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THE LOCAL PICTURE THEOREM ON THE SCALE OF TOPOLOGY

William H. Meeks III*, Joaquín Pérez & Antonio Ros

Abstract

We prove a descriptive theorem on the extrinsic geometry of an embedded minimal surface of injectivity radius zero in a homogeneously regular Riemannian three-manifold, in a certain small intrinsic neighborhood of a point of *almost-minimal injectivity radius.* This structure theorem includes a limit object which we call a *minimal parking garage structure* on $\mathbb{R}^3$, whose theory we also develop.

1. Introduction

This paper is devoted to an analysis of the extrinsic geometry of any embedded minimal surface M in small intrinsic balls in a homogeneously regular Riemannian three-manifold[1] , such that the injectivity radius function of M is sufficiently small in terms of the ambient geometry of the balls. We carry out this analysis by blowing-up such an M at a sequence of points with *almost-minimal injectivity radius* (we will define this notion precisely in items 1, 2, 3 of the next theorem), which produces a new sequence of minimal surfaces, a subsequence of which has a natural limit object being either a properly embedded minimal surface in $\mathbb{R}^3$, a minimal parking garage structure on $\mathbb{R}^3$ (we will study this notion in Section 3) or possibly, a particular case of a singular minimal lamination of $\mathbb{R}^3$ with restricted geometry, as stated in item 6 of the next result.

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[1]A Riemannian three-manifold N is *homogeneously regular* if there exists an $\varepsilon > 0$ such that the image by the exponential map of any ε-ball in a tangent space T_xN, $x \in N$, is uniformly close to an ε-ball in $\mathbb{R}^3$ in the C^2-norm. In particular, N has positive injectivity radius. Note that if N is compact, then N is homogeneously regular.

In the sequel, we will denote by $B_M(p,r)$ (resp. $\overline{B}_M(p,r)$) the open (resp. closed) metric ball centered at a point p in a Riemannian manifold N, with radius $r>0$. In the case M is complete, we will let $I_M\colon M\to(0,\infty]$ be the injectivity radius function of M, and given a subdomain $\Omega\subset M$, $I_\Omega=(I_M)|_\Omega$ will stand for the restriction of I_M to Ω. The infimum of I_M is called the *injectivity radius* of M.

Theorem 1.1 (Local Picture on the Scale of Topology)**.** *There exists a smooth decreasing function $\delta\colon(0,\infty)\to(0,1/2)$ with $\lim_{r\to\infty} r\,\delta(r)=\infty$ such that the following statements hold. Suppose M is a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n\in M$ (called "points of almost-minimal injectivity radius") and positive numbers $\varepsilon_n=n\,I_M(p_n)\to 0$ such that:*

1: For all n, the closure M_n of the component of $M\cap B_N(p_n,\varepsilon_n)$ that contains p_n is a compact surface with boundary in $\partial B_N(p_n,\varepsilon_n)$. Furthermore, M_n is contained in the intrinsic open ball $B_M(p_n,\frac{r_n}{2}I_M(p_n))$, where $r_n>0$ satisfies $r_n\,\delta(r_n)=n$.
2: Let $\lambda_n=1/I_M(p_n)$. Then, $\lambda_n I_{M_n}\geq 1-\frac{1}{n}$ on M_n.
3: The metric balls $\lambda_n B_N(p_n,\varepsilon_n)$ of radius $n=\lambda_n\varepsilon_n$ converge uniformly as $n\to\infty$ to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n).

Furthermore, exactly one of the following three possibilities occurs.

4: The surfaces $\lambda_n M_n$ have uniformly bounded Gaussian curvature on compact subsets[2] *of $\mathbb{R}^3$ and there exists a connected, properly embedded minimal surface $M_\infty\subset\mathbb{R}^3$ with $\vec{0}\in M_\infty$, $I_{M_\infty}\geq 1$ and $I_{M_\infty}(\vec{0})=1$, such that for any $k\in\mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ to M_∞ with multiplicity one as $n\to\infty$.*
5: After a rotation in $\mathbb{R}^3$, the surfaces $\lambda_n M_n$ converge to a minimal parking garage structure[3] *on $\mathbb{R}^3$, consisting of a foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes, with columns forming a locally finite set $S(\mathcal{L})$ of vertical straight lines (at least two lines). Moreover, if there exists a bound on the genus of the surfaces $\lambda_n M_n$, then $S(\mathcal{L})$ consists of just two lines l_1, l_2, the associated limiting pair of multivalued graphs*[4] *in $\lambda_n M_n$ nearby l_1,l_2 are oppositely handed*

[2]As $M_n\subset B_N(p_n,\varepsilon_n)$, the convergence $\{\lambda_n B_N(p_n,\varepsilon_n)\}_n\to\mathbb{R}^3$ explained in item 3 allows us to view the rescaled surface $\lambda_n M_n$ as a subset of $\mathbb{R}^3$. The uniformly bounded property for the Gaussian curvature of the induced metric on $M_n\subset N$ rescaled by λ_n on compact subsets of $\mathbb{R}^3$ now makes sense.

[3]For a description of a minimal parking garage structure, see Section 3.

and given $R > 0$*, for* $n \in \mathbb{N}$ *large depending on* R*, the surface* $(\lambda_n M_n) \cap B_{\lambda_n N}(p, \frac{R}{\lambda_n})$ *has genus zero.*

6: There exists a nonempty, closed set $\mathcal{S} \subset \mathbb{R}^3$*, a minimal lamination* $\mathcal{L}$ *of* $\mathbb{R}^3 - \mathcal{S}$ *and a subset* $S(\mathcal{L}) \subset \mathcal{L}$ *which is closed in the subspace topology, such that the surfaces* $(\lambda_n M_n) - \mathcal{S}$ *converge to* $\mathcal{L}$ *outside of* $S(\mathcal{L})$ *and* $\mathcal{L}$ *has at least one nonflat leaf. Furthermore, if we let* $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$ *and let* $\mathcal{P}$ *be the sublamination of flat leaves in* $\mathcal{L}$*, then the following holds.* $\mathcal{P} \neq \emptyset$*, the closure of every such flat leaf is a horizontal plane, and if* $L \in \mathcal{P}$ *then the plane* $\overline{L}$ *intersects* $\Delta(\mathcal{L})$ *in a set containing at least two points, each of which are at least distance 1 from each other in* $\overline{L}$*, and either* $\overline{L} \cap \Delta(\mathcal{L}) \subset \mathcal{S}$ *or* $\overline{L} \cap \Delta(\mathcal{L}) \subset S(\mathcal{L})$.

For a more detailed description of cases 5 and 6 of Theorem 1.1, see Propositions 4.20 and 4.30 below.

The results in the series of papers [**8**, **9**, **10**, **11**, **12**, **13**] by Colding and Minicozzi and the minimal lamination closure theorem by Meeks and Rosenberg [**36**] play important roles in deriving the above compactness result. We conjecture that item 6 in Theorem 1.1 does not actually occur.

A short explanation of the organization of the paper is as follows. In Section 2, we introduce some notation and recall the notion and language of laminations, as well as a chord-arc property for embedded minimal disks previously proven by Meeks and Rosenberg and based on a similar one by Colding and Minicozzi. In Section 3, we develop the theory of parking garage surfaces and limit parking garage structures, a notion that appears in item 5 of the main Theorem 1.1. Section 4, the bulk of this paper, is devoted to the proof of Theorem 1.1. Section 5 includes some applications of Theorem 1.1. We refer the reader to [**23**, **24**, **25**, **26**, **27**, **36**, **38**, **40**] for further applications of Theorem 1.1.

2. Preliminaries

Let M be a Riemannian manifold. Let $B_M(p, r)$ be the open ball centered at a point $p \in M$ with radius $r > 0$, for the underlying metric space structure of M associated to its Riemannian metric. When M is complete, the injectivity radius $I_M(p)$ at a point $p \in M$ is the supremum of the radii $r > 0$ of the open balls $B_M(p, r)$ for which the exponential map at p is a diffeomorphism. This defines the *injectivity*

[4]This means that for $i = 1, 2$ and $k \in \mathbb{N}$ fixed ($k \geq 2$), and for n large enough depending on k, $\lambda_n M_n$ contains around l_i a pair of k-valued graphs (See Definition 4.10 for this concept) with opposite orientations, both spiraling together, and the handedness of these k-graphs nearby l_1 is opposite to the related one around l_2.

radius function, $I_M\colon M \to (0,\infty]$, which is continuous on M (see e.g., Proposition 88 in Berger [1]). The infimum of I_M is called the *injectivity radius* of M.

Definition 2.1. A *codimension-one lamination* of a Riemannian three-manifold N is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L},\mathcal{A})$ satisfying:

1. $\mathcal{L}$ is a closed subset of N.
2. $\mathcal{A} = \{\varphi_\beta\colon \mathbb{D}\times(0,1) \to U_\beta\}_\beta$ is an atlas of coordinate charts of N (here $\mathbb{D}$ is the open unit disk in $\mathbb{R}^2$, $(0,1)$ is the open unit interval and U_β is an open subset of N); note that although N is assumed to be smooth, we only require that the regularity of the atlas (i.e., that of its change of coordinates) is of class C^0, i.e., $\mathcal{A}$ is an atlas for the topological structure of N.
3. For each β, there exists a closed subset C_β of $(0,1)$ such that $\varphi_\beta^{-1}(U_\beta\cap \mathcal{L}) = \mathbb{D}\times C_\beta$.

We will simply denote laminations by $\mathcal{L}$, omitting the charts φ_β in $\mathcal{A}$. A lamination $\mathcal{L}$ is said to be a *foliation of* N if $\mathcal{L}=N$. Every lamination $\mathcal{L}$ naturally decomposes into a collection of disjoint, connected topological surfaces (locally given by $\varphi_\beta(\mathbb{D}\times\{t\})$, $t\in C_\beta$, with the notation above), called the *leaves* of $\mathcal{L}$. As usual, the regularity of $\mathcal{L}$ requires the corresponding regularity on the change of coordinate charts φ_β. A lamination $\mathcal{L}$ of N is said to be a *minimal lamination* if all its leaves are (smooth) minimal surfaces. Since the leaves of $\mathcal{L}$ are pairwise disjoint, we can consider the norm of the second fundamental form $|\sigma_{\mathcal{L}}|$ of $\mathcal{L}$, which is the function defined at every point p in $\mathcal{L}$ as $|\sigma_L|(p)$, where L is the unique leaf of $\mathcal{L}$ passing through p and $|\sigma_L|$ is the norm of the second fundamental form of L.

Definition 2.2. If $\{\Sigma_n\}_n$ is a sequence of complete embedded minimal surfaces in a Riemannian three-manifold N, consider the closed set $A\subset N$ of points $p\in N$ such that for every neighborhood U_p of p and every subsequence of $\{\Sigma_{n(k)}\}_k$, the sequence of norms of the second fundamental forms of $\Sigma_{n(k)}\cap U_p$ is not uniformly bounded. By the arguments in Lemma 1.1 of Meeks and Rosenberg [35] (see also Proposition B.1 in [11]), after extracting a subsequence, the Σ_n converge on compact subsets of $N-A$ to a minimal lamination $\mathcal{L}'$ of $N-A$ that extends to a minimal lamination $\mathcal{L}$ of $N-\mathcal{S}$, where $\mathcal{S}\subset A$ is the (possibly empty) *singular set* of $\mathcal{L}$, i.e., $\mathcal{S}$ is the closed subset of N such that $\mathcal{L}$ does not admit a local lamination structure around any point of $\mathcal{S}$. We will denote by $S(\mathcal{L}) = A-\mathcal{S}$ the *singular set of convergence* of the Σ_n to $\mathcal{L}$, i.e., those points of N around which $\mathcal{L}$ admits a lamination structure but where the second fundamental forms of the Σ_n still blow-up.

In this paper we will apply the *Minimal Lamination Closure Theorem* in [36], which insures that if M is a complete, embedded minimal surface of positive injectivity radius in a Riemannian three-manifold N (not necessarily complete), then the closure $\overline{M}$ of M in N has the structure of a $C^{0,1}$-minimal lamination $\mathcal{L}$ with the components of M being leaves of $\mathcal{L}$. We will also use the following technical result from [36], which generalizes to the manifold setting some of the results in [12].

Definition 2.3. Given a surface Σ embedded in a Riemannian three-manifold N, a point $p \in \Sigma$ and $R > 0$, we denote by $\Sigma(p, R)$ the closure of the component of $\Sigma \cap B_N(p, R)$ that passes through p.

Theorem 2.4 (Theorem 13 in [36])**.** *Suppose that Σ is a compact, embedded minimal disk in a homogeneously regular three-manifold N whose injectivity radius function $I_\Sigma \colon \Sigma \to [0, \infty)$ equals the distance to the boundary function $d_\Sigma(\cdot, \partial\Sigma)$ (d_Σ denotes intrinsic distance in Σ). Then, there exist numbers $\delta' \in (0, 1/2)$ and $R_0 > 0$, both depending only on N, such that if $\overline{B}_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ and $R \leq R_0$, then*

$$\Sigma(x, \delta' R) \subset \overline{B}_\Sigma(x, R/2).$$

Furthermore, $\Sigma(x, \delta' R)$ is a compact, embedded minimal disk in $\overline{B}_N(x, \delta' R)$ with $\partial\Sigma(x, \delta' R) \subset \partial B_N(x, \delta' R)$.

3. Parking garage structures in $\mathbb{R}^3$

For a Riemannian surface M, K_M will stand for its Gaussian curvature function. In our previous paper [33] we proved the Local Picture Theorem on the Scale of Curvature, which is a tool that applies to any complete, embedded minimal surface M of unbounded absolute Gaussian curvature in a homogeneously regular three-manifold N, and produces via a blowing-up process a nonflat, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with normalized curvature (in the sense that $|K_{M_\infty}| \leq 1$ on M_∞ and $\vec{0} \in M_\infty$, $|K_{M_\infty}|(\vec{0}) = 1$). The key ingredient to do this is to find points $p_n \in M$ of *almost-maximal curvature* and then rescale exponential coordinates in N around these points p_n by $\sqrt{|K_M|(p_n)} \to \infty$ as $n \to \infty$. We will devote the next section to obtain a somehow similar result for an M whose injectivity radius is zero, by exchanging the role of $\sqrt{|K_M|}$ by $1/I_M$, where $I_M \colon M \to (0, \infty]$ denotes the injectivity radius function on M. We will consider this rescaling ratio after evaluation at points $p_n \in M$ of *almost-minimal injectivity radius*, in a sense to be made precise in the first paragraph of Section 4. One of the difficulties of this generalization is that the limit objects that we can find after blowing-up might be not only properly embedded minimal surfaces in $\mathbb{R}^3$, but also new objects, namely limit minimal parking garage structures which we study below, and certain kinds of singular minimal laminations of $\mathbb{R}^3$.

