor. 4.5].

By Proposition 2.3, $s_\delta(v) = s(\delta, v)$ is continuous as a function in (δ, v). It converges point-wise to 0 as $\delta \to 0^+$. By the compactness of $\mathbb{S}^{n-1}$, we have that $s_\delta(.)$ converges uniformly to 0 as $\delta \to 0^+$. This implies the convergence of $[h_K - s_\delta]$ to $[h_K] = K$, see e.g. [**43**, Lem. 7.5.2]. We conclude

$$\lim_{\delta \to 0^+} \mathcal{F}^\mu_\delta K = K. \tag{2.6}$$

Our next goal is to show that the convergence of the weighted floating body is locally determined. This fact and, therefore, most of the following lemmas are probably known for the most part. However, since we were only able to find references in particular cases, for instance, see e.g. [**49**] for related results, and also for the convenience of the reader, we include proofs for the following.

We consider a regular boundary point $x \in \operatorname{bd} K$ and investigate the behavior of $\mathcal{F}^\mu_\delta K$ near x for $\delta \to 0^+$. The shape of $\mathcal{F}^\mu_\delta K$ near x is determined by a neighborhood of directions of the unique normal N_x of K at x. For $s < t$, we have that $\mathcal{F}^\mu_s K \supseteq \mathcal{F}^\mu_t K$. In particular, if $0 \in \operatorname{int} \mathcal{F}^\mu_t K$, then for all $\delta \in (0, t)$ we have $0 \in \operatorname{int} \mathcal{F}^\mu_\delta K$. In this case we define x^K_δ as the unique intersection point of $\operatorname{bd} \mathcal{F}^\mu_\delta K$ with the ray $\operatorname{pos}\{x\}$. Hence, $\lim_{\delta \to 0^+} x^K_\delta = x$. We use x_δ to control the limit process $\mathcal{F}^\mu_\delta K \to K$ near x as $\delta \to 0^+$.

The first step is to consider a convergent sequence of Wulff shapes $[f_i] \to [f]$, where $(f_i)_{i \in \mathbb{N}}$ and f are positive continuous functions on $\mathbb{S}^{n-1}$. Thus, $0 \in \operatorname{int} [f_i]$ and $0 \in \operatorname{int} [f]$. We show that, for any regular boundary point $x \in \operatorname{bd} [f]$ and any neighborhood of directions around the normal N_x of $[f]$ at x, there is $i_0 \in \mathbb{N}$ such that, for all $i > i_0$, $x_i := \operatorname{bd} [f_i] \cap \operatorname{pos}\{x\}$ is determined by the values of f_i in that neighborhood.

Lemma 2.4 (Local dependence of a convergent sequence of Wulff shapes)**.** *Let $f_i \colon \mathbb{S}^{n-1} \to (0, \infty)$, $i \in \mathbb{N}$, be a sequence of positive continuous functions uniformly convergent to $f \colon \mathbb{S}^{n-1} \to (0, \infty)$. Then for $x \in \operatorname{reg} [f]$ and $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that, for all $i > i_0$, we* *have that*

$$[f_i] \cap \operatorname{pos}\{x\} = \bigcap \left\{ H^-_{0,v,f_i(v)} : d_s(v, N_x) < \varepsilon \right\} \cap \operatorname{pos}\{x\},$$

where N_x is the unique outer unit normal of $[f]$ at x.

Since the weighted floating body can be viewed as a Wulff shape and converges to K as $\delta \to 0^+$ we obtain the following corollary. We write $\operatorname{conv}(x, y)$ for the convex hull of two points $x, y \in \mathbb{R}^n$, i.e., $\operatorname{conv}(x, y)$ is the closed affine line segment between x and y.

Corollary 2.5 (Locality of the weighted floating body)**.** *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ be such that $0 \in \operatorname{int} K$. Then for $x \in \operatorname{reg} K$ and $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $\delta < \delta_\varepsilon$, we have $0 \in \operatorname{int} \mathcal{F}^\mu_\delta K$ and*

$$\begin{aligned} \operatorname{conv}(x^K_\delta, 0) &= \mathcal{F}^\mu_\delta K \cap \operatorname{pos}\{x\} \\ &= \bigcap \left\{ H^-_{0,v,h_K(v)-s_\delta(v)} : d_s(v, N_x) < \varepsilon \right\} \cap \operatorname{pos}\{x\}, \end{aligned} \tag{2.7}$$

where $s_\delta(v)$ is uniquely determined by

$$\delta^{\frac{n+1}{2}} = \mu\left(K \cap H^+_{0,v,h_K(v)-s_\delta(v)}\right).$$

Before we prove Lemma 2.4, we recall some common notation. For $u \in \mathbb{S}^{n-1}$, $F(K, u) = K \cap H_{0,u,h_K(u)}$ is the *exposed face* of K in direction u. The following is an easy observation.

Lemma 2.6 (Convergence of exposed faces)**.** *Let $K_i \to K$ in $\mathcal{K}_0(\mathbb{R}^n)$ with respect to the Hausdorff distance δ^e. If $u \in \mathbb{S}^{n-1}$ such that $F(K, u) = \{x\}$ is an exposed point of K, then $F(K_i, u) \to \{x\}$.*

Proof. Since K_i converges to K with respect to δ^e, any sequence $x_i \in F(K_i, u) \subseteq K_i$ has a convergent subsequence with limit $y \in K$, see e.g. [**43**, Thm. 1.8.7]. Let $R > 0$ be such that $K \cup \bigcup_{i\in\mathbb{N}} K_i \subseteq B^n_e(0, R)$. Then $x_i \in F(K_i, u) \subseteq H_{0,u,h(K_i,u)} \cap B^n_e(0, R)$ and also $H_{0,u,h(K_i,u)} \cap B^n_e(0, R) \to H_{0,u,h(K,u)} \cap B^n_e(0, R)$. Hence, for the limit point y of the convergent subsequence, we also have $y \in H_{0,u,h(K,u)}$ and, therefore, $y \in K \cap H_{0,u,h(K,u)} = F(K, u) = \{x\}$. q.e.d.

We denote the set of convex bodies with 0 in the interior by $\mathcal{K}_{(0)}(\mathbb{R}^n)$. For $K \in \mathcal{K}_{(0)}(\mathbb{R}^n)$, $K^\circ = \{y \in \mathbb{R}^n : x \cdot y \leq 1\}$ is the *polar body* of K. For $x \in \operatorname{bd} K$, set $\widehat{x} = \{y \in K^\circ : x \cdot y = 1\}$. Then $N(K, x) = \operatorname{pos} \widehat{x}$, see [**43**, Lem. 2.2.3]. For $x \in \operatorname{bd} K$, we have $h_{K^\circ}(x/\|x\|) = 1/\|x\|$. Hence, $\widehat{x} = F(K^\circ, x/\|x\|)$, or equivalently

$$N(K, x) = \operatorname{pos} F(K^\circ, x/\|x\|). \tag{2.8}$$

For a proof of the following fact see, e.g., [**18**, Lem. 2.3.2].

Lemma 2.7 (Continuity of the polar map)**.** *Let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{K}_{(0)}(\mathbb{R}^n)$ converging to $K \in K_{(0)}(\mathbb{R}^n)$ with respect to the Hausdorff distance δ^e. Then also $K_i^\circ \to K^\circ$.*

By (2.8), the normal cone $N(K,x)$ at a boundary point x is related to the exposed face of the polar body K° in direction $x/\|x\|$. Using the continuity of the polar map, Lemma 2.7, and the convergence of the exposed faces, Lemma 2.6, we now obtain the convergence of the normal cones in regular boundary points. Note that for a regular boundary point $x \in \operatorname{reg} K$, $N_x/(N_x \cdot x)$ is an exposed point of K°, i.e., $F(K^\circ, x/\|x\|) = \{N_x/(N_x \cdot x)\}$.

Lemma 2.8 (Convergence of the normal cone)**.** *Let f_i be a sequence of positive continuous functions on $\mathbb{S}^{n-1}$, uniformly convergent to a positive continuous function f. For $x \in \operatorname{reg}[f]$ we set $\{x_i\} = \operatorname{pos}\{x\} \cap \operatorname{bd}[f_i]$. Then*

$$\lim_{i\to\infty} N([f_i], x_i) = N([f], x).$$

In particular, we have that

$$\lim_{i\to\infty} \sigma([f_i], x_i) = \lim_{i\to\infty} N([f_i], x_i) \cap \mathbb{S}^{n-1} = N([f], x) \cap \mathbb{S}^{n-1} = \{N_x\}.$$

Proof. We set $z = x/\|x\| = x_i/\|x_i\|$. By (2.8),

$$\operatorname{pos}\{N_x\} = N([f], x) = \operatorname{pos} F([f]^\circ, z),$$

or equivalently $F([f]^\circ, z) = \{N_x/(N_x \cdot x)\}$. With Lemma 2.7 and Lemma 2.6, we conclude that

$$\lim_{i\to\infty} F([f_i]^\circ, z) = \{N_x/(x \cdot N_x)\}. \tag{2.9}$$

Since $0 \in \operatorname{int}[f_i]^\circ$ the exposed face in direction z has a positive distance a_i from the origin. Therefore, $a_i = h_{F([f_i]^\circ, z)}(z) > 0$, $F([f_i]^\circ, z) \subseteq H_{0,z,a_i}$. Also $h_{F([f]^\circ,z)}(z) = 1/\|x\| > 0$ and $\lim_{i\to\infty} a_i = 1/\|x\|$. Hence, the convex cone generated by the exposed face does not contain any linear subspace, or equivalently, $S_i := (\operatorname{pos} F([f_i]^\circ, z)) \cap \mathbb{S}^{n-1}$ as well as $(\operatorname{pos} F([f]^\circ, z)) \cap \mathbb{S}^{n-1} = \{N_x\}$ are contained in the open hemisphere with center in z.

We have to show that S_i converges to $\{N_x\}$ with respect to spherical Hausdorff distance δ^s. This is equivalent to the convergence of $g_z(S_i)$ to $g_z(N_x)$ in $z^\perp$ with respect to the Euclidean Hausdorff distance. Here, g_z is the gnomonic projection in z, see (2.5). We obtain $g_z(S_i) = (1/a_i)F([f_i]^\circ, z) - z$ and $g_z(N_x) = N_x/(N_x \cdot z) - z$. Since $a_i \to 1/\|x\|$ and $F([f_i]^\circ, z) \to F([f]^\circ, z) = \{N_x/(N_x \cdot x)\}$, we conclude that $g_z(S_i) \to g_z(N_x)$. This yields that $S_i \to \{N_x\}$, or equivalently, the convex cones generated by the exposed faces converge, i.e.,

$$\lim_{i\to\infty} \operatorname{pos} F([f_i]^\circ, z) = \lim_{i\to\infty} \operatorname{pos} S_i = \operatorname{pos}\{N_x\} = \operatorname{pos} F([f]^\circ, z).$$

By (2.8), this concludes the proof. q.e.d.

We are now ready to prove Lemma 2.4.

Proof of Lemma 2.4. Assume the opposite. Then there exists $x \in \operatorname{reg}[f]$ and $\varepsilon > 0$ such that, for all $i \in \mathbb{N}$, we have

$$\left(\bigcap\left\{H^-_{0,v,f_i(v)} : v \in \mathbb{S}^{n-1}, d_s(v, N_x) < \varepsilon\right\} \cap \operatorname{pos}\{x\}\right) \setminus ([f_i] \cap \operatorname{pos}\{x\}) \neq \emptyset.$$

By definition, $[f_i] = \{y \in \mathbb{R}^n : y \cdot v \leq f_i(v) \, \forall v \in \mathbb{S}^{n-1}\}$, therefore, for $\{x_i\} = \operatorname{bd}[f_i] \cap \operatorname{pos}\{x\}$, there exists $z_i \in \mathbb{S}^{n-1}$ such that $x_i \cdot z_i = f_i(z_i)$. This yields $z_i \in \sigma([f_i], x_i)$ and we conclude

$$\begin{aligned}[f_i] \cap \operatorname{pos}\{x\} &= \operatorname{conv}(x_i, 0) = H^-_{x_i,z_i,0} \cap \operatorname{pos}\{x\} \\ &= H^-_{0,z_i,z_i\cdot x_i} \cap \operatorname{pos}\{x\} = H^-_{0,z_i,f_i(z_i)} \cap \operatorname{pos}\{x\}.\end{aligned}$$

Thus, $d_s(z_i, N_x) \geq \varepsilon > 0$ for all i. By compactness of $\mathbb{S}^{n-1}$, there is a convergent subsequence of $(z_i)_{i\in\mathbb{N}}$ with limit $z \neq N_x$. This is a contradiction, since $\sigma([f_i], x_i) \to \sigma([f], x) = \{N_x\}$ by Lemma 2.8. q.e.d.

By Corollary 2.5, the weighted floating body is locally determined near any regular boundary point x. If x is also exposed, then a neighborhood of x in K already determines the shape of $\mathcal{F}^\mu_\delta K$ near x for $\delta \to 0^+$.

Lemma 2.9 (Approximation of the weighted floating body)**.** *Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ and $x \in \operatorname{bd} K$ be a regular and exposed point, that is, there is a unique outer unit normal N_x and $K \cap H_{x,N_x,0} = \{x\}$. For $\varepsilon > 0$ set $K' = K \cap B^n_e(x, \varepsilon)$. Then $x \in \operatorname{bd} K'$ is a regular and exposed point of K'. Furthermore:*

(i) *There exists $\Delta_\varepsilon > 0$ such that for all $\Delta < \Delta_\varepsilon$ we have*

$$K' \cap H^+_{x,N_x,-\Delta} = K \cap H^+_{x,N_x,-\Delta}.$$

(ii) *There exists $\xi_\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that, for all $v \in \mathbb{S}^{n-1}$ with $d_s(v, N_x) < \xi_\varepsilon$ and $\Delta < \eta_\varepsilon$, we have*

$$K' \cap H^+_{x,v,-\Delta} = K \cap H^+_{x,v,-\Delta}.$$

(iii) *Let $0 \in \operatorname{int} K'$. There exists $\delta_\varepsilon > 0$ such that, for all $\delta < \delta_\varepsilon$, we have $0 \in \operatorname{int}(\mathcal{F}^\mu_\delta K' \cap \mathcal{F}^\mu_\delta K)$ and $x^{K'}_\delta = x^K_\delta$, where $\{x^{K^*}_\delta\} = \operatorname{bd} \mathcal{F}^\mu_\delta K^* \cap \operatorname{pos}\{x\}$.*

Proof. (i): Assume that the statement is false. Then there exists $\varepsilon > 0$ such that for all $\Delta > 0$, we have

$$\begin{aligned}\emptyset \neq \left(K \cap H^+_{x,N_x,-\Delta}\right) \setminus \left(K' \cap H^+_{x,N_x,-\Delta}\right) &= (K \backslash B^n_e(x,\varepsilon)) \cap H^+_{x,N_x,-\Delta} \\ \subseteq (K \backslash \operatorname{int} B^n_e(x,\varepsilon)) \cap H^+_{x,N_x,-\Delta}.\end{aligned}$$

For $\Delta_1 \leq \Delta_2$, we have

$$(K \backslash \operatorname{int} B^n_e(x,\varepsilon)) \cap H^+_{x,N_x,-\Delta_1} \subseteq (K \backslash \operatorname{int} B^n_e(x,\varepsilon)) \cap H^+_{x,N_x,-\Delta_2}.$$

By compactness, we conclude that $\emptyset \neq (K \backslash \operatorname{int} B^n_e(x,\varepsilon)) \cap H^+_{x,N_x,0}$. This is a contradiction, since $K \cap H^+_{x,N_x,0} = \{x\}$.

(ii): Let $\varepsilon > 0$. By (i), there exists $\Delta_{\varepsilon/2}$ such that $\Delta_{\varepsilon/2} < \varepsilon/2$ and

$$(K \backslash \operatorname{int} B_e^n(x, \varepsilon/2)) \cap H^+_{x,N_x,-\Delta_{\varepsilon/2}} = \emptyset. \tag{2.10}$$

We set $\eta_\varepsilon = (1/2)\Delta_{\varepsilon/2}$ and

$$\xi_\varepsilon = \sqrt{2 + (2/\varepsilon)\Delta_{\varepsilon/2}} - \sqrt{2 + (1/\varepsilon)\Delta_{\varepsilon/2}}.$$

Then $\eta_\varepsilon < \varepsilon/4$ and $\xi_\varepsilon > 0$. We show that for all $v \in \mathbb{S}^{n-1}$ with $d_s(v, N_x) < \xi_\varepsilon$, we have that

$$(K \backslash \operatorname{int} B_e^n(x, \varepsilon)) \cap H^+_{x,v,-\eta_\varepsilon} = \emptyset, \tag{2.11}$$

which yields $K' \cap H^+_{x,v,-\Delta} = K \cap H^+_{x,v,-\Delta}$ for all $\Delta < \eta_\varepsilon$.

Assume that (2.11) is not true. Then there exists $z \in (K \backslash \operatorname{int} B_e^n(x, \varepsilon)) \cap H^+_{x,v,-\eta_\varepsilon}$. Since K is convex, the segment $\operatorname{conv}(z, x)$ is contained in K. Furthermore, since $\|z - x\| > \varepsilon$, there exists $z' \in \operatorname{bd} B_e^n(x, \varepsilon) \cap \operatorname{conv}(z, x)$.

We will show that $z' \in H^+_{x,N_x,-\Delta_{\varepsilon/2}}$, i.e., $z' \cdot N_x \geq x \cdot N_x - \Delta_{\varepsilon/2}$. This will be a contradiction to (2.10), since $z' \in K \backslash \operatorname{int} B_e^n(x, \varepsilon/2)$.

Since $z' \in \operatorname{bd} B_e^n(x, \varepsilon)$, we have $z' = x + \varepsilon \|z' - x\|^{-1}(z' - x)$ and, therefore, $z' \cdot N_x = x \cdot N_x + \varepsilon \|z' - x\|^{-1}(z' - x) \cdot N_x$. Since $z' \in H^+_{x,v,-\eta_\varepsilon}$, we obtain $z' \cdot v \geq x \cdot v - \eta_\varepsilon$ or $\|z' - x\|^{-1}(z' - x) \cdot v \geq -\varepsilon^{-1}\eta_\varepsilon$. Put $w = \|z' - x\|^{-1}(z' - x)$. Then

$$\|w - v\|^2 = 2 - 2(w \cdot v) \leq 2\left(1 + \varepsilon^{-1}\eta_\varepsilon\right).$$

Note that $d_s(v, N_x) \leq \xi_\varepsilon$ implies that $\|v - N_x\| \leq \xi_\varepsilon$. This, together with the definition of ξ_ε, implies that

$$\begin{aligned}
w \cdot N_x &= 1 - \frac{\|w - N_x\|^2}{2} \geq 1 - \frac{(\|w - v\| + \|v - N_x\|)^2}{2} \\
&= w \cdot v - \|w - v\|\|v - N_x\| - \frac{\|v - N_x\|^2}{2} \\
&\geq -\frac{\eta_\varepsilon}{\varepsilon} - \sqrt{2 + \frac{2\eta_\varepsilon}{\varepsilon}}\xi_\varepsilon - \frac{\xi_\varepsilon^2}{2} = 1 - \left(\sqrt{1 + \frac{\eta_\varepsilon}{\varepsilon}} + \frac{\xi_\varepsilon}{\sqrt{2}}\right)^2 = -\frac{\Delta_{\varepsilon/2}}{\varepsilon}.
\end{aligned}$$

Hence, $z' \cdot N_x \geq x \cdot N_x - \Delta_{\varepsilon/2}$.

(iii): By Proposition 2.3, we can write $\mathcal{F}_\delta^\mu K' = \bigcap_{v \in \mathbb{S}^{n-1}} H^-_{x,v,-s^{K'}(\delta,v)}$. Here $-s^{K'}(\delta, v)$ is uniquely determined by

$$\delta^{(n+1)/2} = \mu\left(K' \cap H^+_{x,v,-s^{K'}(\delta,v)}\right)$$

and is continuous in both arguments. By (ii), there exist ξ_ε and η_ε such that for all $v \in \mathbb{S}^{n-1}$ with $d_s(v, N_x^K) < \xi_\varepsilon$, we have

$$\mu\left(K' \cap H^+_{x,v,-\Delta}\right) = \mu\left(K \cap H^+_{x,v,-\Delta}\right),$$

for all $\Delta < \eta_\varepsilon$. Hence, there exists $\delta_1 > 0$ such that for all $\delta < \delta_1$ and $d_s(v, N_x^K) < \xi_\varepsilon$ we have $s^K(\delta, v) = s^{K'}(\delta, v)$.

By Corollary 2.5 applied to K and K' with $\varepsilon = \xi_\varepsilon$ there exist δ_2 and δ_3 such that for all $\delta < \min\{\delta_1, \delta_2, \delta_3\}$ we have $0 \in \operatorname{int} \mathcal{F}_\delta^\mu K'$, $0 \in \operatorname{int} \mathcal{F}_\delta^\mu K$ and

$$\begin{aligned}
(\mathcal{F}_\delta^\mu K) \cap \operatorname{pos}\{x\} &= \bigcap \left\{ H^-_{x,v,-s^K(\delta,v)} : v \in \mathbb{S}^{n-1}, d_s(v, N_x^K) < \xi_\varepsilon \right\} \cap \operatorname{pos}\{x\} \\
&= \bigcap \left\{ H^-_{x,v,-s^{K'}(\delta,v)} : v \in \mathbb{S}^{n-1}, d_s(v, N_x^K) < \xi_\varepsilon \right\} \cap \operatorname{pos}\{x\} \\
&= (\mathcal{F}_\delta^\mu K') \cap \operatorname{pos}\{x\}.
\end{aligned}$$

This implies in particular that $x_\delta^{K'} = x_\delta^K$ since it is the unique intersection point of $\operatorname{pos}\{x\}$ with the boundary of the floating body $\mathcal{F}_\delta^\mu K'$, or $\mathcal{F}_\delta^\mu K$. q.e.d.

3. Hyperbolic convex geometry

In the theory of Riemannian manifolds, hyperbolic n-space $\mathbb{H}^n$ is the simply-connected, complete Riemannian manifold of constant sectional curvature -1. Hyperbolic convex bodies are compact subsets such that for any two points in the set, the geodesic segment between them is contained in the set. Hyperbolic convex geometry is the study of intrinsic notions of hyperbolic convex bodies.

In his famous Erlangen program Felix Klein characterized geometries based on their symmetry groups. In the spirit of this approach, we may view Euclidean convex geometry as the study of notions on Euclidean convex bodies that are invariant under the group of rigid motions. In the projective model (also known as Beltrami–Cayley–Klein model) of hyperbolic space, that is, in the open unit ball $\mathbb{B}^n$, hyperbolic convex geometry can be viewed as the study of notions on Euclidean convex bodies $K \in \mathcal{K}(\mathbb{B}^n)$, invariant under hyperbolic motions.

In the following we recall basic facts about the projective model of hyperbolic space. For a rigorous exposition see, e.g., [**1**] or [**41**].