Roughly speaking, a *minimal parking garage structure* is a limit object for a sequence of embedded minimal surfaces which converges C^α, $\alpha \in (0,1)$, to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes, with singular set of convergence $S(\mathcal{L})$ being a locally finite set of lines orthogonal to $\mathcal{L}$, called the *columns* of the limit parking garage structure, along which the limiting surfaces have the local appearance of a highly-sheeted double staircase. For example, the sequence of homothetic shrinkings $\frac{1}{n}H$ of a vertical helicoid H converges to a minimal parking garage structure that consists of the minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ being the x_3-axis.

We remark that some of the language associated to minimal parking garage structures, such as columns, appeared first in a paper of Traizet and Weber [**47**], and the first important application of this type of structure appeared in [**28**] where we applied it to derive curvature estimates for certain complete embedded minimal planar domains in $\mathbb{R}^3$. In [**47**], Traizet and Weber produced an analytic method for constructing a one-parameter family of properly embedded, periodic minimal surfaces in $\mathbb{R}^3$, by analytically untwisting via the implicit function theorem a limit configuration given by a finite number of regions on vertical helicoids in $\mathbb{R}^3$ that have been glued together in a consistent way. They referred to the limiting configuration as a parking garage structure on $\mathbb{R}^3$ with columns corresponding to the axes of the helicoids that they glued together. Most of the area of these surfaces, just before the limit, consists of very flat horizontal levels (almost-horizontal densely packed horizontal planes) joined by the vertical helicoidal columns. One can travel quickly up and down the horizontal levels of the limiting surfaces only along the helicoidal columns in much the same way that some parking garages are configured for traffic flow; hence, the name parking garage structure. Parking garage structures also appear as natural objects in the main results of the papers [**11**, **13**, **39**].

We now describe in more detail the notion of a *parking garage surface.* Consider a possibly infinite, nonempty, locally finite set of points $P \subset \mathbb{R}^2$ and a collection $\mathcal{D}$ of open round disks centered at the points of P such that the closures of these disks form a pairwise disjoint collection. Let $\mu\colon H_1(\mathbb{R}^2 - \mathcal{D}) \to \mathbb{Z}$ be a group homomorphism such that μ takes the values ± 1 on the homology classes represented by the boundary circles of the disks in $\mathcal{D}$. Let $\Pi\colon M \to \mathbb{R}^2 - \mathcal{D}$ be the infinite cyclic covering space associated to the kernel of the composition of the natural map from $\pi_1(\mathbb{R}^2 - \mathcal{D})$ to $H_1(\mathbb{R}^2 - \mathcal{D})$ with μ. It is straightforward to embed M into $\mathbb{R}^3$ so that under the natural identification of $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$, the map Π is the restriction to M of the orthogonal projection of $\mathbb{R}^3$ to $\mathbb{R}^2 \times \{0\}$. Furthermore, in this embedding, we may assume that the covering transformation of M corresponding to an $n \in \mathbb{Z}$ is given geometrically by translating M vertically by $(0,0,n)$. In

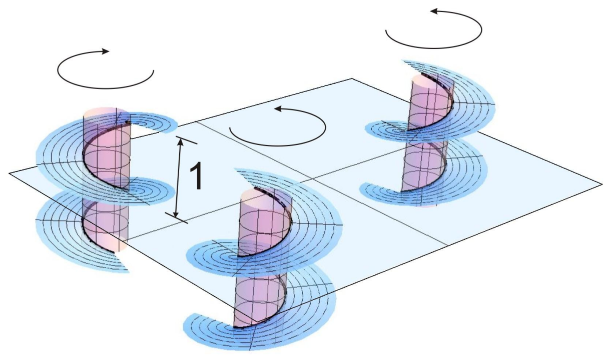

Figure 1. Schematic representation of one of the "two halves" M of a parking garage surface G with three columns, two right-handed and one left-handed. The entire surface G, not represented in the figure, is obtained after gluing M with $M + (0, 0, \frac{1}{2})$ and with an infinite helicoidal strip inside each of the columns.

particular, M is a singly-periodic surface with boundary in $\partial\mathcal{D} \times \mathbb{R}$. M has exactly one boundary curve Γ on each vertical cylinder over the boundary circle of each disk in $\mathcal{D}$. We may assume that every such curve Γ is a helix, see Figure 1. Let $M(\frac{1}{2})$ be the vertical translation of M by $(0, 0, \frac{1}{2})$. $M \cup M(\frac{1}{2})$ is an embedded, disconnected periodic surface in $(\mathbb{R}^2 - \mathcal{D}) \times \mathbb{R}$ with a double helix on each boundary cylinder in $\partial\mathcal{D} \times \mathbb{R}$.

Definition 3.1. In the above situation, we will call a (periodic) *parking garage surface* corresponding to the surjective homomorphism μ to the connected topological surface $G \subset \mathbb{R}^3$ obtained after attaching to $M \cup M(\frac{1}{2})$ an infinite helicoidal strip in each of the solid cylinders in $\mathcal{D} \times \mathbb{R}$. Note that by choosing M appropriately, the resulting surface G can be made smooth.

Since in minimal surface theory we only see the parking garage structure in the limit of a sequence of minimal surfaces, when the helicoidal strips in the cylinders of $\mathcal{D} \times \mathbb{R}$ become arbitrarily densely packed, it is useful in our construction of G to consider parking garages $G(t)$ invariant under translation by $(0, 0, t)$ with $t \in (0, 1]$ tending to zero. For $t \in (0, 1]$, consider the affine transformation $F_t(x_1, x_2, x_3) = (x_1, x_2, tx_3)$. Then $G(t) = F_t(G)$. Note that our previously defined surface G is $G(1)$ in this new setup. As $t \to 0$, the $G(t)$ converge to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ consisting of the vertical lines in $P \times \mathbb{R}$. Also, note that M depends on the epimorphism μ, so to be more specific, we could also denote $G(t)$ by $G(t, \mu)$.

Definition 3.2. In the sequel, we will call the above limit object $\lim_{t\to 0} G(t,\mu)$ a *limit parking garage structure* of $\mathbb{R}^3$ associated to the surjective homomorphism μ.

Next we remark on the topology of the ends of the periodic parking garage surface G in the case that P is a finite set, where $G = G(t,\mu)$ for some t and μ. Suppose $\mathcal{D} = \{D_1, \dots, D_n\}$. Then we associate to G an integer index:

$$I(G) = \sum_{i=1}^{n} \mu([\partial D_i]) \in \mathbb{Z}.$$

Note that the index $I(G) = I(G(t))$ does not depend on the parameter t, and, thus, it makes sense to speak about this index for a limit parking garage structure $\lim_{t\to 0} G(t)$.

Consider the quotient orientable surface $G/\mathbb{Z}$ in $\mathbb{R}^3/\mathbb{Z}$, where $\mathbb{Z}$ is generated by translation by $(0,0,t)$. The ends of $G/\mathbb{Z}$ are annuli and there are exactly two of them. If $I(G) = 0$, then these annular ends of $G/\mathbb{Z}$ lift to graphical annular ends of G. If $I(G) \neq 0$, then the universal cover of an end of $G/\mathbb{Z}$ has $|I(G)|$ orientation preserving lifts to G, each of which gives rise to an infinite-valued graph over its projection to the end of $\mathbb{R}^2 \times \{0\}$.

Lemma 3.3. *Suppose that G is a periodic parking garage surface in $\mathbb{R}^3$ with a finite set $P \times \mathbb{R}$ of n columns. Then, the following properties are equivalent.*

1) *G has genus zero.*
2) *G has finite genus.*
3) $n = 1$ (*in which case G is simply connected and* $I(G) = \pm 1$), *or* $n = 2$ *and* $I(G) = 0$ (*in which case G has an infinite number of annular ends with two limit ends*).

Proof. The equivalence between items 1 and 2 holds since G is periodic. Clearly item 3 implies item 1. Finally, if item 3 does not hold then either $n \geq 3$ or G has two columns with the same handedness. In any of these cases there exist at least two points $x_1, x_2 \in P$ with associated values $\mu([\partial D_1]) = \mu([\partial D_2])$ for the corresponding disks D_1, D_2 in $\mathcal{D}$ around x_1, x_2 (up to reindexing). Consider an embedded arc γ in $\mathbb{R}^2 - P$ joining x_1 to x_2. Then one can lift γ to two arcs in consecutive levels of the parking garage G joined by short vertical segments on the columns over x_1 and x_2. Let $\widetilde{\gamma}$ denote this associated simple closed curve on G. Observe that if $\widetilde{\gamma}'$ is the related simple closed curve obtained by translating $\widetilde{\gamma}$ up exactly one level in G (this means that $\widetilde{\gamma}' = \widetilde{\gamma} + (0,0,t/2)$ if $G = G(t,\mu)$), then $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ have intersection number one. Thus, a small regular neighborhood of $\widetilde{\gamma} \cup \widetilde{\gamma}'$ on G has genus one, which implies that item 1 does not hold. q.e.d.

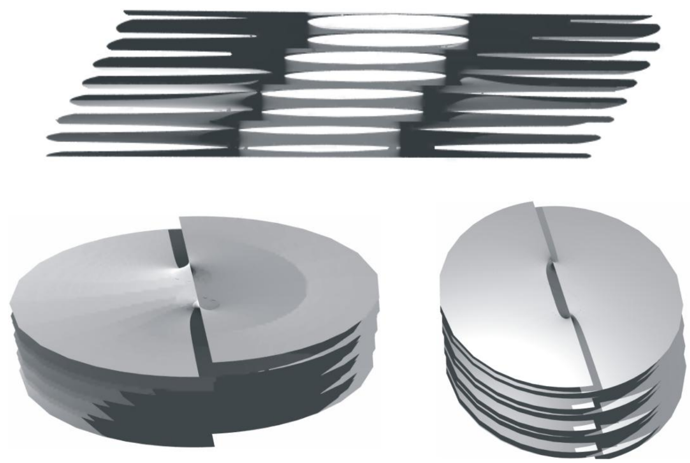

Figure 2. Three views of a minimal parking garage surface, constructed on a Riemann minimal example.

3.1. Examples of parking garage structures.

(B1) Consider the limit of homothetic shrinkings of a vertical helicoid. One obtains in this way the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with a single column, or singular curve of convergence $S(\mathcal{L})$, being the x_3-axis. The related limit *minimal* parking garage surface G has invariant $I(G) = \pm 1$; the word "minimal" is used because in this case, the surface G is a minimal surface.

(B2) Let R_t, $t > 0$, be the classical Riemann minimal examples. These are properly embedded, singly-periodic minimal surfaces with genus zero and infinitely many planar ends asymptotic to horizontal planes. Consider the limit of the R_t when the flux vector of R_t along a compact horizontal section converges to $(2, 0, 0)$. Note that the surfaces R_t are invariant under a translation that only becomes vertical in the limit; in spite of this slight difference with the theoretical framework explained above, where the surface $G(t, \mu)$ is invariant under a vertical translation, we still consider the limit minimal parking structure in this case. This limit minimal parking garage structure G has two columns with opposite handedness (see [**28**] for a proof of these properties) and so, G has invariant $I(G) = 0$, see Figure 2. These examples (B1), (B2) correspond to Case 3 of Lemma 3.3.

(B3) Consider the Scherk doubly-periodic minimal surfaces S_θ, $\theta \in (0, \frac{\pi}{2}]$, with horizontal lattice of periods $\{((m + n)\cos\theta, (m - n)\sin\theta, 0) \mid m, n \in \mathbb{Z}\}$. The limit as $\theta \to 0$ of the surfaces S_θ is a foliation of $\mathbb{R}^3$ by planes parallel to the (x_1, x_3)-plane, with columns of the same orientation being the horizontal lines parallel to the x_2-axis and passing through $\mathbb{Z} \times \{0\} \times \{0\}$. The related minimal parking garage structure of $\mathbb{R}^3$ has an infinite number of columns, all of which are oriented the same way.

We refer the interested reader to [**47**] for further details and more examples of parking garage structures that occur in minimal surface theory.

Lemma 3.4. *Every parking garage surface in $\mathbb{R}^3$ with a finite number of columns is recurrent for Brownian motion.*

Proof. Note that if $G \subset \mathbb{R}^3$ is a parking garage surface and we consider the natural action of $\mathbb{Z}$ over G by vertical orientation preserving translations, then the quotient surface $G/\mathbb{Z}$ has finite topology, exactly two annular ends and quadratic area growth. In particular, $G/\mathbb{Z}$ is conformally a twice punctured compact Riemann surface. On the other hand, since the covering $\Pi\colon G \to G/\mathbb{Z}$ is a normal covering, then there is a natural homomorphism τ from the fundamental group of $G/\mathbb{Z}$ onto the group of automorphisms $\mathrm{Aut}(\Pi)$ of this covering. Since $\mathrm{Aut}(\Pi)$ is abelian, then τ factorizes through the first homology group $H_1(G/\mathbb{Z})$ to a surjective homomorphism $\widetilde{\tau}\colon H_1(G/\mathbb{Z}) \to \mathrm{Aut}(\Pi)$. Furthermore, each one of the two homology classes in $H_1(G/\mathbb{Z})$ given by loops on $G/\mathbb{Z}$ around the ends applies via $\widetilde{\tau}$ on a generator of $\mathrm{Aut}(\Pi)$. In this setting, Theorem 2 in Epstein [**17**] implies that G is recurrent; see the first paragraph in Section 4 of [**30**] for details on this application of the result by Epstein. q.e.d.

Remark 3.5. In Example (B3) above, the surface S_θ for any θ is not recurrent for Brownian motion, but it is close to that condition, in the sense that it does not admit positive nonconstant harmonic functions, see [**30**].

We have already introduced the notation $B_N(p,r)$ for the open metric ball centered at the point p with radius $r > 0$ in a Riemannian three-manifold N. In the case $N = \mathbb{R}^3$, we will simplify $\mathbb{B}(p,r) = B_{\mathbb{R}^3}(p,r)$ and $\mathbb{B}(r) = \mathbb{B}(\vec{0}, r)$.

Note that it also makes sense for a sequence of compact, embedded minimal surfaces $M_n \subset \mathbb{B}(R_n)$ with boundaries $\partial M_n \subset \partial\mathbb{B}(R_n)$ such that $R_n \to \infty$ as $n \to \infty$, to converge on compact subsets of $\mathbb{R}^3$ to a minimal parking garage structure on $\mathbb{R}^3$ consisting of a foliation $\mathcal{L}$ of $\mathbb{R}^3$ by planes with a locally finite set of lines $S(\mathcal{L})$ orthogonal to the planes in $\mathcal{L}$, where $S(\mathcal{L})$ corresponds to the singular set of convergence of the M_n to $\mathcal{L}$. We note that each of the lines in $S(\mathcal{L})$ has an associated $+$ or $-$ sign corresponding to whether or not the associated forming helicoid in M_n along the line is right or left handed. For instance, Theorem 0.9 in [**13**] illustrates a particular case of this convergence to a limit parking garage structure on $\mathbb{R}^3$ when $S(\mathcal{L})$ consists of two lines with associated double staircases of opposite handedness.