We consider $\mathbb{B}^n$ together with the Riemannian metric tensor g^h which defines a scalar product in tangent space $T_p\mathbb{B}^n$ for any point $p \in \mathbb{B}^n$ by

$$g_p^h(X_p, Y_p) = \frac{X_p \cdot Y_p}{1 - \|p\|^2} + \frac{(X_p \cdot p)\,(Y_p \cdot p)}{(1 - \|p\|^2)^2}, \quad X_p, Y_p \in T_p\mathbb{B}^n.$$

Here and in the following we use the natural identification of $T_p\mathbb{B}^n = T_p\mathbb{R}^n$ with $\mathbb{R}^n$. Then $(\mathbb{B}^n, g^h)$ is a simply-connected, complete Riemannian manifold with constant sectional curvature -1 and, therefore, isometric to $\mathbb{H}^n$.

The Euclidean metric tensor is g^e and it is induced naturally by $g_p^e(X_p, Y_p) = X_p \cdot Y_p$ for $X_p, Y_p \in T_p\mathbb{R}^n \cong \mathbb{R}^n$. When $p = 0$, then

$$g^h(X_0, Y_0) = X_0 \cdot Y_0 = g^e(X_0, Y_0) \tag{3.1}$$

and, therefore, the Euclidean metric tensor at the origin agrees with the hyperbolic metric tensor. Geodesic curves in $(\mathbb{B}^n, g^h)$ are straight lines in $\mathbb{R}^n$ intersected with $\mathbb{B}^n$ and the geodesic distance, or *hyperbolic distance*, $d_h(p,q)$ between $p,q \in \mathbb{B}^n$ is, see, for example, [**1**, Sec. 1.5] and [**41**, Ch. 6],

$$\cosh d_h(p,q) = \frac{1 - p\cdot q}{\sqrt{1-\|p\|^2}\sqrt{1-\|q\|^2}}. \tag{3.2}$$

Note that

$$\tanh d_h(p,0) = \|p\|. \tag{3.3}$$

Isometries of the projective model are also called motions and the *group of motions* is $\mathfrak{M}(\mathbb{B}^n)$. The group of motions $\mathfrak{M}(\mathbb{B}^n)$ is isomorphic to the restricted Lorentz group $\mathrm{SO}^+(n,1)$ and hyperbolic n-space is characterized by $\mathfrak{M}(\mathbb{B}^n)$ in the following sense: the homogeneous space $\mathrm{SO}^+(n,1)/\mathrm{SO}(n)$ is isomorphic to $\mathbb{H}^n$, see e.g. [**1**, Ch. 1, §2]. Note that, in the projective model, a hyperbolic motion extends to a uniquely determined collineation of the projective closure of $\mathbb{R}^n$ and conversely any collineation that maps $\mathbb{B}^n$ to $\mathbb{B}^n$ restricts to an hyperbolic motion on $\mathbb{B}^n$.

Geodesics in $(\mathbb{B}^n, g^h)$ are the chords of $\mathbb{B}^n$. More general, any totally geodesic subspace of dimension k, called a *k-plane* of $(\mathbb{B}^n, g^h)$, is the intersection of an affine subspace of $\mathbb{R}^n$ with $\mathbb{B}^n$. Hence, a line is a 1-plane or chord of $\mathbb{B}^n$ and a hyperplane is a $(n-1)$-plane.

The (hyperbolic) exponential map $\exp_p^h$ in a point $p \in \mathbb{B}^n$ maps any tangent vector $X_p \in T_p\mathbb{B}^n$ to the uniquely determined point q in $\mathbb{B}^n$ such that $d_h(p,q) = \|X_p\|$. For the unit speed geodesic path $\gamma\colon [0, \|X_p\|] \to \mathbb{B}^n$ from p to q we have $\gamma'(0) = \|X_p\|^{-1}X_p$. By (3.3),

$$\exp_0^h(X_0) = \frac{\tanh\|X_0\|}{\|X_0\|}X_0, \quad X_0 \in T_0\mathbb{B}^n \cong \mathbb{R}^n. \tag{3.4}$$

An affine hyperplane H restricted to $\mathbb{B}^n$ can be viewed as an object of hyperbolic space or Euclidean space, depending on whether we choose the hyperbolic metric tensor g^h or the Euclidean metric tensor g^e. The normal vector in any point $p \in H$ also depends on the metric we choose and, therefore, we distinguish between the hyperbolic unit normal vector N_p^h and the Euclidean unit normal vector N_p^e. To be more precise, $N_p^h \in T_p\mathbb{B}^n$ is a unit vector with respect to g^h and N_p^e is a unit vector with respect to g^e. The normal vectors are related and we include a proof of the following fact for the reader's convenience.

Lemma 3.1. *Let $p \in \mathbb{B}^n$, $X_1, \dots, X_{n-1} \in T_p\mathbb{B}^n$ be linearly independent and let H be the linear subspace that is spanned by $\{X_1,\dots,X_{n-1}\}$. Then there are unique $N_p^h, N_p^e \in T_p\mathbb{B}^n$ such that:*

(a) *N_p^h is orthonormal to H with respect to g^h and N_p^e is orthonormal to H with respect to g^e.*

(b) *The frames* $(X_1,\dots,X_{n-1},N_p^h)$ *and* $(X_1,\dots,X_{n-1},N_p^e)$ *are positive oriented.*

(c) *We have that*

$$N_p^h = \sqrt{\frac{1-\|p\|^2}{1-(p\cdot N_p^e)^2}}\,(N_p^e-(p\cdot N_p^e)p). \tag{3.5}$$

Proof. Let g be a positive definite linear form on $T_p\mathbb{B}^n$. Then there is a uniquely determined vector $v\in T_p\mathbb{B}^n$, up to sign, such that H is orthonormal to v with respect to g, i.e., for $i=1,\dots,n-1$, we have that $g(X_i,v)=0$ and $g(v,v)=1$. The sign of v is determined by the condition that $(X_1,\dots,X_{n-1},v)$ is a positive frame, which means that $\det(X_1,\dots,X_{n-1},v)>0$.

To conclude the proof, we only need to show (c). We define v as the vector obtained by the right-hand side of (3.5) and verify that v satisfies (a) and (b) for g^h. Since N_p^h is uniquely determined by these properties we conclude $N_p^h=v$. q.e.d.

A *hyperbolic ball* $B_h^n(p,r)$, is the set of all points $q\in\mathbb{B}^n$ with $d_h(q,p)\le r$. For $p=0$ and by (3.3), we have

$$B_h^n(0,r)=\{q\in\mathbb{B}^n: d_h(q,0)\le r\}=B_e^n(0,\tanh r). \tag{3.6}$$

Hence, the hyperbolic balls with center in the origin are also Euclidean balls in the projective model. For a hyperbolic motion m such that $m(0)=x$ we have $B_h^n(p,r)=m(B_h^n(0,r))=m(B_e^n(0,\tanh r))$. So hyperbolic balls in $\mathbb{B}^n$ are images of Euclidean balls with center 0 under hyperbolic motions and, therefore, ellipsoids.

A subset $C\subseteq\mathbb{B}^n$ is hyperbolic convex if and only if C is convex in the Euclidean sense as a subset of $\mathbb{R}^n$. Planes and open, as well as closed, half-spaces are hyperbolic convex subsets. A *(hyperbolic) convex body* is a compact convex subset of $\mathbb{B}^n$. Recall that $\mathcal{K}_0(\mathbb{B}^n)$ denotes set of convex bodies with non-empty interior contained in $\mathbb{B}^n$.

For a measurable subset $A\subset\mathbb{B}^n$, the *hyperbolic volume* $\mathrm{vol}_n^h(A)$ in the projective model is

$$\mathrm{vol}_n^h(A)=\int_A (1-\|x\|^2)^{-\frac{n+1}{2}}\,d\mathrm{vol}_n^e(x). \tag{3.7}$$

The *hyperbolic Hausdorff distance* δ^h between hyperbolic convex bodies $K,L\in\mathcal{K}(\mathbb{B}^n)$ is defined by

$$\delta^h(K,L)=\max\left\{\sup_{p\in K}\inf_{q\in L} d_h(p,q),\sup_{q\in L}\inf_{p\in K} d_h(p,q)\right\}$$

and the *hyperbolic volume difference metric* θ^h is

$$\theta^h(K,L)=\mathrm{vol}_n^h(K\backslash L)+\mathrm{vol}_n^h(L\backslash K),\quad K,L\in\mathcal{K}_0(\mathbb{B}^n).$$

Hyperbolic convex geometry can be viewed as study of notions on convex bodies $\mathcal{K}_0(\mathbb{B}^n)$ that are invariant under the group of motions $\mathfrak{M}(\mathbb{B}^n)$. For instance, the hyperbolic volume vol_n^h, the hyperbolic Hausdorff distance δ^h and the volume difference metric θ^h, are all invariant with respect to hyperbolic motions and are, therefore, intrinsic notions of hyperbolic convex geometry.

The Euclidean support function $h_K(.)$ of a convex body K measures for any direction $u \in \mathbb{S}^{n-1}$ the signed distance of a supporting hyperplane in direction u to K and the origin. Equivalently, one can use the orthogonal projection $K|\ell_u$ of K to the line ℓ_u through the origin in direction u. Then

$$K|\ell_u = \mathrm{conv}(-h_K(-u)u, h_K(u)u). \tag{3.8}$$

We will define the hyperbolic support function in a similar way, but first we have to recall some further facts about hyperbolic space and the projective model. For a closed convex subset $A \subset \mathbb{H}^n$ and a point $p \in \mathbb{H}^n$ there is a unique point $q \in A$ that minimizes the distance $d_h(p,q)$. The metric projection $p_A \colon \mathbb{H}^n \to A$ assigns to each point p this unique point. Hence,

$$d_h(p, p_A(p)) = \min_{q \in A} d_h(p,q).$$

If $p \notin A$, then $p_A(p) \in \mathrm{bd}\, A$ and the line spanned by $p_A(p)$ and p is perpendicular to the boundary of A in $p_A(p)$. In particular, in the projective model $\mathbb{B}^n$ the projection $K|L$ of a convex body $K \in \mathcal{K}(\mathbb{B}^n)$ to a k-plane L through 0 is given by the Euclidean projection of K to L. This follows, since for any point $p \in \mathbb{B}^n \cong T_p\mathbb{B}^n$ the normal directions $Y_p \in T_p\mathbb{B}^n$ are determined by

$$0 = g_p(p, Y_p) = \frac{p \cdot Y_p}{1 - \|p\|^2} + \frac{\|p\|^2 (p \cdot Y_p)}{(1 - \|p\|^2)^2} = \frac{p \cdot Y_p}{(1 - \|p\|^2)^2}. \tag{3.9}$$

Definition 3.2 (hyperbolic support function)**.** Let $p \in \mathbb{H}^n$ be a fixed point and identify the set of unit vectors in $T_p\mathbb{H}^n$ with $\mathbb{S}^{n-1}$. For any hyperbolic convex body $K \subset \mathbb{H}^n$, the *hyperbolic support function* $h_p^h(K,.) \colon \mathbb{S}^{n-1} \to \mathbb{R}$ of K with respect to p is defined by

$$K|\ell_{p,u} = \exp_p^h\left(\mathrm{conv}\left(-h_p^h(K,-u)u, h_p^h(K,u)u\right)\right), \tag{3.10}$$

where $\ell_{p,u} = \exp_p^h(\mathbb{R}u)$, i.e., the uniquely determined geodesic line in p in direction u.

In the projective model the hyperbolic support function for $p = 0$ is related to the Euclidean support function in the following way.

Lemma 3.3. *Let $K \subset \mathbb{B}^n$ be a convex body. For $u \in \mathbb{S}^{n-1}$, we have that*

$$\tanh h_0^h(K,u) = h_K(u). \tag{3.11}$$

Proof. Since $0 \in \ell_{0,u}$ we have that the hyperbolic projection of K to $\ell_{0,u}$ is the same as the Euclidean projection, see (3.9). Therefore, by (3.8) and the definition of the hyperbolic support function, (3.10), we have that

$$\operatorname{conv}\left(-h_K(-u)u, h_K(u)u\right) = \exp_0^h\left(\operatorname{conv}\left(-h_p^h(K,-u)u, h_p^h(K,u)u\right)\right).$$

Using (3.4), we obtain (3.11). q.e.d.