Remark 3.6. To study other aspects of how minimal parking garage structures appear as the limit of a sequence of minimal surfaces in $\mathbb{R}^3$, see Meeks [21, 22].

4. The Proof of Theorem 1.1

Let $M \subset N$ be a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N. As N is homogeneously regular, its injectivity radius is positive. After a fixed constant scaling of the metric of N, we may assume that the injectivity radius of N is greater than 1. The first step in the proof of Theorem 1.1 is to obtain special points $p'_n \in M$, called *points of almost-minimal injectivity radius.* To do this, first consider a sequence of points $q_n \in M$ such that $I_M(q_n) \leq \frac{1}{n}$ (such a sequence $\{q_n\}_n$ exists since the injectivity radius of M is zero). Consider the continuous function $h_n\colon \overline{B}_M(q_n, 1) \to \mathbb{R}$ given by

$$h_n(x) = \frac{d_M(x, \partial B_M(q_n, 1))}{I_M(x)}, \quad x \in \overline{B}_M(q_n, 1), \tag{1}$$

where d_M is the distance function associated to its Riemannian metric. As h_n is continuous and vanishes on $\partial B_M(q_n, 1)$, then there exists $p'_n \in B_M(q_n, 1)$ where h_n achieves its maximum value.

We define $\lambda'_n = I_M(p'_n)^{-1}$. Note that

$$\lambda'_n \geq \lambda'_n d_M(p'_n, \partial B_M(q_n, 1)) = h_n(p'_n) \geq h_n(q_n) = I_M(q_n)^{-1} \geq n. \tag{2}$$

Fix $t > 0$. Consider exponential coordinates centered at p'_n in the extrinsic ball $B_N(p'_n, \frac{t}{\lambda'_n})$ (this can be done if n is sufficiently large). After rescaling the ambient metric by the factor $\lambda'_n \to \infty$ and identifying p'_n with the origin $\vec{0}$, we conclude that the sequence $\{\lambda'_n B_N(p'_n, \frac{t}{\lambda'_n})\}_n$ converges to the open ball $\mathbb{B}(t)$ of $\mathbb{R}^3$ with its usual metric. Similarly, we can consider $\{\lambda'_n \overline{B}_M(p'_n, \frac{t}{\lambda'_n})\}_n$ to be a sequence of embedded minimal surfaces with boundary, all passing through $p'_n = \vec{0}$ with injectivity radius 1 at this point.

Lemma 4.1. *The injectivity radius function of $\lambda'_n M$ (i.e., of M endowed with the rescaled metric by the factor λ'_n) restricted to $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ is greater than some positive constant independent of n large.*

Proof. Pick a point $z_n \in B_M(p'_n, \frac{t}{\lambda'_n})$. Since for n large enough, z_n belongs to $B_M(q_n, 1)$, we have

$$\begin{aligned} \frac{1}{\lambda'_n I_M(z_n)} &= \frac{h_n(z_n)}{\lambda'_n d_M(z_n, \partial B_M(q_n, 1))} \\ &\leq \frac{h_n(p'_n)}{\lambda'_n d_M(z_n, \partial B_M(q_n, 1))} = \frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z_n, \partial B_M(q_n, 1))}. \end{aligned} \tag{3}$$

By the triangle inequality, $d_M(p'_n, \partial B_M(q_n,1)) \leq \frac{t}{\lambda'_n} + d_M(z_n, \partial B_M(q_n,1))$ and so,

$$\frac{d_M(p'_n, \partial B_M(q_n,1))}{d_M(z_n, \partial B_M(q_n,1))} \leq 1 + \frac{t}{\lambda'_n d_M(z_n, \partial B_M(q_n,1))}$$

$$\leq 1 + \frac{t}{\lambda'_n \left(d_M(p'_n, \partial B_M(q_n,1)) - \frac{t}{\lambda'_n} \right)} = 1 + \frac{t}{\lambda'_n d_M(p'_n, \partial B_M(q_n,1)) - t}$$

$$\overset{(2)}{\leq} 1 + \frac{t}{n-t}, \tag{4}$$

which tends to 1 as $n \to \infty$, thereby proving the lemma. q.e.d.

4.1. A chord-arc property and the proof of items 1, 2, 3 of Theorem 1.1. With the notation above, we define

$$t_n = \frac{\sqrt{n}}{2}, \quad \widetilde{M}(n) = \lambda'_n \overline{B}_M(p'_n, \tfrac{t_n}{\lambda'_n}). \tag{5}$$

Since by (2) $h_n(p'_n) \geq n > t_n$, then $\frac{t_n}{\lambda'_n} < \frac{h_n(p'_n)}{\lambda'_n} = d_M(p'_n, \partial B_M(q_n,1))$. Therefore, given any $z \in B_M(p'_n, \frac{t_n}{\lambda'_n})$, we have

$$d_M(z, q_n) \leq d_M(z, p'_n) + d_M(p'_n, q_n) < \frac{t_n}{\lambda'_n} + d_M(p'_n, q_n)$$
$$< d_M(p'_n, \partial B_M(q_n,1)) + d_M(p'_n, q_n) = 1,$$

that is, $z \in B_M(q_n, 1)$. This last property lets us apply (3) and (4) to conclude that for n large and $z \in B_M(p'_n, \frac{t_n}{\lambda'_n})$, we have

$$\frac{1}{\lambda'_n I_M(z)} \leq 1 + \frac{t_n}{n - t_n} \leq 2; \tag{6}$$

hence, the injectivity radius function of the complete surface $\lambda'_n M$ is greater than $\frac{1}{2}$ at any point in $\widetilde{M}(n)$. This clearly implies that

(Inj) $\widetilde{M}(n)$ has injectivity radius at least $\frac{1}{2}$ at points of distance greater than $\frac{1}{2}$ from its boundary.

Proposition 4.2. *Given $R_1 > 0$, there exists $\widetilde{\delta} = \widetilde{\delta}(R_1) \in (0, \frac{1}{2})$ such that for any $R \in (0, R_1]$ and for n sufficiently large, the closure $\Sigma(n, \widetilde{\delta}R)$ of the component of $\widetilde{M}(n) \cap B_{\lambda'_n N}(p'_n, \widetilde{\delta}R)$ passing through p'_n has $\partial\Sigma(n, \widetilde{\delta}R) \subset \partial B_{\lambda'_n N}(p'_n, \widetilde{\delta}R)$ and satisfies*

$$\Sigma(n, \widetilde{\delta}R) \subset B_{\widetilde{M}(n)}(p'_n, \tfrac{R}{2}). \tag{7}$$

Furthermore, the function $r \in (0,\infty) \mapsto \widetilde{\delta}(r) \in (0, \frac{1}{2})$ can be chosen so that $\widetilde{\delta}(r)$ is nonincreasing and $r\,\widetilde{\delta}(r) \to \infty$ as $r \to \infty$.

Proof. We will start by proving the following property.

(C) Given $R_1 > 0$, there exists $\widetilde{\delta} = \widetilde{\delta}(R_1) \in (0, \frac{1}{2})$ such that with the notation of the lemma, the inclusion in (7) holds for all $R \in (0, R_1]$.

Arguing by contradiction, suppose there exists a sequence $\delta_n \searrow 0$ so that $\Sigma(n, \delta_n R_n)$ intersects the boundary of $B_{\widetilde{M}(n)}(p'_n, \frac{R_n}{2})$ for some $R_n \leq R_1$. Observe that we can assume that the number $R_0 > 0$ appearing in Theorem 2.4 is not greater than $\frac{1}{2}$. For n sufficiently large, all points in $B_{\widetilde{M}(n)}(p'_n, R_0)$ are at intrinsic distance greater than $\frac{1}{2}$ from the boundary of $\widetilde{M}(n)$, and, thus, the injectivity radius property (Inj) implies that $B_{\widetilde{M}(n)}(p'_n, R_0)$ is topologically a disk whose injectivity radius function coincides with the distance to its boundary function. This property together with the fact that $\delta_n < \delta'$ for n large (here δ' is the positive constant that appears in Theorem 2.4) allow us to apply Theorem 2.4 to conclude that $R_n > R_0$.

As $\Sigma(n, \delta_n R_n) \cap \partial B_{\widetilde{M}(n)}(p'_n, \frac{R_n}{2}) \neq \emptyset$, then there exists a curve $\gamma_n \colon [0,1] \to \Sigma(n, \delta_n R_n)$ such that $\gamma_n(0) = p'_n$, $\gamma_n(1) \in \partial B_{\widetilde{M}(n)}(p'_n, \frac{R_n}{2})$ and $\gamma_n(t) \in B_{\widetilde{M}(n)}(p'_n, \frac{R_n}{2})$ for all $t \in [0,1)$. In particular, the length of γ_n is at least $\frac{R_n}{2}$.

Consider the positive numbers $\tau_n = \dfrac{1}{\delta_n R_n} \geq \dfrac{1}{\delta_n R_1} \to \infty$ and note that the scaled surfaces $\tau_n \Sigma(n, \delta_n R_n)$ can be viewed as the closure of the component of

$$\tau_n[\widetilde{M}(n) \cap B_{\lambda'_n N}(p'_n, \delta_n R_n)] = \tau_n \widetilde{M}(n) \cap B_{\tau_n \lambda'_n N}(p'_n, 1)$$

that passes through p'_n; recall that $B_{\tau_n \lambda'_n N}(p'_n, 1)$ can be taken arbitrarily close to $\mathbb{B}(1)$ with its standard flat metric. Let $\widetilde{\gamma}_n \subset \tau_n \Sigma(n, \delta_n R_n)$ be the related scaling of γ_n. Since the intrinsic distance from $\widetilde{\gamma}_n(0) = \vec{0} = p'_n$ to $\widetilde{\gamma}_n(1)$ in $\tau_n \Sigma(n, \delta_n R_n)$ is $\frac{\tau_n R_n}{2} = \frac{1}{2\delta_n} \to \infty$, then for n large there exists a collection of points $Q_n = \{q^n_1, \ldots, q^n_{k(n)}\} \subset \widetilde{\gamma}_n \subset \mathbb{B}(1)$ whose intrinsic distances from each other in $\tau_n \widetilde{M}(n)$ are diverging to infinity and with $k(n) \to \infty$ as $n \to \infty$; here we are viewing $\mathbb{B}(1)$ as being exponential coordinates for $B_{\tau_n \lambda'_n N}(p'_n, 1)$ and so, we can consider all of the curves $\widetilde{\gamma}_n$ to lie in the open unit ball in $\mathbb{R}^3$ with metrics converging to the usual flat one. In particular, there are positive numbers $r_n \to \infty$ such that

$$\{B_{\tau_n \widetilde{M}(n)}(q^n_k, r_n) \mid k \in \{1, \ldots, k(n)\}\}$$

forms a pairwise disjoint collection of intrinsic balls contained in the interior of $\tau_n \widetilde{M}(n)$, and property (Inj) implies that

(Inj1) The intrinsic distance from each $B_{\tau_n \widetilde{M}(n)}(q^n_k, r_n)$ to the boundary of $\tau_n \widetilde{M}(n)$ is at least 1 for n sufficiently large (this holds because $\tau_n \to \infty$ and $t_n \to \infty$), and the injectivity radius of $\tau_n \widetilde{M}(n)$ is at least 2 at points of distance at least 2 from the boundary of $\tau_n \widetilde{M}(n)$.

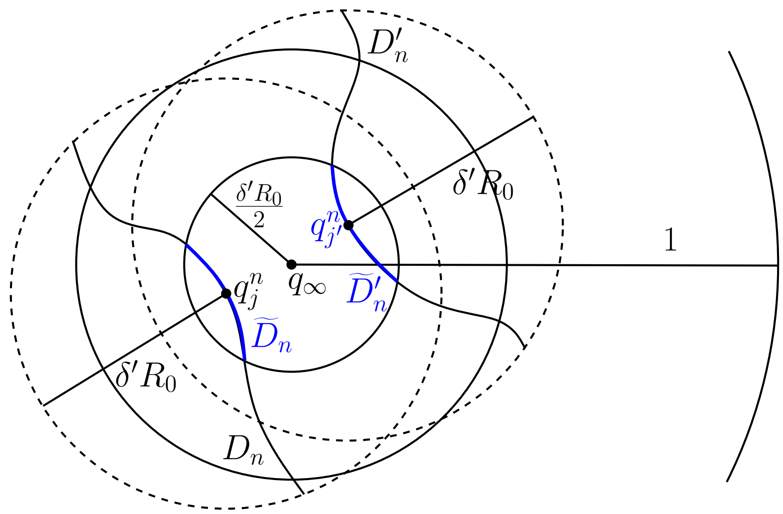

Figure 3. The disks $\widetilde{D}_n, \widetilde{D}'_n$ are disjoint and become close to q_∞.

Since the number of points in Q_n is diverging to infinity as $n \to \infty$, then after replacing by a subsequence, there exists a sequence of pairs of points $q^n_j \neq q^n_{j'} \in Q_n$ such that $\{q^n_j\}_n, \{q^n_{j'}\}_n$ converge to the same point $q_\infty \in \overline{\mathbb{B}}(1)$. By property (Inj1), $B_{\tau_n \widetilde{M}(n)}(q^n_j, 1)$, $B_{\tau_n \widetilde{M}(n)}(q^n_{j'}, 1)$ are minimal disks that satisfy the hypotheses of Theorem 2.4. Therefore, the closures D_n, D'_n of the components of $B_{\tau_n \widetilde{M}(n)}(q^n_j, 1) \cap B_{\tau_n \lambda'_n N}(q^n_j, \delta' R_0)$, $B_{\tau_n \widetilde{M}(n)}(q^n_{j'}, 1) \cap B_{\tau_n \lambda'_n N}(q^n_{j'}, \delta' R_0)$ passing respectively through q^n_j, $q^n_{j'}$ are disks with their boundaries in the respective ambient boundary spheres, where δ', R_0 are defined in Theorem 2.4. For n large enough, we may assume that the boundaries of extrinsic balls of radius at most 1 in $\tau_n \lambda'_n N$ are spheres of positive mean curvature with respect to the inward pointing normal vector. The mean curvature comparison principle implies that the respective components $\widetilde{D}_n, \widetilde{D}'_n$ of $D_n \cap B_{\tau_n \lambda'_n N}(q_\infty, \delta' R_0/2)$, $D'_n \cap B_{\tau_n \lambda'_n N}(q_\infty, \delta' R_0/2)$ passing through the points $q^n_j, q^n_{j'}$ are disks with their boundary curves in $B_{\tau_n \lambda'_n N}(q_\infty, \delta' R_0/2)$, see Figure 3.