3.1. Boundary structure of a convex body. Let $K \in \mathcal{K}_0(\mathbb{B}^n)$. The boundary $\operatorname{bd} K$ is a hypersurface that is endowed with a Riemannian structure depending on the metric used in $\mathbb{B}^n$, i.e. either the Euclidean metric tensor g^e or the hyperbolic metric tensor g^h.

The hyperbolic surface area element $d\mathrm{vol}^h_{\operatorname{bd} K}$ is related to the Euclidean surface area element $d\mathrm{vol}^e_{\operatorname{bd} K}$ in the following way: The tangent space $T_x\mathrm{bd}K$ at a boundary point x is a linear subspace of $T_x\mathbb{B}^n$ and by our identification of $T_x\mathbb{B}^n$ with $\mathbb{R}^n$ it does not depend on the underling metric tensor. By (3.5) and (3.7), we find that the Riemannian volume form induced by g^h and g^e on the boundary of K are related, for $X_1, \dots, X_{n-1} \in T_x\mathrm{bd}\, K$, by

$$\begin{aligned}
d\mathrm{vol}^h_{\operatorname{bd} K}\left(X_1,\dots,X_{n-1}\right) &= d\mathrm{vol}^h_n\left(X_1,\dots,X_{n-1},N^h_x\right)\\
&= (1-\|x\|^2)^{-(n+1)/2} d\mathrm{vol}^e_n\left(X_1,\dots,X_{n-1},(N^h_x\cdot N^e_x)N^e_x\right)\\
&= \sqrt{\frac{1-(x\cdot N^e_x)^2}{(1-\|x\|^2)^n}}\, d\mathrm{vol}^e_{\operatorname{bd} K}\left(X_1,\dots,X_{n-1}\right).
\end{aligned}$$

In particular, for $K \in \mathcal{K}_0(\mathbb{B}^n)$ and a measurable function $f\colon \operatorname{bd} K \to \mathbb{R}$, we have that

$$\int\limits_{\operatorname{bd} K} f(x)\, d\mathrm{vol}^h_{\operatorname{bd} K}(x) = \int\limits_{\operatorname{bd} K} f(x)\sqrt{\frac{1-(x\cdot N^e_x)^2}{(1-\|x\|^2)^n}}\, d\mathrm{vol}^e_{\operatorname{bd} K}(x). \tag{3.12}$$

The Riemannian metric induced on the boundary of K is denoted by $\hat{g}^h_p = g^h_p|_{\operatorname{bd} K}$ or $\hat{g}^e_p = g^e_p|_{\operatorname{bd} K}$. If $0 \in \operatorname{bd} K$, then, by (3.1), $\hat{g}^h_0 = \hat{g}^e_0$. Therefore, in the projective model the hyperbolic curvature of $\operatorname{bd} K$ in 0 is the same as the Euclidean curvature. In the following theorem we collect the relations between the hyperbolic notions at a boundary point and the Euclidean ones in the projective model. This is definitely well-known and we again include a proof for convenience.

Theorem 3.4. *Let M be a smooth orientable manifold of dimension $n-1$ immersed in $\mathbb{B}^n$. We denote the metric induced by g^h, resp. g^e, on M by $\hat{g}^h$, resp. $\hat{g}^e$. The unique unit normal vector field along M is denoted by N^h, resp. N^e. For $x \in M$, we have*

$$\det \hat{g}^h_x = (1-\|x\|^2)^{-n}(1-(N^e_x\cdot x)^2).$$

Denoting the covariant derivative on $\mathbb{B}^n$ by ∇^h, resp. ∇^e, the second fundamental form $\hat{h}^h$, resp. $\hat{h}^e$, is determined by

$$\nabla^*_X Y = \overline{\nabla}^*_X Y + \hat{h}^*(X,Y)N^*,$$

where $\overline{\nabla}^h$, resp. $\overline{\nabla}^e$, denotes the induced covariant derivative on M. Then

$$\hat{h}^h_x = \hat{h}^e_x \left(1-\|x\|^2\right)^{-1/2} \left(1-(N^e_x\cdot x)^2\right)^{-1/2}.$$

Let $S^h_x = (\hat{g}^h_x)^{-1}\hat{h}^h_x$ be the shape operator, i.e., the $(1,1)$-tensor equivalent to $\hat{h}^h_x$ and obtained by raising an index. For the Gauss–Kronecker curvature $H^*_{n-1}(M,x) = \det S^*_x$, we obtain

$$H^h_{n-1}(M,x) = H^e_{n-1}(M,x)\left(1-\|x\|^2\right)^{\frac{n+1}{2}}\left(1-(N^e_x\cdot x)^2\right)^{-\frac{n+1}{2}}. \tag{3.13}$$

Proof. Let g^* be a Riemannian metric tensor of $\mathbb{B}^n$. We identify T_xM with the $n-1$ dimensional subspace $\{X_x \in T_x\mathbb{B}^n \cong \mathbb{R}^n : X_x\cdot N^e_x = 0\}$. For $X_x, Y_x \in T_xM$, the induced metric tensor $\hat{g}^*$ is determined by $\hat{g}^*_x(X_x,Y_x) = g^*_x(X_x,Y_x)$. In particular, for g^h we have

$$\hat{g}^h_x(X_x,Y_x) = \frac{X_x\cdot Y_x}{1-\|x\|^2} + \frac{(x\cdot X_x)(x\cdot Y_x)}{(1-\|x\|^2)^2}.$$

We put $\overline{x} = x-(N^e_x\cdot x)N^e_x$. Then, for $X_x\in T_xM$, we have that $x\cdot X_x = \overline{x}\cdot X_x$ and $\|\overline{x}\| = \sqrt{\|x\|^2-(N^e_x\cdot x)^2}$. We define the matrix $A = (1-\|x\|^2)^{-1}\mathrm{Id}_n + (1-\|x\|^2)^{-2}\overline{x}\,\overline{x}^\top$ and obtain

$$A\frac{\overline{x}}{\|\overline{x}\|} = \frac{\overline{x}}{\|\overline{x}\|(1-\|x\|^2)} + \frac{\|\overline{x}\|^2\overline{x}}{\|\overline{x}\|(1-\|x\|^2)^2} = \frac{1-(N^e_x\cdot x)^2}{(1-\|x\|^2)^2}\frac{\overline{x}}{\|\overline{x}\|}.$$

For $v\in\overline{x}^\perp$, we have that $Av = (1-\|p\|^2)^{-1}v$. By definition of A, $\hat{g}^h_x(X_x,Y_x) = X_x^\top A Y_x$. We conclude that

$$\det\hat{g}^h_x = \det A = (1-\|x\|^2)^{-n}(1-(N^e_x\cdot x)^2).$$

We know that $(\mathbb{B}^n, g^h)$ and $(\mathbb{B}^n, g^e)$ have the same (pre-)geodesics. This implies, see e.g. [**15**, (40.7)], that there is a function ψ such that

$$\frac{\partial\log\det(g^h)}{\partial x^i} = \frac{\partial\log\det(g^e)}{\partial x^i} + 2(n+1)\frac{\partial\psi}{\partial x^i}.$$

Since $\log\det(g^h) = -(n+1)\log\left(1-\|x\|^2\right)$ and $\det(g^e)=1$, we conclude that $\frac{\partial\psi}{\partial x^i} = (1-\|x\|^2)^{-1}x_i$. Consequently, see e.g. [**15**, (40.6)], for the 1-form ρ defined by

$$\rho(X_x) = (1-\|x\|^2)^{-1}(x\cdot X_x), \tag{3.14}$$

the covariant derivative with respect to g^h can be written as

$$\nabla^h_X Y = \nabla^e_X Y + \rho(X)Y + \rho(Y)X. \tag{3.15}$$

Combining (3.5), (3.14) and (3.15), a straightforward calculation shows that

$$g_x^h\left((\nabla_X^h Y)_x, N_x^h\right) = \left(1-\|x\|^2\right)^{-1/2}\left(1-(x\cdot N_x^e)^2\right)^{-1/2}\left((\nabla_X^e Y)_x \cdot N_x^e\right).$$

This concludes the proof, since $\hat{h}_x^*(X_x, Y_x) = g_x^*((\nabla_X^* Y)_x, N_x^*)$ and (3.13) follows from $\det S_x^h = \det(\hat{h}_x^h)/\det(\hat{g}_x^h)$. q.e.d.

An immediate consequence of this theorem is, that for smooth convex bodies the hyperbolic Gauss–Kronecker curvature and the Euclidean Gauss–Kronecker curvature are related by (3.13). This can be generalized to general convex bodies with the usual methods: For $K \in \mathcal{K}_0(\mathbb{B}^n)$ we call a boundary point $x \in \operatorname{bd} K$ *normal*, if $\operatorname{bd} K$ at x can locally be expressed as the graph of a convex function that is second order differentiable in x, see e.g. [**23**, p. 4]. Hence, in a normal boundary point the Gauss–Kronecker curvature is defined and since almost all boundary points are normal, see e.g. [**43**, Thm. 2.5.5], we obtain a generalized notion of hyperbolic Gauss–Kronecker curvature $H_{n-1}^h(K,x)$ for arbitrary convex bodies $K \in \mathcal{K}_0(\mathbb{B}^n)$.

Corollary 3.5. *Let $K \in \mathcal{K}_0(\mathbb{B}^n)$. In a normal boundary point $x \in \operatorname{bd} K$, we have*

$$H_{n-1}^h(K,x) = H_{n-1}^e(K,x)\left(\frac{1-\|x\|^2}{1-(N_x^e\cdot x)^2}\right)^{(n+1)/2}. \tag{3.16}$$

The following proposition is well-known, see e.g. [**47**, Lem. 3]. It is a change of variables formula, where we switch from integration in Cartesian coordinates to integration along rays from the origin with the directions parametrized by the boundary of a convex body.

Proposition 3.6 (Euclidean cone volume formula, see [**47**, Lem. 3])**.** *Let $K, L \in \mathcal{K}_0(\mathbb{R}^n)$ such that $L \subseteq K$ and $0 \in \operatorname{int} L$. For $x \in \operatorname{bd} K$ we set $\{x_L\} = \operatorname{bd} L \cap \operatorname{pos}\{x\}$. Then*

$$\operatorname{vol}_n^e(K\backslash L) = \int_{\operatorname{bd} K} \frac{h_K(N_x^e)}{\|x\|^n} \int_{\|x_L\|}^{\|x\|} t^{n-1}\, dt\, d\operatorname{vol}_{\operatorname{bd} K}^e(x).$$

There is an analog of the above in hyperbolic convex geometry.

Proposition 3.7 (Hyperbolic cone volume formula)**.** *Let $K, L \in \mathcal{K}_0(\mathbb{B}^n)$ such that $L \subseteq K$ and $0 \in \operatorname{int} L$. For $x \in \operatorname{bd} K$ we set $\{x_L\} = \operatorname{bd} L \cap \operatorname{pos}\{x\}$. Then*

$$\operatorname{vol}_n^h(K\backslash L) = \int_{\operatorname{bd} K} \frac{\sinh(h_0^h(K, N_x^e))}{\sinh(d_h(x,0))^n} \int_{d_h(0,x_L)}^{d_h(0,x)} \sinh(t)^{n-1}\, dt\, d\operatorname{vol}_{\operatorname{bd} K}^h(x). \tag{3.17}$$

Proof. By Lemma 3.3, Proposition 3.6 and (3.3),

$$\begin{aligned}\operatorname{vol}_n^h(K\backslash L) &= \int\limits_{\operatorname{bd} K} \frac{\tanh(h_0^h(K,N_x^e))}{\tanh(d_h(x,0))^n} \int\limits_{\tanh d_h(x_L,0)}^{\tanh d_h(x,0)} \frac{t^{n-1}}{(1-t^2)^{(n+1)/2}}\, dt\, d\operatorname{vol}_{\operatorname{bd} K}^e(x)\\ &= \int\limits_{\operatorname{bd} K} \frac{\tanh(h_0^h(K,N_x^e))}{\tanh(d_h(x,0))^n} \int\limits_{d_h(x_L,0)}^{d_h(x,0)} \sinh(t)^{n-1}\, dt\, d\operatorname{vol}_{\operatorname{bd} K}^e(x).\end{aligned}$$

Since $h_K(N_x^e) = x\cdot N_x^e$ and by (3.3) and (3.12), we have that

$$d\operatorname{vol}_{\operatorname{bd} K}^h(x) = \frac{\cosh(d_h(x,0))^n}{\cosh(h_0^h(K,N_x^e))}\, d\operatorname{vol}_{\operatorname{bd} K}^e. \qquad \text{q.e.d.}$$

3.2. A Euclidean model for real space forms. Similar to the projective model, we may define a Euclidean model for space forms $\operatorname{Sp}^n(\lambda)$ of arbitrary curvature λ. Let

$$\mathbb{B}^n(\lambda) := \begin{cases} \left(1/\sqrt{-\lambda}\right)\mathbb{B}^n & \text{if } \lambda < 0,\\ \mathbb{R}^n & \text{else.}\end{cases}$$

Further, define a Riemannian metric g^λ on $\mathbb{B}^n(\lambda)$ by

$$g^\lambda(X_p,Y_p) = \frac{X_p\cdot Y_p}{1+\lambda\|p\|^2} - \lambda\frac{(X_p\cdot p)(Y_p\cdot p)}{(1+\lambda\|p\|^2)^2}, \quad X_p,Y_p\in T_p\mathbb{B}^n(\lambda). \tag{3.18}$$

Then $(\mathbb{B}^n(\lambda), g^\lambda)$ is a Riemannian manifold of constant sectional curvature λ. By the Killing–Hopf Theorem there is, up to isometry, only one simply-connected and complete Riemannian manifold $\operatorname{Sp}^n(\lambda)$ of constant sectional curvature $\lambda\in\mathbb{R}$, see e.g. [**27**, Ch. 6], [**28**, Thm. 1.9] or [**38**, Ch. 8, Cor. 25]. Thus, for $\lambda \le 0$, $(\mathbb{B}^n(\lambda), g^\lambda)$ is isometric to $\operatorname{Sp}^n(\lambda)$ and for $\lambda > 0$, $(\mathbb{B}^n(\lambda), g^\lambda)$ is isometric to an open hemisphere of $\operatorname{Sp}^n(\lambda)$.

Geodesics in $(\mathbb{B}^n(\lambda), g^\lambda)$ are Euclidean straight lines intersected with $\mathbb{B}^n(\lambda)$. Therefore, the set of geodesically convex bodies in $(\mathbb{B}^n(\lambda), g^\lambda)$ is equivalent to $\mathcal{K}^n(\mathbb{B}^n(\lambda))$, i.e. the Euclidean convex bodies contained in $\mathbb{B}^n(\lambda)$. Note that in the spherical setting, $\lambda > 0$, we define proper convex bodies as convex bodies contained in an open hemisphere. Hence, when investigating a fixed proper convex body $K\in\mathcal{K}(\operatorname{Sp}^n(\lambda))$, we may use the model $(\mathbb{B}^n(\lambda), g^\lambda)$ and identify K with a convex body in $\mathcal{K}(\mathbb{B}^n(\lambda))$.

It is useful to define

$$\tan^\lambda \alpha = \begin{cases} \tanh\left(\sqrt{-\lambda}\alpha\right)/\sqrt{-\lambda} & \text{if } \lambda < 0,\\ \alpha & \text{if } \lambda = 0,\\ \tan\left(\sqrt{\lambda}\alpha\right)/\sqrt{\lambda} & \text{if } \lambda > 0.\end{cases}$$

Then the geodesic distance d_λ between a point $p\in\mathbb{B}^n(\lambda)$ and the origin is given by

$$\tan^\lambda d_\lambda(p,0) = d^e(p,0) = \|p\|. \tag{3.19}$$

For a geodesic ball $B^n_\lambda(0,\alpha)$ with center at the origin and geodesic radius α we have $B^n_\lambda(0,\alpha) = B^n_e(0,\tan^\lambda \alpha)$, i.e., geodesic balls with center at the origin are Euclidean balls.

The volume element in $(\mathbb{B}^n, g^\lambda)$ is

$$d\mathrm{vol}_n^\lambda(p) = (1+\lambda\|p\|^2)^{-(n+1)/2}\, d\mathrm{vol}_n^e(p). \tag{3.20}$$

For a convex body $K \subset \mathbb{B}^n(\lambda)$ we define a support function $h^\lambda_p(K,.)$ with respect to a fixed point $p \in \mathbb{B}^n(\lambda)$ similar to Definition 3.2. If $p=0$, then

$$\tan^\lambda h_0^\lambda(K,u) = h_K(u), \quad u \in \mathbb{S}^{n-1}. \tag{3.21}$$

For a fixed convex body $K \in \mathcal{K}_0(\mathbb{B}^n(\lambda))$ and a regular boundary point $x \in \operatorname{bd} K$ we can compare the outer unit vector N^λ_x with respect to g^λ with the Euclidean outer unit normal. Analogous to Lemma 3.1 we find that

$$N_p^\lambda = \sqrt{\frac{1+\lambda\|x\|^2}{1+\lambda(x\cdot N_x^e)^2}}\,\left(N_x^e + \lambda(x\cdot N_x^e)x\right). \tag{3.22}$$

This implies that

$$d\mathrm{vol}^\lambda_{\operatorname{bd} K}(x) = \sqrt{\frac{1+\lambda(N_x^e\cdot x)^2}{(1+\lambda\|x\|^2)^n}}\, d\mathrm{vol}^e_{\operatorname{bd} K}(x). \tag{3.23}$$

Finally, we can also adapt Theorem 3.4 and conclude that for normal boundary points $x \in \operatorname{bd} K$,

$$H^\lambda_{n-1}(K,x) = H^e_{n-1}(K,x)\left(\frac{1+\lambda\|x\|^2}{1+\lambda(N_x^e\cdot x)^2}\right)^{(n+1)/2}. \tag{3.24}$$

For $\lambda = 1$, we already obtained (3.20), (3.21), (3.23) and (3.24) in [**7**, (4.8), (4.3), (4.11) and (4.14)].

4. The floating body in real space forms

For a convex body $K \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$ and $\delta > 0$, we define the λ-floating body by

$$\mathcal{F}^\lambda_\delta K = \bigcap \left\{ H^- : \mathrm{vol}_n^\lambda(K\cap H^+) \le \delta^{\frac{n+1}{2}} \right\}. \tag{4.1}$$

In the Euclidean model $(\mathbb{B}^n(\lambda), g^\lambda)$, the λ-floating body is a weighted floating body, that is, by (3.20), we have $\mathcal{F}^\lambda_\delta K = \mathcal{F}^\mu_\delta K$ for $\mu = \mathrm{vol}^\lambda_n$. Note that for $\lambda = 0$, we obtain the well known Euclidean (convex) floating body $\mathcal{F}^0_\delta K$, see e.g. [**47**]. For $\lambda = 1$, we obtain the spherical floating body $\mathcal{F}^1_\delta K$ introduced in [**7**]. Finally, for $\lambda = -1$ we obtain the new notion of hyperbolic floating body $\mathcal{F}^{-1}_\delta K$.

By Proposition 2.3 we have that

$$\mathcal{F}_\delta^\lambda K = \left[h_K - s_\delta^\lambda\right] = \bigcap_{v\in\mathbb{S}^{n-1}} H^-_{0,v,h_K(v)-s_\delta^\lambda(v)}, \tag{4.2}$$

where s_δ^λ is determined by

$$\delta^{\frac{n+1}{2}} = \mathrm{vol}_n^\lambda\left(K\cap H^+_{0,v,h_K(v)-s_\delta^\lambda(v)}\right).$$

The λ-floating body can be bounded by the Euclidean (convex) floating body in the following way.

Lemma 4.1. *Let $K\in\mathcal{K}_0(\mathbb{B}^n(\lambda))$, $p\in\operatorname{int}K$ and $0\leq\alpha<\beta$ be such that $B_\lambda^n(p,\alpha)\subset K\subseteq B_\lambda^n(p,\beta)$. We set*

$$\delta_1 := \delta\left(1+\lambda\tan^\lambda\left(d_\lambda(0,p)-\alpha\right)^2\right),$$
$$\delta_2 := \delta\left(1+\lambda\tan^\lambda\left(d_\lambda(0,p)+\beta\right)^2\right).$$

If $\delta>0$ is small enough so that $B_\lambda^n(p,\alpha)\subseteq\mathcal{F}_\delta^\lambda K$, then

$$\begin{cases}\mathcal{F}^0_{\delta_1}K\subseteq\mathcal{F}_\delta^\lambda K\subseteq\mathcal{F}^0_{\delta_2}K & \text{if } \lambda<0,\\ \mathcal{F}^0_{\delta_2}K\subseteq\mathcal{F}_\delta^\lambda K\subseteq\mathcal{F}^0_{\delta_1}K & \text{if } \lambda>0.\end{cases} \tag{4.3}$$

Proof. It will be convenient to use the substitution $\tan^\lambda s(v,\delta) = h_K(v)-s_\delta^\lambda(v)$ in (4.2) to obtain

$$\mathcal{F}_\delta^\lambda K = \bigcap_{v\in\mathbb{S}^{n-1}} H^-_{0,v,\tan^\lambda s(v,\delta)}, \tag{4.4}$$

where $s(v,\delta)$ is determined by

$$\delta^{\frac{n+1}{2}} = \mathrm{vol}_n^\lambda\left(K\cap H^+_{0,v,\tan^\lambda s(v,\delta)}\right). \tag{4.5}$$

Since $K\subseteq B_\lambda^n(p,\beta)$ and by (3.19), we conclude that, for all $q\in K$, $\|q\|\leq\tan^\lambda\left(d_\lambda(0,p)+\beta\right)$. This implies that

$$\begin{cases}(1+\lambda\|q\|^2)^{-1}\leq\cosh\left(\sqrt{-\lambda}(d_\lambda(0,p)+\beta)\right)^2 & \text{if } \lambda<0,\\ (1+\lambda\|q\|^2)^{-1}\geq\cos\left(\sqrt{\lambda}(d_\lambda(0,p)+\beta)\right)^2 & \text{if } \lambda>0.\end{cases}$$

Using this, (3.19) and (4.5), we obtain

$$\begin{cases}\delta^{\frac{n+1}{2}}\leq\cosh\left(\sqrt{-\lambda}(d_\lambda(0,p)+\beta)\right)^{n+1}\mathrm{vol}_n^e\left(K\cap H^+_{0,v,\tan^\lambda s(v,\delta)}\right) & \text{if } \lambda<0,\\ \delta^{\frac{n+1}{2}}\geq\cos\left(\sqrt{\lambda}(d_\lambda(0,p)+\beta)\right)^{n+1}\mathrm{vol}_n^e\left(K\cap H^+_{0,v,\tan^\lambda s(v,\delta)}\right) & \text{if } \lambda>0.\end{cases}$$

For $\lambda<0$, let $t\left(v,\delta_2\right)$ be such that

$$\delta_2^{\frac{n+1}{2}} = \left(\delta\cosh\left(\sqrt{-\lambda}(d_\lambda(0,p)+\beta)\right)^{-2}\right)^{\frac{n+1}{2}} = \mathrm{vol}_n^e\left(K\cap H^+_{0,v,t(v,\delta_2)}\right).$$

Then $\tan^\lambda s(v,\delta) \leq t\,(v,\delta_2)$ and, therefore,

$$\mathcal{F}_\delta^\lambda\, K = [\tan^\lambda s(.,\delta)] \subseteq [t\,(.,\delta_2)] = \mathcal{F}_{\delta_2}^0\, K.$$

For $\lambda > 0$, an analogous argument gives $\tan^\lambda s(v,\delta) \geq t\,(v,\delta_2)$, which yields $\mathcal{F}_\delta^\lambda\, K \supseteq \mathcal{F}_{\delta_2}^0\, K$.