As described in the proof of the minimal lamination closure theorem in [**36**], the extrinsic[5] one-sided curvature estimates for minimal disks of Colding–Minicozzi (Corollary 0.4 in [**11**]) imply that there exists a constant $C > 0$ only depending on N such that the norm of the second fundamental forms of the subdisks of $\widetilde{D}_n, \widetilde{D}'_n$ in the smaller ball $B_{\tau_n \lambda'_n N}(q_\infty, \delta' R_0/4)$ containing the respective points $q^n_j, q^n_{j'}$, are

[5]One could instead use the intrinsic version of the one-sided curvature estimates (Corollary 0.8 in Colding and Minicozzi [**12**]) to shorten this argument.

bounded by C (see Theorem 7 in [**36**] for an exact statement of this result). Since these subdisks have uniformly bounded second fundamental forms, a subsequence of these subdisks converges to a compact minimal disk $D(q_\infty)$ passing through q_∞ with boundary in the boundary of the ball $\mathbb{B}(q_\infty, \delta' R_0/4)$ and the norm of the second fundamental form of $D(q_\infty)$ is everywhere bounded from above by C. A prolongation argument (see, for instance, the proof of Theorem 4.37 in Pérez and Ros [**42**]) implies that $D(q_\infty)$ lies in a complete, embedded minimal surface $M(\infty)$ in $\mathbb{R}^3$ with its flat metric and with the norm of the second fundamental form of $M(\infty)$ bounded from above by C; furthermore, $M(\infty)$ must be proper in $\mathbb{R}^3$ by Theorem 2.1 in [**37**].

The above arguments also prove that for any fixed $T > 0$, for n sufficiently large, the norms of the second fundamental forms of the intrinsic balls $B_{\tau_n \widetilde{M}(n)}(q_j^n, T)$, $B_{\tau_n \widetilde{M}(n)}(q_{j'}^n, T)$, are bounded from above by $2C$. By Lemma 3.2 in [**33**], for T sufficiently large, the boundary of the component of $B_{\tau_n \widetilde{M}(n)}(q_j^n, T) \cap B_{\tau_n \lambda_n' N}(q_j^n, 3)$ that passes through q_j^n lies on the boundary of the ball $B_{\tau_n \lambda_n' N}(q_j^n, 3)$. This is a contradiction since for n sufficiently large, the curve $\widetilde{\gamma}_n$ intersects this component, $\widetilde{\gamma}_n$ does not intersect the boundary of the component and $\widetilde{\gamma}_n$ passes through the point $q_{j'}^n \notin B_{\tau_n \widetilde{M}(n)}(q_j^n, T)$. This contradiction proves Property (C).

Note that given $R_1 > 0$, we can assume that the number t_n given by (5) satisfies $t_n > \frac{R_1}{2}$ for n sufficiently large; this implies that given $R \in (0, R_1]$, by definition of $\Sigma(n, \widetilde{\delta} R)$, no points in $\Sigma(n, \widetilde{\delta} R)$ are in the boundary of $\widetilde{M}(n)$. Thus, the boundary $\partial \Sigma(n, \widetilde{\delta} R)$ is contained in the boundary of $B_{\lambda_n' N}(p_n', \widetilde{\delta} R)$, as stated in the first sentence of Proposition 4.2.

To finish the proof of Proposition 4.2, it remains to show that $\widetilde{\delta} = \widetilde{\delta}(R_1)$ can be chosen so $\widetilde{\delta}(r)$ is nonincreasing and that $r\,\widetilde{\delta}(r) \to \infty$ as $r \to \infty$. Property (C) lets us define for each $r \in (0, \infty)$, $\widehat{\delta}(r)$ as the supremum of the values $\widetilde{\delta} \in (0, \frac{1}{2})$ such that for n sufficiently large, the inclusion in (7) holds for this value of $\widetilde{\delta}$, for all $R \leq r$. Note that the function $r \mapsto \widehat{\delta}(r)$ is nonincreasing. In order to complete the proof of the proposition it suffices to show that $\lim_{r\to\infty} r\,\widehat{\delta}(r) = \infty$.

Arguing by contradiction, suppose that there exists a sequence $r_n \to \infty$ such that $r_n \widehat{\delta}(r_n) \leq K$, for some $K \in (1, \infty)$; in particular, $\lim_{r\to\infty} \widehat{\delta}(r) = 0$. By the definition of $\widehat{\delta}(r)$, it follows that after choosing a subsequence,

$$\Sigma(n, 2\widehat{\delta}(r_n) R_n) \not\subset B_{\widetilde{M}(n)}(p_n', \tfrac{R_n}{2}), \text{ for some } R_n \in (R_0, r_n]. \tag{8}$$

We claim that

$$\lim_{n\to\infty} R_n = \infty. \tag{9}$$

Otherwise, after choosing a subsequence, we have $R_n \leq C'$ for some $C' > 1$. Note that we may also assume that $\widehat{\delta}(r_n) < \frac{1}{2}\widehat{\delta}(C')$ for all $n \in \mathbb{N}$. But then,

(10)
$$\Sigma(n, 2\widehat{\delta}(r_n)R_n) \subset \Sigma(n, \widehat{\delta}(C')R_n) \subset \Sigma(n, \widehat{\delta}(R_n)R_n) \overset{(7)}{\subset} B_{\widetilde{M}(n)}(p_n', \tfrac{R_n}{2}),$$

which contradicts (8). Therefore, (9) holds.

Now define $\tau_n = \dfrac{1}{\widehat{\delta}(r_n)R_n}$. By property (Inj), the rescaled surfaces $\tau_n\widetilde{M}(n)$ have injectivity radius bounded from below by

$$\frac{\tau_n}{2} = \frac{1}{2\widehat{\delta}(r_n)R_n} \geq \frac{1}{2\widehat{\delta}(r_n)r_n} \geq \frac{1}{2K} > 0$$

at points of distance at least $\tau_n/2$ from its boundary. Furthermore, the scaled surfaces

$$\tau_n\Sigma(n, 2\widehat{\delta}(r_n)R_n)$$

can be viewed to be contained in the ball $\overline{\mathbb{B}}(2)$. As in the previous case where $r = R_1$ was fixed, one can define curves γ_n in $\Sigma(n, 2\widehat{\delta}(r_n)R_n)$ such that the associated scaled curves $\widetilde{\gamma}_n$ in $\tau_n\Sigma(n, 2\widehat{\delta}(r_n)R_n)$ have intrinsic distances between the end points of $\widetilde{\gamma}_n$ diverging to infinity and $n \to \infty$. From straightforward modifications of the arguments in the first part of the proof of this proposition, one arrives to a contradiction; for these modifications one does not need that the injectivity radii of the surfaces $\tau_n\widetilde{M}(n)$ are at least 2 but just that they are bounded from below by a uniform constant. This contradiction proves that $\lim_{r\to\infty} r\,\widehat{\delta}(r) = \infty$ and completes the proof of the proposition after redefining $\widetilde{\delta}(r)$ by $\widehat{\delta}(r)$.

q.e.d.

Remark 4.3. In Proposition 4.2, the value of $r \mapsto \widetilde{\delta}(r)$ might depend *a priori* on the homogeneously regular ambient manifold N where the blow-up process on the scale of topology was performed or on the complete minimal surface M with injectivity radius zero. In fact, this $\widetilde{\delta}(r)$ can be chosen independent upon N because the inclusion in (7) is invariant under rescaling once the metric on the scaled manifold $\lambda_n' N$ is sufficiently C^2-close to a flat metric and the injectivity radius of $\lambda_n' N$ is at least 1. A similar argument shows that $\widetilde{\delta}(r)$ can be also chosen independent of the minimal surface M.

We next continue with the proof of Theorem 1.1. Consider the nonincreasing function $\widetilde{\delta}\colon (0,\infty) \to (0, \frac{1}{2})$ given by Proposition 4.2. The function $\delta\colon (0,\infty) \to (0,\frac{1}{2})$ described in the statement of Theorem 1.1 can be defined as any smooth decreasing function such that $\frac{1}{2}\widetilde{\delta}(r) \leq \delta(r) \leq \widetilde{\delta}(r)$ for any $r > 0$. In particular, (7) holds true after replacing $\widetilde{\delta}$ by δ, and $r\,\delta(r) \to \infty$ as $r \to \infty$.

Lemma 4.4. *Items 1, 2, 3 of Theorem 1.1 hold.*

Proof. By the last statement in Proposition 4.2, for $k \in \mathbb{N}$, we can pick values $r_k > 0$ such that $r_k\, \delta(r_k) = k$. In particular, Proposition 4.2 ensures that $\partial\Sigma(n,k) \subset \partial B_{\lambda'_n N}(p'_n, k)$ and

$$\Sigma(n,k) \subset B_{\widetilde{M}(n)}(p'_n, \frac{r_k}{2}), \tag{11}$$

for all $n \geq n(k)$, where $n(k) \in \mathbb{N}$ that can be assumed to tend to infinity as $k \to \infty$. Furthermore, we can also assume $n(k)$ is chosen so that $\frac{k}{n(k)} \to 0$ as $k \to \infty$. Defining

$$\varepsilon_k = \frac{k}{\lambda'_{n(k)}} = k\, I_M(p'_{n(k)}) \overset{(2)}{\leq} \frac{k}{n(k)},$$

then $\varepsilon_k \to 0$ as $k \to \infty$. Finally, we define for every $k \in \mathbb{N}$

$$p_k = p'_{n(k)} \quad \text{and} \quad M_k = \frac{1}{\lambda'_{n(k)}} \Sigma(n(k), k). \tag{12}$$

Then, item 1 of Theorem 1.1 follows directly from (11). Item 2 of the theorem also holds from (6) after replacing by a further subsequence, and item 3 of the theorem hold trivially since N is homogeneously regular and $\lambda_k = 1/I_M(p_k) = \lambda'_{n(k)}$ tends to ∞ as $k \to \infty$. This completes the proof of the lemma, after replacing k by n. q.e.d.

From this point on in the proof, we will assume that the first three items of Theorem 1.1 hold for the sequence of points $p_n \in M$ and we will discuss two cases in distinct subsections, depending on whether or not a subsequence of the surfaces $\lambda_n B_M(p_n, \frac{t}{\lambda_n})$ has uniformly bounded Gaussian curvature (the bound could depend on $t > 0$). Before doing this, we will state a property which will be useful in both cases.

Assertion 4.5. *For n large, there exists an embedded geodesic loop $\beta_n \subset \lambda_n M_n$ of length two based at p_n (smooth except possibly at p_n) which is homotopically nontrivial in $\lambda_n M_n$.*

Proof. Since the surfaces $\lambda_n M_n$ are minimal and the sectional curvatures of the ambient spaces $\lambda_n N$ are converging uniformly to zero, then the Gauss equation implies that the exponential map $\exp_{p_n}$ of $T_{p_n}(\lambda_n M_n)$ restricted to the closed metric ball of radius 2 centered at the origin is a local diffeomorphism. As the injectivity radius of $\lambda_n M_n$ at p_n is 1, then $\exp_{p_n}$ is a diffeomorphism when restricted to the open disk of radius 1, and it fails to be injective on the boundary circle of radius 1. Now it is standard to deduce the existence of a geodesic loop β_n as in the statement of the assertion, except for the property that β_n is homotopically nontrivial which we prove next.

Arguing by contradiction and after extracting a subsequence, assume that β_n is homotopically trivial in $\lambda_n M_n$. Thus, β_n bounds a disk $D_n \subset \lambda_n M_n$. Observe that $\lambda_n M_n$ is contained in $B_{\lambda_n N}(p_n, n)$ (this follows from (12) and from Definition 2.3). Since the extrinsic spheres $\partial B_{\lambda_n N}(p_n, R)$ have positive mean curvature for n large with respect to the inward pointing normal vector (because they are produced by rescaling of extrinsic balls of radius $R/\lambda_n \to 0$ in the homogeneously regular manifold N) and $D_n \subset B_{\lambda_n N}(p_n, n)$ with $\partial D_n = \beta_n \subset B_{\lambda_n M_n}(p_n, 2) \subset B_{\lambda_n N}(p_n, 2)$, then the mean curvature comparison principle implies $D_n \subset B_{\lambda_n N}(p_n, 2)$. As the balls $B_{\lambda_n N}(p_n, 2)$ are converging uniformly to the flat ball $\mathbb{B}(2)$, and $\mathbb{B}(2)$ contains no closed minimal surfaces without boundary, then the isoperimetric inequality in [**48**] implies that there exists an upper bound A_0 for the areas of the disks D_n (here we are using Theorem 2.1 in [**48**] on the mean convex balls $B_{\lambda_n N}(p_n, 2)$ for n large, hence, the constant A_0 does in principle depend on n; the fact that A_0 can be taken independently of n large follows from the upper semicontinuous dependence of A_0 on the ambient Riemannian manifold with mean convex boundary, see the sentence just before Corollary 2.4 in [**48**]). As the limsup of the sequence of numbers $\max_{D_n} K_{\lambda_n M_n}$ is nonpositive, then the Gauss–Bonnet formula gives

$$2\pi = \int_{D_n} K_{\lambda_n M_n} + \alpha \leq \int_{D_n} K_{\lambda_n M_n} + \pi,$$

where α is the angle of β_n at p_n. The above inequality is impossible, since as $n \to \infty$,

$$\int_{D_n} K_{\lambda_n M_n} \leq \max_{D_n}(K_{\lambda_n M_n})\, A_0 \leq \limsup \left(\max_{D_n} K_{\lambda_n M_n} \right) A_0 \leq 0.$$

This contradiction proves that the embedded geodesic loop β_n is homotopically nontrivial in $\lambda_n M_n$. q.e.d.

4.2. The case of uniformly bounded Gaussian curvature on compact subsets of $\mathbb{R}^3$.

Proposition 4.6. *In the situation above, suppose that for every $t > 0$, the surfaces $\lambda_n B_M(p_n, \frac{t}{\lambda_n})$ have uniformly bounded Gaussian curvature. Then, item 4 of Theorem 1.1 holds.*

Proof. In the special case that there exists $C > 0$ so that for every $t > 0$ there exists $n(t) \in \mathbb{N}$ such that the surfaces $\lambda_n B_M(p_n, \frac{t}{\lambda_n})$, $n \geq n(t)$, have absolute Gaussian curvature bounded by C, then a complete proof of this proposition can be found in Section 3 of [**33**]. We will next modify some of those arguments in order to deal with the more general current situation, where the bound C might depend on $t > 0$.

Fix $t > 0$. As by hypothesis the surfaces $\lambda_n B_M(p_n, \frac{t}{\lambda_n})$ have uniformly bounded Gaussian curvature, then they also have uniformly bounded area by comparison theorems in Riemannian geometry

(Bishop's second theorem, see e.g., Theorem III.4.4 in Chavel [4]). After extracting a subsequence, the compact surfaces $\lambda_n \overline{B}_M(p_n, \frac{t}{\lambda_n})$ converge on compact sets of $\mathbb{R}^3$ (possibly with integer nonconstant multiplicities) to an embedded, compact minimal surface with boundary $M_\infty(t) \subset \overline{\mathbb{B}}(t)$, with bounded Gaussian curvature, such that $\vec{0}$ lies in the interior of $M_\infty(t)$. We claim that the intrinsic distance from $\vec{0}$ to $\partial M_\infty(t)$ is t. To see this, first note that this intrinsic distance is clearly at most t, as t is the radius of $\lambda_n B_M(p_n, \frac{t}{\lambda_n})$. Let α be a minimizing geodesic from $\vec{0}$ to $\partial M_\infty(t)$. For n large, one can lift α normally to nearby arcs α_n on $\lambda_n B_M(p_n, \frac{t}{\lambda_n})$, each of which starts at $\vec{0}$ and has one end point in $\partial[\lambda_n B_M(p_n, \frac{t}{\lambda_n})]$, so that their lengths converge as $n \to \infty$ to the length of α. Clearly the length of α_n is at least t; hence, after taking limits we deduce that the length of α is at least t, and our claim is proved.

Consider an increasing sequence $1 = t_1 < t_2 < \dots$ with $t_m \to \infty$ as $m \to \infty$. For $m = 1$, consider the compact surface $M_\infty(1)$ together with the sequence of surfaces $\lambda_n \overline{B}_M(p_n, \frac{1}{\lambda_n})$ that converges to it (after passing to a subsequence). For this sequence, consider the corresponding intrinsic balls $\lambda_n \overline{B}_M(p_n, \frac{t_2}{\lambda_n})$. After extracting a subsequence, these surfaces converge to $M_\infty(t_2)$; in particular, $M_\infty(1) \subset M_\infty(t_2)$. Repeating this argument and using a diagonal subsequence, one can construct the surface

$$M_\infty = \bigcup_{m=1}^{\infty} M_\infty(t_m).$$

As the intrinsic distance from $\vec{0}$ to $\partial M_\infty(t_m)$ is t_m for every $m \in \mathbb{N}$, then M_∞ is a complete, injectively immersed minimal surface in $\mathbb{R}^3$ without boundary. Observe that for every m, the convergence of the limit of the surfaces $\lambda_n \overline{B}_M(p_n, \frac{t_m}{\lambda_n})$ to $M_\infty(t_m)$ is one, as follows, for example, from the arguments in the proof of Lemma 3.1 in [33] (higher multiplicity produces a positive Jacobi function on M_∞, hence, M_∞ is stable and so, M_∞ is a plane, which contradicts the following assertion).

Assertion 4.7. *M_∞ is not a plane.*

Proof. Consider for each $n \in \mathbb{N}$ large, the embedded geodesic loop $\beta_n \subset \lambda_n M_n$ of length two based at p_n given by Assertion 4.5. Clearly, $\beta_n \subset \lambda_n \overline{B}_M(p_n, \frac{1}{\lambda_n})$. As the surfaces $\lambda_n \overline{B}_M(p_n, \frac{1}{\lambda_n})$ converge on compact subsets of $\mathbb{R}^3$ to $M_\infty(1)$, then the β_n converge after passing to a subsequence to an embedded geodesic loop $\beta_\infty \subset M_\infty(1)$, which is impossible if M_∞ were a plane. Hence, the assertion follows. q.e.d.

We next analyze the injectivity radius function of M_∞. Fix $m \in N$. Since $M_\infty(t_m)$ is compact and injectively immersed, then there exists $\mu(t_m) > 0$ such that $M_\infty(t_m)$ admits a regular neighborhood $U(t_m) \subset$

$\mathbb{R}^3$ of radius $\mu(t_m)$ and we have a related normal projection

$$\Pi_{t_m}: U(t_m) \to M_\infty(t_m).$$

In this setting, Lemma 3.1 in [**33**] applies[6] to give the following property:

Assertion 4.8. *Given $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that if $n \geq k$, then $\lambda_n B_M(p_n, \frac{t_m}{\lambda_n})$ is contained in $U(t_{m+1})$ and $\lambda_n B_M(p_n, \frac{t_m}{\lambda_n})$ is a small normal graph over its projection to $M_\infty(t_{m+1})$, i.e.,*

$$(\Pi_{t_{m+1}})|_{\lambda_n B_M(p_n, \frac{t_m}{\lambda_n})} : \lambda_n B_M(p_n, \tfrac{t_m}{\lambda_n}) \to \Pi_{t_{m+1}} \left(\lambda_n B_M(p_n, \tfrac{t_m}{\lambda_n}) \right)$$

is a diffeomorphism.

We now remark on some properties of the minimal surface $M_\infty \subset \mathbb{R}^3$. By Assertion 4.8, given $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that we can induce the metric of $\lambda_n \overline{B}_M(p_n, \frac{t_m}{\lambda_n})$ to its projected image $\Pi_{t_{m+1}} \left(\lambda_n B_M(p_n, \frac{t_m}{\lambda_n}) \right)$ through the diffeomorphism $\Pi_{t_{m+1}}$. As the sequence $\{\lambda_n \overline{B}_M(p_n \frac{t_m}{\lambda_n})\}_n$ converges to $M_\infty(t_m)$ with multiplicity one as $n \to \infty$, then using the continuity of the injectivity radius function with respect to the Riemannian metric on a given compact surface (see Ehrlich [**16**] and Sakai [**44**]) and inequality (4), we deduce that

(13)
$$(I_{M_\infty})\,|_{M_\infty(t_m)} = \lim_{n\to\infty} (I_{\lambda_n M})\,|_{\lambda_n \overline{B}_M(p_n, \frac{t_m}{\lambda_n})} \geq \lim_{n\to\infty} \frac{1}{1 + \frac{t_m}{n - t_m}} = 1.$$

Hence, we conclude that $I_{M_\infty} \geq 1$ everywhere on M_∞, with $I_{M_\infty}(\vec{0}) = 1$.

Since $M_\infty \subset \mathbb{R}^3$ is a complete embedded minimal surface in $\mathbb{R}^3$ with positive injectivity radius, the minimal lamination closure theorem [**36**] insures that M_∞ is properly embedded in $\mathbb{R}^3$. This finishes the proof of Proposition 4.6. q.e.d.

4.3. The case of Gaussian curvature not uniformly bounded. Suppose now that the uniformly bounded Gaussian curvature hypothesis in Proposition 4.6 fails to hold. It follows, after extracting a subsequence, that for some fixed positive number $t_1 > 0$, the maximum absolute Gaussian curvature of the surfaces $\lambda_n' B_M(p_n, \frac{t_1}{\lambda_n})$ diverges to infinity as $n \to \infty$. To finish the proof of Theorem 1.1, it remains to show that under this condition, items 5 or 6 hold.

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[6]In Section 3 of [**33**], we had the additional hypothesis that M_∞ has globally bounded Gaussian curvature, hence, it is proper; nevertheless, the proof of Lemma 3.1 in [**33**] only uses that $M_\infty(t_m)$ is compact for each $m \in \mathbb{N}$ and that M_∞ is not a plane (or more precisely, that the convergence of the limit $\{\lambda_n \overline{B}_M(p_n, \frac{t_m}{\lambda_n})\}_n \to M_\infty(t_m)$ is one, which in turn follows from the fact that M_∞ is not a plane), conditions which are satisfied in our current setting.

Definition 4.9. A sequence of compact embedded minimal surfaces Σ_n in $\mathbb{R}^3$ with boundaries diverging in space, is called *uniformly locally simply connected*, if there is an $\varepsilon > 0$ such that for any ambient ball $\mathbb{B}$ of radius $\varepsilon > 0$ and for n sufficiently large, $\mathbb{B}$ intersects Σ_n in compact disks with boundaries in the boundary of $\mathbb{B}$ (this definition is more restrictive than the similarly defined notion in the introduction of [**13**], where ε might depend on $\mathbb{B}$).

We next return to the proof of Theorem 1.1. By the discussion in the proof of Proposition 4.2 (also see Theorem 2.4), the sequence of minimal surfaces $\lambda_n M_n$ can be considered to be uniformly locally simply connected, as the metric balls containing the surfaces are converging to $\mathbb{R}^3$ with the usual metric. Thus,

(ULSC) There exists $\varepsilon_1 \in (0, 1/2)$ such that for every ball $\mathbb{B} \subset \mathbb{R}^3$ of radius ε_1, and for n sufficiently large, $(\lambda_n M_n) \cap \mathbb{B}$ consists of disks with boundaries in $\partial\mathbb{B}$.

In this situation, several results by Colding and Minicozzi [**8**, **9**, **11**, **13**] apply to describe the nature of both the surfaces $\lambda_n M_n$ in the sequence and their limit objects after passing to a subsequence. We next briefly explain this description, which can also be modified to work in the setting of a homogeneously regular manifold (see, for instance, page 33 of [**8**] and [**36**]).

Definition 4.10. In polar coordinates (ρ, θ) on $\mathbb{R}^2 - \{0\}$ with $\rho > 0$ and $\theta \in \mathbb{R}$, a *k-valued graph on an annulus of inner radius r and outer radius R,* is a single-valued graph of a real-valued function $u(\rho, \theta)$ defined over

$$S_{r,R}^{-k,k} = \{(\rho, \theta) \mid r \leq \rho \leq R,\ |\theta| \leq k\pi\}, \tag{14}$$

k being a positive integer.

By the one-sided curvature estimates for minimal disks as applied in the proof of Theorem 0.1 in [**11**] (also see the proof of Theorem 0.9 in [**13**]), there exists a closed set $\mathcal{S} \subset \mathbb{R}^3$, a nonempty minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ which cannot be extended across any proper closed subset of $\mathcal{S}$, and a subset $S(\mathcal{L}) \subset \mathcal{L}$ which is closed in the subspace topology, such that after replacing by a subsequence, $\{\lambda_n M_n\}_n$ has uniformly bounded second fundamental form on compact subsets of $\mathbb{R}^3 - \Delta(\mathcal{L})$ where $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$, and $\{\lambda_n M_n\}_n$ converges C^α, $\alpha \in (0, 1)$, on compact subsets of $\mathbb{R}^3 - \Delta(\mathcal{L})$ to $\mathcal{L}$.

Around each point $p \in \Delta(\mathcal{L})$, the surfaces $\lambda_n M_n$ have the following local description. By (ULSC) and Theorem 5.8 in [**9**], there exists $\varepsilon \in (0, \varepsilon_1)$ such that after a rotation of $\mathbb{R}^3$ and extracting a subsequence, each of the disks $(\lambda_n M_n) \cap \overline{\mathbb{B}}(p, \varepsilon)$ contains a 2-valued minimal graph defined on an annulus $\{(x_1, x_2, 0) \mid r_n^2 \leq x_1^2 + x_2^2 \leq R^2\}$ with inner radius $r_n \searrow 0$, for certain $R \in (r_n, \varepsilon)$ small but fixed.

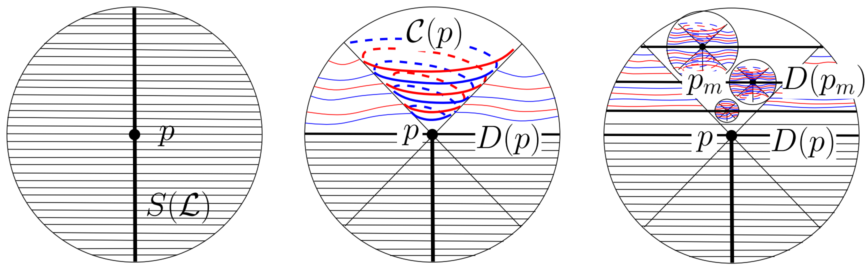

Figure 4. Left: Case (D1), in a neighborhood of a point $p \in S(\mathcal{L})$. Center: Case (D2-A) for an isolated point $p \in \mathcal{S}$; in the picture, p is the end point of an arc contained in $S(\mathcal{L})$, although $D(p,*)$ could also be the limit of two pairs of multivalued graphical leaves, one pair on each side. Right: Case (D2-B) for a nonisolated point $p \in \mathcal{S}$.

(D1) If $p \in S(\mathcal{L})$ (in particular, $\mathcal{L}$ admits a local lamination structure around p), then after possibly choosing a smaller $\varepsilon > 0$, there exists a neighborhood of p in $\overline{\mathbb{B}}(p,\varepsilon)$ which is foliated by compact disks in $\mathcal{L} \cap \overline{\mathbb{B}}(p,\varepsilon)$, and $S(\mathcal{L})$ intersects this family of disks transversely in a connected Lipschitz arc. This case corresponds to case (P) described in Section II.2 of [**13**]. In fact, the Lipschitz curve $S(\mathcal{L})$ around p is a $C^{1,1}$-curve orthogonal to the local foliation (Meeks [**21**, **22**]). See Figure 4 left.

(D2) If $p \in \mathcal{S}$, then after possibly passing to a smaller ε, a subsequence of the surfaces $\{(\lambda_n M_n) \cap \mathbb{B}(p,\varepsilon)\}_n$ (denoted with the same indexes n) converges C^α, $\alpha \in (0,1)$, on compact subsets of $\mathbb{B}(p,\varepsilon) - \Delta(\mathcal{L})$ to the (regular) lamination $\mathcal{L}_p = \mathcal{L} \cap \mathbb{B}(p,\varepsilon)$ of $\mathbb{B}(p,\varepsilon) - \mathcal{S}_p$, where $\mathcal{S}_p = \mathcal{S} \cap \mathbb{B}(p,\varepsilon)$. Furthermore, $\mathcal{L}_p$ contains a limit leaf $D(p,*)$ which is a stable, minimal punctured disk with $\partial D(p,*) \subset \partial \mathbb{B}(p,\varepsilon)$ and $\overline{D(p,*)} \cap \mathcal{S}_p = \{p\}$, and $D(p,*)$ extends to a stable, embedded minimal disk $D(p)$ in $\mathbb{B}(p,\varepsilon)$ (this is Lemma II.2.3 in [**13**]). By Corollary I.1.9 in [**11**], there is a solid double cone[7] $\mathcal{C}_p \subset \mathbb{B}(p,\varepsilon)$ with vertex at p and axis orthogonal to the tangent plane $T_pD(p)$, that intersects $D(p)$ only at the point p (i.e., $D(p,*) \subset \mathbb{B}(p,\varepsilon) - \mathcal{C}_p$) and such that $[\mathbb{B}(p,\varepsilon) - \mathcal{C}_p] \cap \Delta(\mathcal{L}) = \emptyset$. Furthermore, for n large, $(\lambda_n M_n) \cap \mathbb{B}(p,\varepsilon)$ has the appearance outside $\mathcal{C}_p$ of a highly-sheeted double multivalued graph over $D(p,*)$, see Figure 5.

In this local description of this case (D2), it is worth distinguishing two subcases:

[7] A *solid double cone* in $\mathbb{R}^3$ is a set that after a rotation and a translation, can be written as $\{(x_1,x_2,x_3) \mid x_1^2 + x_2^2 \leq \delta^2 x_3^2\}$ for some $\delta > 0$. A solid double cone in a ball is the intersection of a solid double cone with a ball centered at its vertex.