For the other inclusions we first note that $B_\lambda^n(p,\alpha) \subseteq \mathcal{F}_\delta^\lambda\, K \subseteq K$ implies $\|q\| \geq \left|\tan^\lambda (d_\lambda(0,p)-\alpha)\right|$, for all $q \in K\backslash\mathcal{F}_\delta^\lambda\, K$. By an argument analogous to the above, we find that

$$\begin{cases} \delta^{\frac{n+1}{2}} \geq \cosh\left(\sqrt{-\lambda}(d_\lambda(0,p)-\alpha)\right)^{n+1} \mathrm{vol}_n^e\left(K \cap H^+_{0,v,\tan^\lambda s(v,\delta)}\right) & \text{if } \lambda < 0,\\ \delta^{\frac{n+1}{2}} \leq \cos\left(\sqrt{\lambda}(d_\lambda(0,p)-\alpha)\right)^{n+1} \mathrm{vol}_n^e\left(K \cap H^+_{0,v,\tan^\lambda s(v,\delta)}\right) & \text{if } \lambda > 0. \end{cases}$$

Hence, we have that $\mathcal{F}_\delta^\lambda\, K \supseteq \mathcal{F}_{\delta_1}^0\, K$, for $\lambda < 0$, respectively, $\mathcal{F}_\delta^\lambda\, K \subseteq \mathcal{F}_{\delta_1}^0\, K$, for $\lambda > 0$. q.e.d.

A special case of Lemma 4.1 for $\lambda = 1$ has been obtained in [**7**, Thm. 5.2].

Let $K \in \mathcal{K}_0(\mathbb{B}^n(\lambda))$ be such that $0 \in \operatorname{int} K$. For $x \in \operatorname{bd} K$ we denote by x_δ^K the uniquely determined intersection point of $\operatorname{bd}\mathcal{F}_\delta^\lambda\, K$ with the ray $\operatorname{pos}\{x\}$. We obtain the following corollary to Lemma 2.9.

Corollary 4.2. *Let $K \in \mathcal{K}_0(\mathbb{B}^n(\lambda))$ be such that $0 \in \operatorname{int} K$ and let $x \in \operatorname{bd} K$ be a regular and exposed point. For $\varepsilon > 0$ set $K' = K \cap B_\lambda^n(x,\varepsilon)$. Then $x \in \operatorname{bd} K'$ is a regular and exposed point of K'. Moreover, there exists δ_ε such that for all $\delta < \delta_\varepsilon$, we have that $x_\delta^{K'} = x_\delta^K$.*

Proof. We may move K by an isometry of $(\mathbb{B}^n(\lambda), g^\lambda)$ so that $0 \in \operatorname{int} K'$. Since the geodesic balls $B_\lambda^n(p,\alpha)$ are ellipsoids, there exists a small Euclidean ball with the same center p that is contained in $B_\lambda^n(p,\alpha)$. Hence, without loss of generality, there is $\eta := \eta(\varepsilon,x,K) > 0$ such that $0 \in \operatorname{int}(K \cap B_e^n(x,\eta)) \subseteq \operatorname{int} K'$. We set $K'' = K \cap B_e^n(x,\eta)$.

We apply Lemma 2.9 for $\mu = \mathrm{vol}_n^\lambda$ and $\varepsilon = \eta$, and obtain $(\mathcal{F}_\delta^\lambda\, K) \cap \operatorname{pos}\{x\} = (\mathcal{F}_\delta^\lambda\, K'') \cap \operatorname{pos}\{x\}$, for all $\delta < \delta_\eta$. Note that $K \subseteq L$ implies $F_\delta^\mu\, K \subseteq F_\delta^\mu\, L$. This yields

$$\begin{aligned}(\mathcal{F}_\delta^\lambda\, K) \cap \operatorname{pos}\{x\} &= (\mathcal{F}_\delta^\lambda\, K'') \cap \operatorname{pos}\{x\}\\ &\subseteq (\mathcal{F}_\delta^\lambda\, K') \cap \operatorname{pos}\{x\} \subseteq (\mathcal{F}_\delta^\lambda\, K) \cap \operatorname{pos}\{x\}.\end{aligned}$$

Hence, $(\mathcal{F}_\delta^\lambda\, K) \cap \operatorname{pos}\{x\} = (\mathcal{F}_\delta^\lambda\, K') \cap \operatorname{pos}\{x\}$ and, therefore, $x_\delta^K = x_\delta^{K'}$ for all $\delta < \delta_\eta =: \delta_\varepsilon$. q.e.d.

4.1. Proof of Theorem 1.2. We are now ready to prove Theorem 1.2. For a (proper) convex body $K \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$ we consider the Euclidean model $(\mathbb{B}^n(\lambda), g^\lambda)$ for $\mathrm{Sp}^n(\lambda)$ and identify K with an Euclidean convex body in $\mathbb{B}^n(\lambda)$ such that $0 \in \operatorname{int} K$.

Analogous to Proposition 3.7 we obtain the following.

Proposition 4.3. *Let* $K, L \in \mathcal{K}_0(\mathbb{B}^n(\lambda))$ *be such that* $L \subseteq K$ *and* $0 \in \operatorname{int} L$. *For* $x \in \operatorname{bd} K$ *we set* $\{x_L\} = \operatorname{bd} L \cap \operatorname{pos}\{x\}$. *Then*

$$\operatorname{vol}_n^\lambda (K \backslash L) = \int\limits_{\operatorname{bd} K} \frac{x \cdot N_x^e}{\|x\|^n} \int\limits_{\|x_L\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt\, d\operatorname{vol}_{\operatorname{bd} K}^e(x).$$

Let $\delta > 0$ be small enough, so that $0 \in \operatorname{int} \mathcal{F}_\delta^\lambda K$. To prove Theorem 1.2 we have to show that

$$\lim_{\delta \to 0^+} \frac{\operatorname{vol}_n^\lambda \left(K \backslash \mathcal{F}_\delta^\lambda K\right)}{\delta} = c_n \int\limits_{\operatorname{bd} K} H_{n-1}^\lambda (K, x)^{\frac{1}{n+1}}\, d\operatorname{vol}_{\operatorname{bd} K}^\lambda (x).$$

By Proposition 4.3, we have

$$\frac{\operatorname{vol}_n^\lambda \left(K \backslash \mathcal{F}_\delta^\lambda K\right)}{\delta} = \int\limits_{\operatorname{bd} K} \frac{x \cdot N_x^e}{\delta \|x\|^n} \int\limits_{\|x_\delta^\lambda\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt\, d\operatorname{vol}_{\operatorname{bd} K}^e(x). \tag{4.6}$$

We will first show that the integrand is uniformly bounded in δ by an integrable function.

Lemma 4.4. *Let* $K \in \mathcal{K}_0(\mathbb{B}^n(\lambda))$ *and* $0 \in \operatorname{int} K$. *Then there exists* $\alpha, \beta > 0$ *and* $\delta_0 > 0$ *such that* $B_\lambda^n(0, \alpha) \subseteq \operatorname{int} \mathcal{F}_\delta^\lambda K$ *for all* $\delta \le \delta_0$ *and* $K \subset B_\lambda^n(0, \beta)$. *Furthermore, for regular boundary points* $x \in \operatorname{bd} K$ *and for* $0 < \delta < \delta_0$, *define*

$$f(x, \delta) := \frac{x \cdot N_x^e}{\delta \|x\|^n} \int\limits_{\|x_\delta^\lambda\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt. \tag{4.7}$$

Then $f(x, \delta)$ *is bounded from above for all* $\delta < \delta_0$ *by an integrable function* $g(x)$, *for almost all* $x \in \operatorname{bd} K$.

Proof. Since $0 \in \operatorname{int} K$, there is $\delta_0 > 0$ such that $0 \in \operatorname{int} \mathcal{F}_{\delta_0}^\lambda K$. Thus, there exists $\alpha > 0$ such that $B_\lambda^n(0, \alpha) \subseteq \operatorname{int} \mathcal{F}_{\delta_0}^\lambda K$ and, by monotonicity, this yields $B_\lambda^n(0, \alpha) \subseteq \mathcal{F}_\delta^\lambda K$, for all $\delta \le \delta_0$. Furthermore, since K is bounded, there exists $\beta > 0$ such that $K \subseteq B_\lambda^n(0, \beta)$. By (3.19), this implies that $\tan^\lambda \alpha \le \|x_\delta^\lambda\| \le \|x\| \le \tan^\lambda \beta$ for all $\delta < \delta_0$.

We set

$$\widetilde{\delta} := \begin{cases} \delta \cosh(\sqrt{-\lambda}\alpha)^{-2} & \text{if } \lambda < 0, \\ \delta \cos(\sqrt{\lambda}\beta)^{-2} & \text{if } \lambda > 0. \end{cases} \tag{4.8}$$

By Lemma 4.1, we have that $\mathcal{F}^e_{\widetilde{\delta}} K \subseteq \mathcal{F}^\lambda_\delta K$ and, therefore, $\|x^e_{\widetilde{\delta}} - x\| \geq \|x^\lambda_\delta - x\|$. For $\|x^\lambda_\delta\| \leq t \leq \|x\|$, we obtain

$$\frac{1}{(1+\lambda t^2)^{\frac{n+1}{2}}} \leq \begin{cases} \cosh(\sqrt{-\lambda}\beta)^{n+1} & \text{if } \lambda < 0, \\ \cos(\sqrt{\lambda}\alpha)^{n+1} & \text{if } \lambda > 0. \end{cases} \tag{4.9}$$

We conclude that

$$\begin{aligned} \frac{1}{\delta}\left(\frac{x}{\|x\|}\cdot N^e_x\right) \int\limits_{\|x^\lambda_\delta\|}^{\|x\|} \left(\frac{t}{\|x\|}\right)^{n-1} (1+\lambda t^2)^{-\frac{n+1}{2}}\, dt \\ \leq C\left(\frac{x}{\|x\|}\cdot N^e_x\right) \frac{\left\|x^e_{\widetilde{\delta}} - x\right\|}{\widetilde{\delta}}, \end{aligned} \tag{4.10}$$

where we put, for $\lambda < 0$, $C := \cosh(\sqrt{-\lambda}\beta)^{n+1}\cosh(\sqrt{-\lambda}\alpha)^{-2}$, respectively, for $\lambda > 0$, $C := \cos(\sqrt{\lambda}\alpha)^{n+1}\cos(\sqrt{\lambda}\beta)^{-2}$.

This concludes the proof, since the right-hand side of (4.10) is the same integrand we obtain for the Euclidean (convex) floating body and is, therefore, bounded uniformly in $\widetilde{\delta}$ by an integrable function for almost all $x \in \operatorname{bd} K$, by [**47**, Lem. 5 and Lem. 6]. q.e.d.

It only remains to show that (4.7) converges point-wise for almost all boundary points. Since almost all boundary points are normal, it is sufficient to show the following.