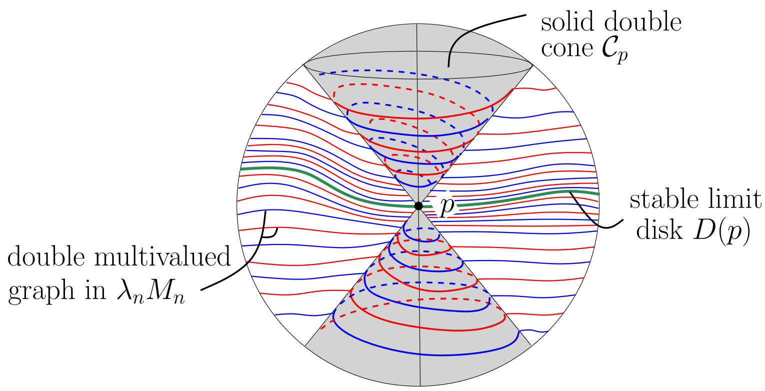

Figure 5. The local picture of disk-type portions of $\lambda_n M_n$ around an isolated point $p \in \mathcal{S}$. The stable minimal punctured disk $D(p,*)$ appears in the limit lamination $\mathcal{L}_p$, and extends smoothly through p to a stable minimal disk $D(p)$ which is orthogonal at p to the axis of the double cone $\mathcal{C}_p$.

(D2-A): If p is an isolated point in $\mathcal{S}$, then the limit leaf $D(p,*)$ of $\mathcal{L}_p$ is either the limit of two pairs of multivalued graphical leaves in $\mathcal{L}_p$ (one pair on each side of $D(p,*)$), or $D(p,*)$ is the limit on one side of just one pair of multivalued graphical leaves in $\mathcal{L}_p$; in this last case, p is the end point of an open arc $\Gamma \subset S(\mathcal{L}) \cap \mathcal{C}_p$, and a neighborhood of p in the closure of the component of $\mathbb{B}(p,\varepsilon) - D(p,*)$ that contains Γ is entirely foliated by disk leaves of $\mathcal{L}_p$. See Figure 4 center.

(D2-B): p is not isolated as a point in $\mathcal{S}$. In this case, p is the limit of a sequence $\{p_m\}_m \subset \mathcal{S} \cap \mathcal{C}_p$. In particular, $D(p)$ is the limit of the related sequence of stable minimal disks $D(p_m)$, and $D(p,*)$ is the limit of a sequence of pairs of multivalued graphical leaves of $\mathcal{L}_p \cap [\mathbb{B}(p,\varepsilon) - (\mathcal{C}_p \cup \{D(p_m)\}_m)]$. Note that these singular points p_m might be isolated or not in $\mathcal{S}$. See Figure 4 right.

We next continue with the proofs of items 5 and 6 of Theorem 1.1. Since the maximum absolute Gaussian curvature of the surfaces $\lambda_n B_M(p_n, \frac{t_1}{\lambda_n})$ diverges to infinity as $n \to \infty$ and $\lambda_n B_M(p_n, \frac{t_1}{\lambda_n})$ is contained in $\lambda_n M_n$ for n sufficiently large, then $\Delta(\mathcal{L})$ is nonempty and contains a point which is at an extrinsic distance at most t_1 from the origin in $\mathbb{R}^3$.

Lemma 4.11. *Let $p \in \Delta(\mathcal{L})$. Then, there exists a limit leaf L_p of $\mathcal{L}$ whose closure $\overline{L_p}$ in $\mathbb{R}^3$ is a plane passing through p. Moreover, the set*

$$\mathcal{P}' = \{\overline{L_p} \mid p \in \Delta(\mathcal{L})\}$$

is a nonempty, closed set of planes.

Proof. The previous description (D1)–(D2) shows that there exists a minimal disk $D(p)$ passing through p such that $D(p,*) = D(p) - \{p\}$ is contained in a limit leaf L of $\mathcal{L}$. As L is a limit leaf of $\mathcal{L}$, then the two-sided cover $\widehat{L}$ of L is stable (Meeks, Pérez and Ros [**32**, **31**]). Consider the union $\widetilde{L}$ of L with all points $q \in \mathcal{S}$ such that the related punctured disk $D(q,*)$ defined in (D2) above is contained in L. Clearly, the two-sided cover of $\widetilde{L}$ is stable and, thus, the classification of complete stable minimal surfaces in $\mathbb{R}^3$ (see e.g. do Carmo and Peng [**15**], Fischer–Colbrie and Schoen [**18**] or Pogorelov [**43**]) implies that to prove the lemma it remains to demonstrate that $\widetilde{L}$ is complete.

Arguing by contradiction, suppose that there exists a shortest unit speed geodesic $\gamma\colon [0,l) \to L$ such that $\gamma(0) \in L$ and $p := \lim_{t\to l^-} \gamma(t) \in \mathcal{S}$. Therefore, there exists $\delta > 0$ such that $\gamma(t) \in \overline{\mathbb{B}}(p,\varepsilon)$ for every $t \in [l-\delta, l)$, where $\overline{\mathbb{B}}(p,\varepsilon)$ is the closed ball that appears in (D2). Note that by construction, $\gamma(t) \notin D(p,*)$ for every $t \in [l-\delta,l)$. As $D(p)$ separates $\overline{\mathbb{B}}(p,\varepsilon)$, then $\gamma([l-\delta,l))$ is contained in one of the two half-balls of $\overline{\mathbb{B}}(p,\varepsilon) - D(p)$, say in the upper half-ball (we can choose orthogonal coordinates in $\mathbb{R}^3$ centered at p so that $T_pD(p)$ is the (x_1,x_2)-plane). In particular, there cannot exist a sequence of $\{p_m\}_m \subset \mathcal{S}$ converging to p in that upper half-ball (otherwise p_m produces a related disk $D(p_m)$ which is proper in the upper half-ball, such that the sequence $\{D(p_m)\}_m$ converges to $D(p)$ as $m \to \infty$; as $\gamma(l-\delta)$ lies above one of these disks $D(p_k)$ for k sufficiently large, then $\gamma([l-\delta,l))$ lies entirely above $D(p_k)$, which contradicts that γ limits to p). Therefore, after possibly choosing a smaller ε, we can assume that there are no points of $\mathcal{S}$ in the closed upper half-ball of $\overline{\mathbb{B}}(p,\varepsilon) - D(p)$ other than p. Now consider the lamination $\mathcal{L}_1$ of $\overline{\mathbb{B}}(p,\varepsilon) - \{p\}$ given by $D(p)$ together with the closure of $L \cap \overline{\mathbb{B}}(p,\varepsilon)$ in $\overline{\mathbb{B}}(p,\varepsilon) - \{p\}$. As the leaves of $\mathcal{L}_1$ are all stable (if L is two-sided; otherwise its two-sided cover is stable), then Corollary 7.1 in [**34**] implies that $\mathcal{L}_1$ extends smoothly across p, which is clearly impossible. This contradiction proves the first statement in the lemma.

We now prove the second statement of the lemma. Suppose that $\{p_m\}_m \subset \Delta(\mathcal{L})$ and the related planes $\overline{L_{p_m}}$ converge to a plane $P \subset \mathbb{R}^3$ as $m \to \infty$. Arguing by contradiction, we suppose that $P \cap \Delta(\mathcal{L}) = \emptyset$. Given $m \in \mathbb{N}$, consider a closed disk $D(q_m, \varepsilon_1)$ in P of radius $\varepsilon_1 > 0$ centered at the orthogonal projection q_m of p_m over P, where $\varepsilon_1 \in (0,1/2)$ was defined in Property (ULSC). As P lies in $\mathcal{L}$ (because $\mathcal{L}$ is closed in $\mathbb{R}^3 - \mathcal{S}$ and $P \cap \mathcal{S} = \emptyset$) and P does not contain points of $\Delta(\mathcal{L})$, then q_m is at positive distance from $\Delta(\mathcal{L})$; in particular, $D(q_m,\varepsilon_1)$ can be arbitrarily well-approximated by almost-flat closed disks $D_{n,m}$ in $\lambda_n M_n$ for n large. For m large, the component $\Omega_n(p_m)$ of $(\lambda_n M_n) \cap \overline{\mathbb{B}}(q_m, \frac{\varepsilon_1}{2})$ that contains p_m is a compact disk which is disjoint from the almost-flat compact disk $D_{n,m} \cap \overline{\mathbb{B}}(q_m, \frac{\varepsilon_1}{2})$ and, thus, $\Omega_n(p_m)$ lies at one side of $D_{n,m} \cap \overline{\mathbb{B}}(q_m, \frac{\varepsilon_1}{2})$. As $p_m \in \Omega_n(p_m)$ is arbitrarily close to $D_{n,m} \cap \overline{\mathbb{B}}(q_m, \frac{\varepsilon_1}{2})$ (for m large), then we contradict the one-sided curvature estimates for embedded minimal disks (Corollary 0.4 in [**11**]). Now the proof of the lemma is complete. q.e.d.

In the sequel, we will assume that the planes in $\mathcal{P}'$ are horizontal.

Recall that our goal in this section is to prove that items 5 or 6 of Theorem 1.1 occur. The key to distinguish which of these options occurs will be based on the singular set $\mathcal{S}$ of $\mathcal{L}$: If $\mathcal{S} = \emptyset$ (hence, $\mathcal{L}$ is a lamination of $\mathbb{R}^3$) then item 5 holds, while if $\mathcal{S} \neq \emptyset$ then item 6 holds. This distinction is equivalent to $\mathcal{L} = \mathcal{P}$ or $\mathcal{L} \neq \mathcal{P}$. The arguments to prove this dichotomy are technical and delicate; we will start by adapting some of the arguments in the last paragraph of the proof of Lemma 4.11 to demonstrate the following result.

Lemma 4.12. *Every flat leaf of $\mathcal{L}$ lies in a plane in $\mathcal{P}'$, and no plane in $\mathbb{R}^3$ is disjoint from $\mathcal{L}$.*

Proof. Arguing by contradiction, suppose L is a flat leaf in $\mathcal{L}$ which does not lie in a plane in $\mathcal{P}'$. Hence, L does not intersect $\Delta(\mathcal{L})$. This implies that L is a plane and arbitrarily large disks in L can be approximated by almost-flat disks in the surfaces $\lambda_n M_n$. Since these surfaces have injectivity radius greater than $1/2$ at points at distance at least $1/2$ from their boundaries, then the one-sided curvature estimates for minimal disks (Corollary 0.4 in [**11**]) imply that there are positive constants δ and C, both independent of L, such that for $R > 0$ and for n sufficiently large, the surface

$$(\lambda_n M_n) \cap \{|x_3 - x_3(L)| < \delta\} \cap \mathbb{B}(\vec{0}, R)$$

has Gaussian curvature less than C. From here, we deduce that the leaves of $\mathcal{L} \cap \{|x_3 - x_3(L)| < \delta\}$ have uniformly bounded Gaussian curvature. From this bounded curvature hypothesis, the proof of Lemma 1.3 in [**35**] implies that $\{|x_3 - x_3(L)| < \delta\} \cap \mathcal{L}$ consists only of planes of $\mathcal{L}$; hence, the distance from L to $\Delta(\mathcal{L})$ is at least δ. Let $\mathcal{L}'$ be the minimal lamination of $\mathbb{R}^3 - \mathcal{S}$ obtained by enlarging $\mathcal{L}$ by adding to it all planes which are disjoint from $\mathcal{L}$. Note that by the one-sided curvature estimates in [**11**], each of the added on planes is also at a fixed distance at least $\delta > 0$ from $\Delta(\mathcal{L})$ and from leaves of $\mathcal{L}$ which are not flat, where δ is the same small number defined previously. Hence, the planes of $\mathcal{L}'$ which are not in $\mathcal{P}'$ form a both open and closed subset of $\mathbb{R}^3$, but $\mathbb{R}^3$ is connected. Hence, this set is empty.

Note that the arguments in the last paragraph also prove that no plane in $\mathbb{R}^3 - \mathcal{L}$ is disjoint from $\mathcal{L}$, which proves the lemma. q.e.d.

Clearly, the closure of every flat leaf of $\mathcal{L}$ is an element of $\mathcal{P}'$ and vice versa, which gives a bijection between $\mathcal{P}'$ and the collection $\mathcal{P}$ that appears in the statement of item 6 of Theorem 1.1.

Lemma 4.13. *Consider a point $x \in \Delta(\mathcal{L})$ and the plane $\overline{L_x} \in \mathcal{P}'$ passing through x. Then, the distance between any two points in $\overline{L_x} \cap \Delta(\mathcal{L})$ is at least 1.*

Proof. Given any $p \in \overline{L_x} \cap \Delta(\mathcal{L})$, $\overline{L_x}$ contains the disk $D(p)$ that appears in description (D2) above. Since the sequence $\{\lambda_n M_n\}_n$ is uniformly locally simply connected, the Colding–Minicozzi local picture (D1)–(D2) of $\lambda_n M_n$ near a point of $\Delta(\mathcal{L})$ implies that there exists an $\eta > 0$ so that for every pair of distinct points $p, q \in \overline{L_x} \cap \Delta(\mathcal{L})$, the distance between p and q is at least η. Fix $p \in \overline{L_x} \cap \Delta(\mathcal{L})$ and take a point $q \in \overline{L_x} \cap \Delta(\mathcal{L})$ closest to p. Hence, $\Delta(\mathcal{L})$ only intersects the closed segment $[p, q] = \{tp + (1-t)q \mid t \in [0, 1]\}$ at the extrema p, q. Using the plane $\overline{L_x}$ as a guide, one can produce homotopically nontrivial simple closed curves γ_n on the approximating surfaces $\lambda_n M_n$ which converge with multiplicity 2 outside p, q to the segment $[p, q]$ as $n \to \infty$ (see, for example the discussion just after Remark 2 in [**29**]). Our injectivity radius assumption implies that the length of the γ_n is greater than or equal to 2, which implies after taking $n \to \infty$ that the distance between p and q is at least 1. This finishes the proof of the lemma. q.e.d.

Definition 4.14. Given $p \in \Delta(\mathcal{L})$, we assign an *orientation number* $n(p) = \pm 1$ according to the following procedure. Outside a solid double vertical cone $\mathcal{C}_p \subset \mathbb{B}(p, \varepsilon)$ based at p, there exists a pair of multivalued graphs contained in the approximating surface $\lambda_n M_n$ for n large, which, after choosing a subsequence, are both right or left handed for n sufficiently large (depending on p). Assign a number $n(p) = \pm 1$ depending on whether these multivalued graphs in $\lambda_n M_n$ occurring nearby p are right $(+)$ or left handed $(-)$.