Lemma 4.5. *Let $K \in \mathcal{K}_0(\mathbb{B}^n(\lambda))$ and $0 \in \operatorname{int} K$. Then, for normal boundary points $x \in \operatorname{bd} K$, we have that*

$$\lim_{\delta\to 0^+} \frac{x\cdot N^e_x}{\delta\|x\|^n} \int\limits_{\|x^\lambda_\delta\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt = c_n \frac{H^e_{n-1}(K,x)^{\frac{1}{n+1}}}{(1+\lambda\|x\|^2)^{\frac{n-1}{2}}}, \tag{4.11}$$

where $c_n = \frac{1}{2}\left((n+1)/\kappa_{n-1}\right)^{2/(n+1)}$.

Proof. A normal boundary point $x \in \operatorname{bd} K$ has a unique outer unit normal N^e_x and the Gauss–Kronecker curvature $H^e_{n-1}(K,x)$ exists. We first consider the case that $H^e_{n-1}(K,x) = 0$ and show that the left-hand side of (4.11) converges to 0, for $\delta \to 0^+$. With $\widetilde{\delta}$ as defined by (4.8) in Lemma 4.1, we find again the upper bound (4.10). The function in the upper bound is the same as the integrand we obtain for the Euclidean convex floating body and, therefore, it converges to 0, for $\widetilde{\delta} \to 0^+$, by [**47**, Lem. 7 and Lem. 10]. This implies that

$$\limsup_{\delta\to 0^+} \frac{x\cdot N^e_x}{\delta\|x\|^n} \int\limits_{\|x^\lambda_\delta\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt \leq C \limsup_{\widetilde{\delta}\to 0^+} \left(\frac{x}{\|x\|}\cdot N^e_x\right) \frac{\left\|x^e_{\widetilde{\delta}} - x\right\|}{\widetilde{\delta}} = 0.$$

Next, let $H^e_{n-1}(K,x) > 0$. Let $\varepsilon > 0$ be arbitrary and set $K' = K \cap B^n_\lambda(x,\varepsilon)$. Furthermore, let p be a point inside K' and on the segment spanned by x and the origin, that is, $p \in \operatorname{int} K' \cap \operatorname{pos}\{x\}$. For $\alpha = 0$ and $\beta = \varepsilon$ define δ_1 and δ_2 as in Lemma 4.1. Then, for δ small enough, we have $p \in \operatorname{int} \mathcal{F}^\lambda_\delta K'$. Thus, for $\lambda < 0$, $\mathcal{F}^0_{\delta_1} K' \subseteq \mathcal{F}^\lambda_\delta K' \subseteq \mathcal{F}^0_{\delta_2} K'$, respectively, for $\lambda > 0$, $\mathcal{F}^0_{\delta_1} K' \supseteq \mathcal{F}^\lambda_\delta K' \supseteq \mathcal{F}^0_{\delta_2} K'$. Corollary 4.2 implies that

$$\{x^\lambda_\delta\} = \operatorname{bd} \mathcal{F}^\lambda_\delta K \cap \operatorname{conv}(x,p) = \operatorname{bd} \mathcal{F}^\lambda_\delta K' \cap \operatorname{conv}(x,p).$$

This yields

$$\begin{cases} \|x - x^e_{\delta_1}\| \geq \|x - x^\lambda_\delta\| \geq \|x - x^e_{\delta_2}\| & \text{if } \lambda < 0, \\ \|x - x^e_{\delta_1}\| \leq \|x - x^\lambda_\delta\| \leq \|x - x^e_{\delta_2}\| & \text{if } \lambda > 0. \end{cases} \tag{4.12}$$

Hence, for $\lambda < 0$, we have

$$\frac{x \cdot N^e_x}{\delta \|x\|^n} \int_{\|x^\lambda_\delta\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt \geq \frac{\frac{x}{\|x\|} \cdot N^e_x}{(1+\lambda \|x^\lambda_\delta\|^2)^{\frac{n+1}{2}}} \left(\frac{\|x^\lambda_\delta\|}{\|x\|}\right)^{n-1} \frac{\delta_2}{\delta} \frac{\|x - x^e_{\delta_2}\|}{\delta_2},$$

$$\frac{x \cdot N^e_x}{\delta \|x\|^n} \int_{\|x^\lambda_\delta\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt \leq \frac{\frac{x}{\|x\|} \cdot N^e_x}{(1+\lambda \|x\|^2)^{\frac{n+1}{2}}} \frac{\delta_1}{\delta} \frac{\|x - x^e_{\delta_1}\|}{\delta_1}.$$

Conversely, if $\lambda > 0$, then

$$\frac{x \cdot N^e_x}{\delta \|x\|^n} \int_{\|x^\lambda_\delta\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt \geq \frac{\frac{x}{\|x\|} \cdot N^e_x}{(1+\lambda \|x\|^2)^{\frac{n+1}{2}}} \left(\frac{\|x^\lambda_\delta\|}{\|x\|}\right)^{n-1} \frac{\delta_1}{\delta} \frac{\|x - x^e_{\delta_1}\|}{\delta_1},$$

$$\frac{x \cdot N^e_x}{\delta \|x\|^n} \int_{\|x^\lambda_\delta\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\, dt \leq \frac{\frac{x}{\|x\|} \cdot N^e_x}{(1+\lambda \|x^\lambda_\delta\|^2)^{\frac{n+1}{2}}} \frac{\delta_2}{\delta} \frac{\|x - x^e_{\delta_2}\|}{\delta_2}.$$

To finish the proof we first notice that the functions that appear on the right-hand side of the above inequalities are again related to the integrand that is obtained for the Euclidean convex floating body. Hence, by [**47**, Lem. 7 and Lem. 11], for $\delta_* \in \{\delta_1, \delta_2\}$,

$$\lim_{\delta_* \to 0^+} \frac{x}{\|x\|} \cdot N^e_x \frac{\|x - x^e_{\delta_*}\|}{\delta_*} = c_n H^e_{n-1}(K,x)^{\frac{1}{n+1}}.$$

By the choice of p, we have $|\|x\| - \varepsilon| \leq \|p\| \leq \|x\|$. For $\lambda < 0$, by the definition of δ_1 and δ_2, there exist positive constants $C_1, C_2 > 0$, such that

$$\frac{\delta_1}{\delta} \leq (1+\lambda\|x\|^2)(1+C_1\varepsilon) \text{ and } \frac{\delta_2}{\delta} \geq (1+\lambda\|x\|^2)(1-C_2\varepsilon).$$

Therefore,

$$\limsup_{\delta\to 0^+} \frac{x\cdot N_x^e}{\delta\|x\|^n}\int\limits_{\|x_\delta^\lambda\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\,dt \le \limsup_{\delta\to 0^+} \frac{\frac{x}{\|x\|}\cdot N_x^e}{(1+\lambda\|x\|^2)^{\frac{n+1}{2}}}\frac{\delta_1}{\delta}\frac{\|x-x_{\delta_1}^e\|}{\delta_1}$$

$$\le c_n \frac{H_{n-1}^e(K,x)^{\frac{1}{n+1}}}{(1+\lambda\|x\|^2)^{\frac{n-1}{2}}}(1+C_1\varepsilon),$$

and, similarly,

$$\liminf_{\delta\to 0^+} \frac{x\cdot N_x^e}{\delta\|x\|^n}\int\limits_{\|x_\delta^\lambda\|}^{\|x\|} \frac{t^{n-1}}{(1+\lambda t^2)^{\frac{n+1}{2}}}\,dt \ge c_n \frac{H_{n-1}^e(K,x)^{\frac{1}{n+1}}}{(1+\lambda\|x\|^2)^{\frac{n-1}{2}}}(1-C_2\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we obtain (4.11), for $\lambda < 0$. For $\lambda > 0$ the argument is analogous. q.e.d.

Combining Lemma 4.4, Lemma 4.5, (3.23) and (3.24), we conclude that

$$\lim_{\delta\to 0^+} \frac{\mathrm{vol}_n^\lambda\left(K\backslash \mathcal{F}_\delta^\lambda K\right)}{\delta} = c_n \int\limits_{\mathrm{bd}\,K} \frac{H_{n-1}^e(K,x)^{\frac{1}{n+1}}}{(1+\lambda\|x\|^2)^{\frac{n-1}{2}}}\,d\mathrm{vol}_{\mathrm{bd}\,K}^e(x)$$
$$= c_n \int\limits_{\mathrm{bd}\,K} H_{n-1}^\lambda(K,x)^{\frac{1}{n+1}}\,d\mathrm{vol}_{\mathrm{bd}\,K}^\lambda(x). \tag{4.13}$$

This finishes the proof of Theorem 1.2.

5. The floating area in real space forms

We denote the Borel σ-algebra of a metric space (X,d) by $\mathcal{B}(X)$. For $K\in\mathcal{K}_0(\mathbb{B}^n(\lambda))$ and $\omega\in\mathcal{B}(\mathbb{B}^n(\lambda))$ we conclude, by Theorem 1.3 and (4.13), that

$$\lim_{\delta\to 0^+} \frac{\mathrm{vol}_n^\lambda\left((K\backslash \mathcal{F}_\delta^\lambda K)\cap\omega\right)}{\delta} = \int\limits_{(\mathrm{bd}\,K)\cap\omega} H_{n-1}^\lambda(K,x)^{\frac{1}{n+1}}\,d\mathrm{vol}_{\mathrm{bd}\,K}^\lambda(x). \tag{5.1}$$

Definition 5.1. The λ-*floating measure* $\Omega^\lambda(.,.)$ is defined, for $K\in\mathcal{K}(\mathrm{Sp}^n(\lambda))$ and $\omega\in\mathbb{B}(\mathrm{Sp}^n(\lambda))$, by

$$\Omega^\lambda(K,\omega) = \int\limits_{(\mathrm{bd}\,K)\cap\omega} H_{n-1}^\lambda(K,x)^{\frac{1}{n+1}}\,d\mathrm{vol}_{\mathrm{bd}\,K}^\lambda(x), \tag{5.2}$$

if $\mathrm{int}\,K\neq\emptyset$ and $\Omega^\lambda(K,\omega)=0$ else. The λ-floating area $\Omega^\lambda(.)$ of a convex body K is $\Omega^\lambda(K)=\Omega^\lambda(K,\mathrm{Sp}^n(\lambda))$.

For $\lambda > 0$, we distinguish between proper and non-proper convex bodies. Recall that a convex body is proper, if and only if it does not contain two antipodal points. Equivalently, a convex body is proper if and only if it is contained in an open half-space (open hemisphere). By (5.1), definition (5.2) makes sense for proper convex bodies. Non-proper convex bodies K with non-empty interior are either the whole space or a *lune*. A k-lune is the convex hull $\mathrm{conv}(S, L)$ of a k-dimensional totally geodesic subspace (k-sphere) S and a proper convex body L in an $(n-k-1)$-dimensional totally geodesic subspace polar to S. Thus, for non-proper convex bodies we either have $K = \mathrm{Sp}^n(\lambda)$ and, therefore, $\mathrm{bd}\, K = \emptyset$ or K is a lune and the boundary is "flat", that is, $H^\lambda_{n-1}(K, x) = 0$ for almost all boundary points $x \in \mathrm{bd}\, K$. Therefore, we set $\Omega^\lambda(K, \omega) = 0$ for non-proper convex bodies. See also [**7**] for more details.

Finally, from the definition (5.2) it is obvious that the λ-floating area vanishes for "flat" bodies. In particular, the λ-floating area for polytopes is zero.