We also define, given a plane $P \in \mathcal{P}'$,

$$|I|(P) = \sum_{p \in P \cap \Delta(\mathcal{L})} |n(p)| = \text{ Cardinality}(P \cap \Delta(\mathcal{L})) \in \mathbb{N} \cup \{\infty\}.$$

By Lemma 4.13, the set $P \cap \Delta(\mathcal{L})$ is a closed, discrete countable set. After enumerating this set and applying a diagonal argument, it is possible to choose a subsequence of the $\lambda_n M_n$ so that locally around each point $p \in P \cap \Delta(\mathcal{L})$, there exists $n(p) \in \mathbb{N}$ such that $\lambda_n M_n$ contains a pair of multivalued graphs around p with a fixed handedness for all $n \geq n(p)$. This allows us to define consistently the number

$$I(P) = \sum_{p \in P \cap \Delta(\mathcal{L})} n(p) \in \mathbb{Z}, \quad \text{provided that } P \cap \Delta(\mathcal{L}) \text{ is finite}.$$

Our next step will be to study the local constancy of the quantities $|I|(P)$ and $I(P)$ when we vary the plane $P \in \mathcal{P}'$. Recall that the sequence $\{\lambda_n M_n\}_n$ is uniformly locally simply connected close, see Property (ULSC).

Lemma 4.15. *Given $P \in \mathcal{P}'$, there exists $\mu_0 = \mu_0(P)$ such that if $x \in P \cap \Delta(\mathcal{L})$, then for every $P' \in \mathcal{P}'$ with $|x_3(P) - x_3(P')| < \mu_0$ there exists a unique point $x' \in P' \cap \Delta(\mathcal{L}) \cap \mathbb{B}(x, \varepsilon)$ (this number $\varepsilon > 0$ was defined in Description (D1)–(D2)), and the handedness of the two multivalued graphs occurring in (a subsequence of) the $\lambda_n M_n$ nearby x coincides with the handedness of the two multivalued graphs occurring in $\lambda_n M_n$ nearby x' (note that, in particular, we do not need to change the subsequence of the $\lambda_n M_n$ to produce a well-defined handedness at x'). In particular, if $|I|(P) < \infty$, then $|I|(P') = |I|(P)$ and $I(P') = I(P)$.*

Proof. We will start by proving the following simplified version of the first sentence in the statement of the lemma.

Claim 4.16. *Suppose $P = \{x_3 = 0\} \in \mathcal{P}'$ and $x = \vec{0} \in P \cap \Delta(\mathcal{L})$. Then, there exists $\mu_0 > 0$ such that if $P' \in \mathcal{P}'$ and $|x_3(P')| < \mu_0$, then there exists a unique point $x' \in P' \cap \Delta(\mathcal{L}) \cap \mathbb{B}(\varepsilon)$.*

Proof of the Claim. Arguing by contradiction, assume that there exists a sequence $P(m)$ of horizontal planes in $\mathcal{P}'$ of heights $x_{3,m}$ converging to 0 as $m \to \infty$, so that for each $m \in \mathbb{N}$, the open disk $P(m) \cap \mathbb{B}(\varepsilon)$ does not contain any point in $\Delta(\mathcal{L})$. After passing to a subsequence, we can assume that the $P(m)$ converge to P on one of its sides, say from above. By definition of $\mathcal{P}'$, every plane $P(m)$ is the closure in $\mathbb{R}^3$ of a limit leaf L_m of $\mathcal{L}$. As for m fixed the intersection $P(m) \cap \mathbb{B}(\varepsilon) \cap \Delta(\mathcal{L})$ is empty, then the convergence of portions of the surfaces $\lambda_n M_n$ to $P(m) \cap \mathbb{B}(\varepsilon) = L_m \cap \mathbb{B}(\varepsilon)$ as $n \to \infty$ is smooth; in particular, for n, m large (but m fixed), $(\lambda_n M_n) \cap \overline{\mathbb{B}}(\varepsilon/2)$ contains a component $\Omega_1(m, n)$ which is a compact, almost-horizontal disk of height arbitrarily close to $x_{3,m}$, with $\partial\Omega_1(m, n) \subset \partial\mathbb{B}(\varepsilon/2)$. On the other hand, as $\vec{0} \in \Delta(\mathcal{L})$ then there exists a sequence of points $y_n \in \lambda_n M_n$ converging to the origin where the absolute Gaussian curvature of $\lambda_n M_n$ tends to infinity. Let $\Omega_2(n)$ be the component of $(\lambda_n M_n) \cap \overline{\mathbb{B}}(\varepsilon/2)$ that contains y_n. In particular, $\Omega_2(n) \cap \Omega_1(m, n) = \emptyset$ for m fixed and large, and for all n sufficiently large depending on m. Note that $\Omega_2(n)$ is topologically a disk and Theorem 2.4 ensures that $\Omega_2(n)$ is compact with boundary contained in $\partial\mathbb{B}(\varepsilon/2)$. As $x_{3,m} \to 0$ but ε is fixed, then the one-sided curvature estimates for disks in [**11**] (see also Theorem 7 in [**36**]) applied to $\Omega_1(m, n)$, $\Omega_2(n)$ gives a contradiction if m, n are large enough. This contradiction proves the claim, as the uniqueness of the point $x' \in P' \cap \Delta(\mathcal{L}) \cap \mathbb{B}(\varepsilon)$ in the last part of the statement of the claim follows directly from Lemma 4.13. Now the claim is proved. q.e.d.

We next continue the proof of Lemma 4.15. The existence part of the first sentence in the statement of the lemma can be deduced from similar arguments as those in the proof of the last claim; the only difference is that in the argument by contradiction, the plane P cannot be assumed to be $\{x_3 = 0\}$ but instead one assumes that there exist sequences $\{P_m\}_m, \{P'_m\}_m \subset \mathcal{P}'$, $\{\mu_m\}_m \subset \mathbb{R}^+$ with $\mu_m \searrow 0$ and $x_m \in P_m \cap \Delta(\mathcal{L})$ so that $|x_3(P_m) - x_3(P'_m)| < \mu_m$ and $P'_m \cap \Delta(\mathcal{L}) \cap \mathbb{B}(x_m, \varepsilon) = \emptyset$. The desired contradiction also appears in this case from the one-sided curvature estimates for embedded minimal disks and we leave the details to the reader. This finishes the proof of the first sentence in the statement of the lemma.

It remains to show that if $|I|(P) < \infty$, then we can choose $\mu_0 > 0$ sufficiently small so that $|I|(P') < \infty$ and $|I|(P) = |I|(P')$ and $I(P) = I(P')$ for every $P' \in \mathcal{P}'$ with $|x_3(P) - x_3(P')| < \mu_0$. As before, take $x \in P \cap \Delta(\mathcal{L})$. If $x \in S(\mathcal{L})$, then Lemma 4.11 and the description in (D1) of $S(\mathcal{L})$ around x imply that after possibly choosing a smaller $\mu_0 > 0$, $\mathcal{L} \cap \mathbb{B}(x, \mu_0)$ consists of a foliation by horizontal flat disks, and $S(\mathcal{L}) \cap \mathbb{B}(x, \mu_0)$ consists of a Lipschitz curve passing through x which is transverse to this local foliation by flat disks. By the main theorem in Meeks [**21**], this Lipschitz curve is, in fact, a vertical segment with x in its interior. In particular, every horizontal plane P' that intersects $\mathbb{B}(x, \mu_0)$ also intersects $S(\mathcal{L}) \cap \mathbb{B}(x, \mu_0)$ at exactly one point x', and the handedness of the two multivalued graphs occurring in $\lambda_n M_n$ nearby x, x' coincide.

If $x \in \mathcal{S}$, then similar arguments as in the last paragraph give the same conclusion, following the local description in (D2) of $\mathcal{L} \cap \mathbb{B}(x, \mu_0)$ together with the uniform locally simply connected property of $\{\lambda_n M_n\}_n$. This completes the proof of Lemma 4.15. q.e.d.

Lemma 4.17. *For a plane $P \in \mathcal{P}'$, the following properties are equivalent:*

1) *$\mathcal{L}$ does not restrict to a foliation in any μ-neighborhood of P, $\mu > 0$.*
2) *$\Delta(\mathcal{L}) \cap P = \mathcal{S} \cap P$.*
3) *$\mathcal{S} \cap P \neq \emptyset$.*

Proof. That statement 1 implies 2 follows from the observation that if there exists a point p in $S(\mathcal{L}) \cap P$, then there exists a small cylindrical neighborhood of p which is entirely foliated by horizontal disks contained in planes of $\mathcal{P}$, which in turn implies that there exists a slab neighborhood of P which is foliated by planes in $\mathcal{L}$, a contradiction. Statement 2 implies 3 by definition of $\mathcal{P}'$. Finally, the description in (D2-A), (D2-B) give that item 3 implies item 1. q.e.d.

Proposition 4.18. *Let x be a point in $\Delta(\mathcal{L})$ and let $P_x \in \mathcal{P}'$ be the horizontal plane that passes through x. If $|I|(P_x) < \infty$ and $I(P_x) = 0$, then $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by horizontal planes.*

Proof. Consider the largest closed horizontal slab W containing P_x, so that every plane P in W lies in $\mathcal{P}'$ and satisfies $|I|(P) = |I|(P_x)$ and $I(P) = I(P_x)$ (we allow W to be just P_x, to be $\mathbb{R}^3$ or a closed halfspace of $\mathbb{R}^3$). If $W = \mathbb{R}^3$ then the proposition is proved. Arguing by contradiction, suppose W has a boundary plane, which we relabel as P_x (since it has the same numbers $|I|(P_x)$, $I(P_x)$ as the original P_x by Lemma 4.15). Without loss of generality, we will assume P_x is the upper boundary plane of W. Since $\mathbb{R}^3 - \cup_{P' \in \mathcal{P}'} P'$ is a nonempty union of open slabs or halfspaces, then one of the following possibilities occurs:

(F1) P_x is not a limit of planes in $\mathcal{P}'$ from above. In this case, P_x is the boundary of an open slab or halfspace in $\mathbb{R}^3 - \cup_{P' \in \mathcal{P}'} P'$.

(F2) P_x is the limit of a sequence of planes $P_m \in \mathcal{P}'$ from above, such that for every m, $P_{2m} \cup P_{2m+1}$ is the boundary of an open slab component of $\mathbb{R}^3 - \cup_{P' \in \mathcal{P}'} P'$. Thus, Lemma 4.15 implies that for m sufficiently large, P_m satisfies $|I|(P_m) = |I|(P_x)$ and $I(P_m) = I(P_x)$.

In either of the cases (F1), (F2), we can relabel P_x so that P_x is the bottom boundary plane of an open component of $\mathbb{R}^3 - \cup_{P' \in \mathcal{P}'} P'$. By Lemma 4.17, $\Delta(\mathcal{L}) \cap P_x = \mathcal{S} \cap P_x$.

After the translation by vector $-x$, we can assume $x = \vec{0}$ and $P_x = \{x_3 = 0\}$. By the local description of $\mathcal{L}$ in (D2-A), there is a nonflat leaf L of $\mathcal{L}$ which has the origin in its closure. Note that this implies that P_x is contained in the limit set of L. As by hypothesis $P_x \cap \Delta(\mathcal{L})$ is finite, we can choose a large round open disk $D \subset P_x$ such that $P_x \cap \Delta(\mathcal{L}) \subset D$. Let

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2, 0) \notin D\}.$$

Choose $\mu > 0$ small enough so that

$$(A \times [0, \mu]) \cap \mathcal{P}' = A \times \{0\}. \tag{15}$$

Claim 4.19. *The surface $L^\mu = \mathcal{L} \cap \{(x_1, x_2, x_3) \mid 0 < x_3 \leq \mu\}$ is connected. Furthermore, $(A \times (0, \varepsilon]) \cap \mathcal{L} = (A \times (0, \varepsilon]) \cap L$.*

Proof of the claim. The equality (15) together with a standard barrier argument (see the proof of Lemma 1.3 in [**35**]) imply that if the claim fails, then there exists a connected, nonflat, stable minimal surface Σ in $\{(x_1, x_2, x_3) \mid 0 < x_3 \leq \mu\}$, such that its boundary $\partial\Sigma$ is contained in $\mathcal{L} \cap \{x_3 = \mu\}$, Σ is proper in $\{(x_1, x_2, x_3) \mid 0 < x_3 \leq \mu\}$, Σ is not contained in $\{x_3 = \mu\}$ and Σ is complete outside the finite set $\mathcal{S} \cap \{x_3 = 0\}$ in the sense that any proper divergent arc α in Σ of finite length must have its divergent end point contained in $\mathcal{S} \cap \{x_3 = 0\}$. Since $[\Sigma \cap \{(x_1, x_2, x_3) \mid 0 \leq x_3 < \mu\}] \cup (P_x - \mathcal{S})$ is a minimal lamination of $\{(x_1, x_2, x_3) \mid -1 < x_3 < \mu\}$ outside of a finite set of points and this lamination consists of stable leaves, then item 1 of Corollary 7.1 in [**34**] implies that this lamination extends to a minimal lamination of $\{(x_1, x_2, x_3) \mid -1 < x_3 < \mu\}$, which contradicts the maximum principle for minimal surfaces at the origin. This finishes the proof of the claim. q.e.d.

We next continue the proof of Proposition 4.18. Clearly, we may also assume

$$\partial L^\mu \cap \{x_3 = \mu\} \neq \emptyset,$$

by taking $\mu > 0$ smaller. As L^μ is not flat and the injectivity radius function of the surfaces $\lambda_n M_n$ is bounded from below by $1/2$ away from their diverging boundaries, then we can apply the intrinsic version of the one-sided curvature estimates for minimal disks in [**11**] to $\lambda_n M_n$ to conclude that the norm of the second fundamental form of the possibly disconnected surfaces $(A \times [0, \mu]) \cap (\lambda_n M_n)$ is arbitrarily small if we take μ sufficiently small. Thus, by choosing $\mu > 0$ small enough, it follows that each component G of $(A \times [0, \mu]) \cap L$ is locally a graph over its projection to $A \times \{0\}$, with boundary ∂G contained in $((\partial A) \times [0, \mu]) \cup (A \times \{\mu\})$.

Note that so far in the proof of the proposition we have only used that $|I|(P_x) < \infty$. Next we will use the second hypothesis $I(P_x) = 0$ to obtain the desired contradiction, which will follow from the invariance of flux for $\nabla(x_3|_L)$.