5.1. Proof of Theorem 1.3. We first prove the valuation property. The proof is analogously to the proof for the affine surface area in [**45**]. Let $K, L \in \mathcal{K}(\mathrm{Sp}^n(\lambda))$ such that $K \cup L \in \mathcal{K}(\mathrm{Sp}^n(\lambda))$. We have to show

$$\Omega^\lambda(K, \omega) + \Omega^\lambda(L, \omega) = \Omega^\lambda(K \cup L, \omega) + \Omega^\lambda(K \cap L, \omega). \tag{5.3}$$

We first assume that $K, L \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$. We observe

$$\begin{aligned}
\mathrm{bd}\, K &= (\mathrm{bd}\, K \cap \mathrm{bd}\, L) \cup (\mathrm{bd}\, K \cap \mathrm{int}\, L) \cup (\mathrm{bd}\, K \cap L^c),\\
\mathrm{bd}\, L &= (\mathrm{bd}\, K \cap \mathrm{bd}\, L) \cup (\mathrm{int}\, K \cap \mathrm{bd}\, L) \cup (K^c \cap \mathrm{bd}\, L),\\
\mathrm{bd}(K \cap L) &= (\mathrm{bd}\, K \cap \mathrm{bd}\, L) \cup (\mathrm{bd}\, K \cap \mathrm{int}\, L) \cup (\mathrm{int}\, K \cap \mathrm{bd}\, L),\\
\mathrm{bd}(K \cup L) &= (\mathrm{bd}\, K \cap \mathrm{bd}\, L) \cup (\mathrm{bd}\, K \cap L^c) \cup (K^c \cap \mathrm{bd}\, L),
\end{aligned}$$

where $K^c = \mathrm{Sp}^n(\lambda) \backslash K$ and $L^c = \mathrm{Sp}^n(\lambda) \backslash L$. Then (5.3) reduces to

$$\begin{aligned}
&\int\limits_{(\mathrm{bd}\, K \cap \mathrm{bd}\, L) \cap \omega} H^\lambda_{n-1}(K, x)^{\frac{1}{n+1}} + H^\lambda_{n-1}(L, x)^{\frac{1}{n+1}} \, d\mathrm{vol}^\lambda_{\mathrm{bd}\, L}(x)\\
&= \int\limits_{(\mathrm{bd}\, K \cap \mathrm{bd}\, L) \cap \omega} H^\lambda_{n-1}(K \cup L, x)^{\frac{1}{n+1}} + H^\lambda_{n-1}(K \cap L, x)^{\frac{1}{n+1}} \, d\mathrm{vol}^\lambda_{\mathrm{bd}\, L}(x).
\end{aligned} \tag{5.4}$$

Locally around any point $x \in \mathrm{bd} K \cap \mathrm{bd}\, L$, we use the Euclidean model $(\mathbb{B}^n(\lambda), g^\lambda)$. Hence, $H^\lambda_{n-1}(K, x)$ and $H^\lambda_{n-1}(K, x)$ are related by (3.24) at normal boundary points $x \in \mathrm{bd}\, K$. With [**45**, Lem. 5], we conclude that

$$\begin{aligned}
H^\lambda_{n-1}(K \cup L, x) &= \min\left\{ H^\lambda_{n-1}(K, x), H^\lambda_{n-1}(L, x) \right\},\\
H^\lambda_{n-1}(K \cap L, x) &= \max\left\{ H^\lambda_{n-1}(K, x), H^\lambda_{n-1}(L, x) \right\}.
\end{aligned}$$

This verifies (5.4) in the case that $K, L \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$. If $\operatorname{int} K = \operatorname{int} L = \emptyset$, then (5.4) holds trivially. So assume without loss of generality that $\operatorname{int} K = \emptyset$ and $\operatorname{int} L \neq \emptyset$ such that $K \cup L \in \mathcal{K}(\mathrm{Sp}^n(\lambda))$. If $\lambda > 0$ then we may assume that L is proper, because otherwise (5.4) holds trivially. We use a Euclidean model such that L can be identified with a Euclidean compact convex body and K is a, possibly unbounded, closed convex subset of $\mathbb{R}^n$. We want to show, that $K \subseteq L$. Assume that there is $x \in K \backslash L$ and let C be the convex hull of x and L. By the hyperplane separation Theorem there is $v \in \mathbb{S}^{n-1}$ and $c \in \mathbb{R}$ such that

$$x \cdot v > c > \sup_{y \in L} y \cdot v.$$

Hence, $\operatorname{int}(C \backslash L) \neq \emptyset$ and, therefore, $K = \operatorname{conv}(K \cup L) \backslash L \supseteq C \backslash L$ has non-empty interior – a contradiction. So necessarily we have $K \subseteq L$ and, therefore, (5.4) holds. This shows, that $\Omega^\lambda(., \omega)$ is a valuation on $\mathcal{K}(\mathrm{Sp}^n(\lambda))$.

Since $\Omega^\lambda(., \omega)$ can be seen as a curvature measure on $\operatorname{bd} K$, the proof of the upper-semicontinuity of $\Omega^\lambda(., \omega)$ is analogous to the proofs presented in [**31**]. We include the following short argument: Let $(K_\ell)_{\ell \in \mathbb{N}}$ be a sequence of convex bodies converging to $K \in \mathcal{K}(\mathrm{Sp}^n(\lambda))$. By the valuation property we may assume, for $\lambda > 0$, that $K \cup \bigcup_{\ell \in \mathbb{N}} K_\ell$ is contained in an open half-space. We choose a Euclidean model $(\mathbb{B}^n(\lambda), g^\lambda)$ and identify K_ℓ and K with Euclidean convex bodies. Hence,

$$\Omega^\lambda(K, \omega) = \int\limits_{(\operatorname{bd} K) \cap \omega} \frac{H^e_{n-1}(K, x)^{\frac{1}{n+1}}}{(1 + \lambda \|x\|^2)^{\frac{n-1}{2}}} \, d\mathrm{vol}^e_{\operatorname{bd} K}(x).$$

The density $f^\lambda(x) := (1 + \lambda \|x\|^2)^{-(n-1)/2}$ is continuous and

$$\Omega^0(K, \omega) = \int\limits_{(\operatorname{bd} K) \cap \omega} H^e_{n-1}(K, x)^{\frac{1}{n+1}} \, d\mathrm{vol}^e_{\operatorname{bd} K}(x)$$

is the classical affine surface area. Thus, $\Omega^0(., \omega)$ is upper semicontinuous, see e.g., [**35**]. To finish the proof let $\varepsilon > 0$. By compactness of K and continuity of f^λ, we find a finite partition of $\operatorname{bd} K \cap \omega$ into measurable subsets $(\omega_j)_{j=0}^N$ and points $x_j \in \omega_j$ such that $|f^\lambda(x) - f^\lambda(x_j)| < \varepsilon$, for all $x \in \omega_j$. Therefore,

$$\begin{aligned}\limsup_{\ell \in \mathbb{N}} \Omega^\lambda(K_\ell, \omega) &\leq \sum_{j=0}^N (f^\lambda(x_j) + \varepsilon) \limsup_{\ell \in \mathbb{N}} \Omega^0(K_\ell, \omega_j) \\ &= \sum_{j=0}^N (f^\lambda(x_j) + \varepsilon) \Omega^0(K, \omega_j) \leq \Omega^\lambda(K, \omega) + \varepsilon 2N \Omega^0(K).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves the upper semicontinuity of $\Omega^\lambda(., \omega)$.

Finally, the fact that $\Omega^\lambda(.,\omega)$ is invariant under isometries is obvious, since it is a intrinsic notion. For $\lambda = 0$, the *equi-affine* transformations are characterized as bijective automorphisms that map lines to lines, are measurable and preserve volume.

5.2. Isoperimetric inequality. The classical and well-known inequality associated with the affine surface area can be found, for general convex bodies, in e.g., [**35**]. Namely, for $K \in \mathcal{K}_0(\mathbb{R}^n)$ we have that

$$\mathrm{as}_1(K) \leq n\kappa_n^{\frac{2}{n+1}} \mathrm{vol}_n^e(K)^{\frac{n-1}{n+1}}, \tag{5.5}$$

with equality if and only if K is an ellipsoid. A natural question is, whether an extension of this inequality holds for the λ-floating area. Inequality (5.5) can be restated as: For all convex bodies of volume α the ball of radius $(\alpha/\kappa_n)^{1/n}$ maximizes the affine surface area, i.e.,

$$\sup_{K\in\mathcal{K}_0(\mathbb{R}^n)} \{\mathrm{as}_1(K) : \mathrm{vol}_n^e(K) = \alpha\} = \mathrm{as}_1\left(B_e^n\left(0, (\alpha/\kappa_n)^{1/n}\right)\right).$$

Therefore, we define

$$C^\lambda(\alpha) := \sup_{K\in\mathcal{K}_0(\mathrm{Sp}^n(\lambda))} \left\{\Omega^\lambda(K) : \mathrm{vol}_n^\lambda(K) = \alpha\right\}. \tag{5.6}$$

Then, for $\lambda = 0$ and by (5.5), we conclude

$$C^0(\alpha) = n\kappa_n^{\frac{2}{n+1}} \alpha^{\frac{n-1}{n+1}}.$$

For $\lambda > 0$, $\mathcal{K}_0(\mathrm{Sp}^n(\lambda))$ is compact. Since $\Omega^\lambda(.)$ is upper semi-continuous, there exists $K^* \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$ such that $\Omega^\lambda(K^*) = C^\lambda(\alpha)$. We conjecture, that K^* is a geodesic ball, that is, for arbitrary $p \in \mathrm{Sp}^n(\lambda)$, we have

$$C^\lambda(\alpha) \stackrel{?}{=} \Omega^\lambda\left(B_\lambda^n(p,r)\right),$$

where r is determined by $\alpha = \mathrm{vol}_n^\lambda\left(B_\lambda^n(p,r)\right)$.

For $\lambda < 0$, the problem becomes more intricate, since $\mathrm{Sp}^n(\lambda)$ admits unbounded closed convex sets with non-empty interior and finite volume. For example, in hyperbolic space the ideal simplices are among them. Ideal simplices are simplices with vertices at infinity and they have finite hyperbolic volume. In the Euclidean model $(\mathbb{B}^n, g^h)$, such ideal simplices are just Euclidean simplices inscribed in the sphere at infinity $\mathbb{S}^{n-1} = \mathrm{bd}\,\mathbb{B}^n$. More generally, any polytope with vertices at infinity has finite volume. This is immediate by the valuation property of hyperbolic volume and the fact that any polytopes can be partitioned into simplices. By monotonicity of the hyperbolic volume, we also conclude that any closed convex subset that is contained in a polytope with vertices at infinity has finite hyperbolic volume. We denote by $\mathcal{K}_0^\infty(\mathrm{Sp}^n(\lambda))$ the space of closed convex subsets of $\mathrm{Sp}^n(\lambda)$ with non-empty interior and finite volume. Hence, for $\lambda < 0$, the space of convex bodies $\mathcal{K}_0(\mathrm{Sp}^n(\lambda))$ endowed with the volume difference metric is not complete. The closure of $\mathcal{K}_0(\mathrm{Sp}^n(\lambda))$ in the space of all closed convex subsets of $\mathrm{Sp}^n(\lambda)$ with non-empty interior is $\mathcal{K}_0^\infty(\mathrm{Sp}^n(\lambda))$.

Extremizers of (5.6) could appear in $K_0^\infty(\mathrm{Sp}^n(\lambda))$ for $\lambda < 0$, since any unbounded convex set in $\mathcal{K}_0^\infty(\mathrm{Sp}_n(\lambda))$ can be approximated with respect to the volume difference metric by a sequence of convex bodies $(K_\ell)_{\ell\in\mathbb{N}}$ in $\mathcal{K}_0(\mathrm{Sp}^n(\lambda))$ such that $\mathrm{vol}_n^\lambda(\mathcal{K}_\ell) = \alpha$. However, we conjecture that also in the hyperbolic setting geodesic balls will be extremal.

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Department of Mathematics
Case Western Reserve University
Cleveland, Ohio 44106
U.S.A.

E-mail address: florian.besau@case.edu

Department of Mathematics
Case Western Reserve University
Cleveland, Ohio 44106
U.S.A.

E-mail address: elisabeth.werner@case.edu