Since $I(P_x) = 0$, then $P_x \cap \Delta(\mathcal{L}) = P_x \cap \mathcal{S}$ consists of an even positive number of points $p_1, q_1, p_2, q_2, \ldots, p_d, q_d$ with $n(p_i) = -n(q_i)$ for each $i = 1, \ldots, d$. Consider a collection $\{\delta_1, \ldots, \delta_d\}$ of pairwise disjoint embedded planar arcs in P_x such that the end points of δ_i are p_i, q_i. For $i \in \{1, \ldots, d\}$ fixed, construct a sequence $\{\gamma_i(m)\}_m$ of connection loops in L^μ, as indicated in the proof of Lemma 4.13 (note that now the δ_i are not necessarily straight line segments), i.e., each $\gamma_i(m)$ consists of two lifts of $\delta_i - [\mathbb{B}(p_i, \varepsilon_i(m)) \cup \mathbb{B}(q_i, \varepsilon_i(m))]$ to adjacent sheets of L^μ over δ_i joined by short arcs of length at most $3\varepsilon_i(m) > 0$ near p_i and q_i, such that the $\gamma_i(m)$ converge as $m \to \infty$ with multiplicity 2 to δ_i and $\varepsilon_i(m) \to 0$ as $m \to \infty$. It is possible to choose the indexing of these curves $\gamma_i(m)$ so that for every $m \in \mathbb{N}$, the collection $\Gamma_m = \{\gamma_1(m), \gamma_2(m), \ldots, \gamma_d(m)\}$ separates the connected surface L^μ into two components (this property holds because the portion of L^μ sufficiently close to P_x is topologically equivalent to the intersection of a periodic parking garage surface with a closed lower halfspace, hence, it suffices to choose all curves in the collection Γ_m corresponding to closed curves at the same level in the parking garage surface; for instance, in the particular case $d = 1$, the surface L^μ is modeled by a suitable portion of a Riemann minimal example, and each of the connection loops $\gamma_1(m)$ is a generator of the homology of the approximating Riemann minimal example).

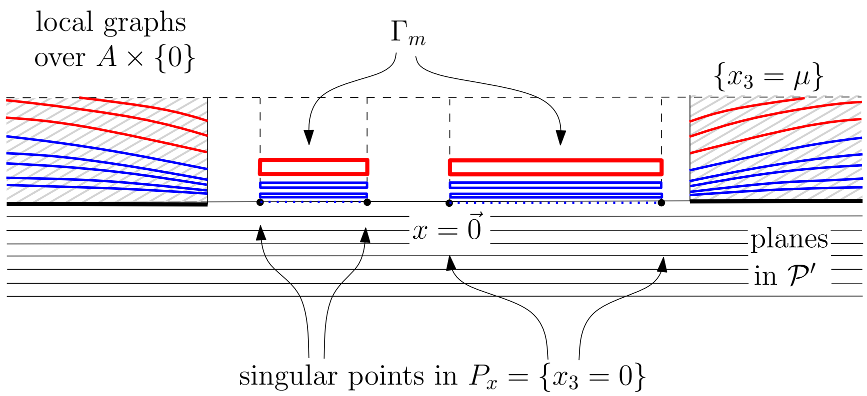

Figure 6. The collection of connection loops Γ_m disconnects L^μ into two components, one of which, denoted by L^μ_m, is proper in $x_3^{-1}([0,\mu])$ (we have sketched L^μ_m in red color). Note that the boundary of L^μ_m contains curves lying in $\{x_3 = \mu\}$, and that $\partial L^\mu_m \cap \{x_3 = \mu\} \subset \partial L^\mu_{m+1} \cap \{x_3 = \mu\}$ for all $m \in \mathbb{N}$.

Recall that $\partial L^\mu \cap \{x_3 = \mu\} \neq \emptyset$, and that L^μ is separated by Γ_m into two components; we will denote by L^μ_m the component of $L^\mu - \Gamma_m$ whose nonempty boundary contains $\partial L^\mu \cap \{x_3 = \mu\}$, see Figure 6. We remark that L^μ_m lies above any horizontal plane $P' \subset \{x_3 > 0\}$ which is strictly below ∂L^μ_m (since otherwise we would contradict that the portion of L below P' is connected by Claim 4.19, as $L - L^\mu_m$ intersects the open slab bounded by P_x and P'); in particular, equation (15) ensures that L^μ_m is properly embedded in $x_3^{-1}([0,\mu])$; this is in contrast with $L - L^\mu_m$, which is nonproper and limits to P_x.

As L^μ_m is a properly immersed minimal surface with boundary (in fact, embedded) in a halfspace, then Theorem 3.1 in Collin, Kusner, Meeks and Rosenberg [**14**] implies that L^μ_m is a parabolic surface, in the sense that the harmonic measure on its boundary is full. Now consider the scalar flux of a smooth tangent vector field X to L^μ_m across a finite collection of compact curves or arcs $\alpha \subset \partial L^\mu_m$, defined as

$$F(X, \alpha) = \int_\alpha \langle X, \eta \rangle,$$

where η is the exterior unit conormal vector to L^μ_m along α. Pick a compact arc $\sigma \subset \partial L^\mu_1 \cap \{x_3 = \mu\}$. Since $\sigma \subset \partial L^\mu_m \cap \{x_3 = \mu\}$ for all $m \in \mathbb{N}$, then we conclude that

$$F(\nabla x_3, \sigma) \leq F(\nabla x_3, \partial L^\mu_m \cap \{x_3 = \mu\}), \tag{16}$$

where the right-hand-side of (16) must be understood as the limit of $F(\nabla x_3, \alpha)$ where α runs along an increasing exhaustion of $\partial L^\mu_m \cap \{x_3 =$

$\mu\}$ by compact curves or arcs. Also note that the left-hand-side of (16) is a positive number by the maximum principle, which does not depend on m.

For each $m \in \mathbb{N}$ fixed, let $\{L^{\mu}_{m,k} \mid k \in \mathbb{N}, k \geq k_0(m)\}$ be a smooth increasing compact exhaustion of L^{μ}_m such that each $L^{\mu}_{m,k}$ contains all points of distance at most k from some previously chosen point in L^{μ}_m and such that $\sigma \cup \Gamma_m \subset \partial L^{\mu}_{m,k_0(m)}$ (hence, $\sigma \cup \Gamma_m \subset \partial L^{\mu}_{m,k}$ for all $k \geq k_0(m)$). The boundary of $L^{\mu}_{m,k}$ is the disjoint union of the following three pieces:

$$\partial L^{\mu}_{m,k} = \partial_{\mu}(m,k) \cup \Gamma_m \cup \partial_*(m,k),$$

where

$$\begin{aligned} \partial_{\mu}(m,k) &= \partial L^{\mu}_{m,k} \cap \{x_3 = \mu\}, \\ \partial_*(m,k) &= \partial L^{\mu}_{m,k} - [\partial_{\mu}(m,k) \cup \Gamma_m]. \end{aligned}$$

Consider the nonnegative harmonic function $u_m = \mu - x_3$ on the surface L^{μ}_m. Let $u_{m,k}$ be the harmonic function on $L^{\mu}_{m,k}$ defined by the boundary values $u_{m,k} = 0$ on $\partial_*(m,k)$, $u_{m,k} = u_m$ on $\partial L^{\mu}_{m,k} - \partial_*(m,k)$. Using that $u_{m,k} = 0$ along $\partial_*(m,k)$, $u_m = u_{m,k} = 0$ along $\partial_{\mu}(m,k)$ and the double Green's formula, we have

$$\begin{aligned} F(u_{m,k}\nabla u_m, \Gamma_m) &= F(u_{m,k}\nabla u_m, \partial L^{\mu}_{m,k}) = F(u_m \nabla u_{m,k}, \partial L^{\mu}_{m,k}) \\ &= F(u_m \nabla u_{m,k}, \Gamma_m) + F(u_m \nabla u_{m,k}, \partial_*(m,k)). \end{aligned} \tag{17}$$

Also note that

$$\begin{aligned} F(u_m \nabla u_{m,k}, \Gamma_m) &= F((u_m - \mu)\nabla u_{m,k}, \Gamma_m) + \mu F(\nabla u_{m,k}, \Gamma_m) \\ &\overset{(\star)}{=} F((u_m-\mu)\nabla u_{m,k}, \Gamma_m) - \mu F(\nabla u_{m,k}, \partial_{\mu}(m,k)) - \mu F(\nabla u_{m,k}, \partial_*(m,k)), \end{aligned} \tag{18}$$

where in $(\star)$ we have used the Divergence Theorem applied to $u_{m,k}$ in $L^{\mu}_{m,k}$. From (17) and (18) we deduce that

$$\begin{aligned} &F(u_{m,k}\nabla u_m, \Gamma_m) \\ &\quad = F((u_m - \mu)\nabla u_{m,k}, \Gamma_m) - \mu F(\nabla u_{m,k}, \partial_{\mu}(m,k)) \\ &\qquad + F((u_m - \mu)\nabla u_{m,k}, \partial_*(m,k)). \end{aligned} \tag{19}$$

Equation (19) leads to a contradiction, as the following properties hold:

(G1) The left-hand-side of (19) tends to zero as $m, k \to \infty$.
(G2) The first term in the right-hand-side of (19) tends to zero as $m, k \to \infty$.
(G3) The second term in the right-hand-side of (19) is at least $\frac{1}{2}\mu\, F(\nabla x_3, \sigma) > 0$ for k large.
(G4) The third term in the right-hand-side of (19) is nonnegative.

We next prove (G1)–(G4). As $u_m - \mu = -x_3 \leq 0$ and $\langle \nabla u_{m,k}, \eta \rangle \leq 0$ along $\partial_*(m,k)$ (this last inequality follows from the facts that $u_{m,k} \geq 0$ in $L^\mu_{m,k}$, $u_{m,k} = 0$ along $\partial_\mu(m,k)$ and η is exterior to L^μ_m along its boundary), then (G4) holds. Similarly, the fact that $\langle \nabla u_{m,k}, \eta \rangle \leq 0$ along $\partial_\mu(m,k)$ implies that $F(\nabla u_{m,k}, \partial_\mu(m,k)) \leq F(\nabla u_{m,k}, \sigma)$. As L^μ_m is a parabolic surface, then the functions $u_{m,k}$ converge as $k \to \infty$ uniformly on compact subsets of L^μ_m to the bounded harmonic function u_m (hence, their gradients converge as well). This implies that $F(\nabla u_{m,k}, \sigma)$ converges as $k \to \infty$ to $F(\nabla u_m, \sigma) = -F(\nabla x_3, \sigma)$, from where (G3) follows. Property (G2) also holds because

(G2.a) $\lim_{k\to\infty} |(\nabla u_{m,k})|_{\Gamma_m}| = |(\nabla u_m)|_{\Gamma_m}| \leq 1$,
(G2.b) Length(Γ_m) is bounded independently of m, and
(G2.c) $u_m - \mu = -x_3$ is arbitrarily small along Γ_m for m large.

Finally, (G1) holds because

$$F(u_{m,k} \nabla u_m, \Gamma_m) \overset{(k\to\infty)}{\longrightarrow} F(u_m \nabla u_m, \Gamma_m) \overset{(m\to\infty)}{\longrightarrow} -\mu \lim_{m\to\infty} F(\nabla x_3, \Gamma_m) = 0,$$

where in the last equality we have used that the tangent plane to L^μ becomes arbitrarily horizontal along Γ_m except along $2d$ subarcs of Γ_m whose total length goes to zero as $m \to \infty$. Now Proposition 4.18 is proved. q.e.d.

As announced above, we next show that if $\mathcal{S} = \emptyset$, then item 5 of Theorem 1.1 holds. In fact, we will obtain a more detailed description in the following result.

Proposition 4.20. *If $\mathcal{L}$ is a regular lamination of $\mathbb{R}^3$, then $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by parallel planes and item 5 of Theorem 1.1 holds. Furthermore:*

(A) $S(\mathcal{L})$ contains a line l_1 which passes through the closed ball of radius 1 centered at the origin, and another line l_2 at distance one from l_1, and all of the lines in $S(\mathcal{L})$ have distance at least one from each other.

(B) There exist oriented closed geodesics $\gamma_n \subset \lambda_n M_n$ with lengths converging to 2, which converge to the line segment γ that joins $(l_1 \cup l_2) \cap \{x_3 = 0\}$ and such that the integrals of the unit conormal vector of $\lambda_n M_n$ along γ_n in the induced exponential $\mathbb{R}^3$-coordinates of $\lambda_n B_N(p_n, \varepsilon_n)$ converge to a horizontal vector of length 2 orthogonal to γ.

Proof. Since we are assuming $\mathcal{S} = \emptyset$ but the uniformly bounded Gaussian curvature hypothesis in Proposition 4.6 fails to hold, then $\Delta(\mathcal{L}) = S(\mathcal{L}) \neq \emptyset$ and Lemma 4.17 implies that the Lipschitz curves in $S(\mathcal{L})$ go from $-\infty$ to $+\infty$ in height and, thus, $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes. By the $C^{1,1}$-regularity theorem for $S(\mathcal{L})$ in [**21**], $S(\mathcal{L})$ consists of vertical lines, precisely one passing through each point in $P_z \cap S(\mathcal{L})$ (here

z is any point in $S(\mathcal{L})$). The surfaces $\lambda_n M_n$ are now seen to converge on compact subsets of $\mathbb{R}^3 - S(\mathcal{L})$ to the minimal parking garage structure on $\mathbb{R}^3$ consisting of horizontal planes, with vertical columns over the points $y \in P_z \cap S(\mathcal{L})$ and with orientation numbers $n(y) = \pm 1$ in the sense of Definition 4.14.

We next show that item (A) of the proposition holds. Note that since the geodesic loops β_n given in Assertion 4.5 all pass through p_n, then the limit set $\mathrm{Lim}(\{\beta_n\}_n)$ of the β_n contains the origin in $\mathbb{R}^3$. Since the surfaces $\lambda_n M_n$ converge to the foliation of $\mathbb{R}^3$ by horizontal planes and this convergence is C^α (actually C^∞ tangentially to the leaves of the limit foliation) away from $S(\mathcal{L})$, for any $\alpha \in (0,1)$, then $\mathrm{Lim}(\{\beta_n\}_n)$ consists of a connected, simplicial complex consisting of finitely many horizontal segments joined by vertical segments contained in lines of $S(\mathcal{L})$; this complex could have dimension zero, in which case it reduces to the origin. Clearly there exists a point $q \in S(\mathcal{L}) \cap \mathrm{Lim}(\{\beta_n\}_n)$, since otherwise $\mathrm{Lim}(\{\beta_n\}_n)$ contains a horizontal segment l with $\vec{0} \in l$, and in this case one of the end points of l lie in $S(\mathcal{L})$. As $q \in S(\mathcal{L})$, then the vertical line l_q passing through q lies in $S(\mathcal{L})$.

We claim that there exists a vertical line l contained in $S(\mathcal{L})$ at distance one from l_q. As $S(\mathcal{L})$ is a closed set, Lemma 4.13 implies that if our claim fails to hold then there exists $\eta > 0$ such that the vertical cylinder of radius $1 + 2\eta$ with axis l_q only intersects $S(\mathcal{L})$ at l_q. Consider a sequence of points $q_n \in \beta_n$ limiting to q and consider the related extrinsic balls $B_{\lambda_n N}(q_n, 1+\eta)$. By the triangle inequality, the loop β_n is contained in $(\lambda_n M_n) \cap B_{\lambda_n N}(q_n, 1+\eta)$ (as the length of β_n is two and the intrinsic distance between any two points in β_n is at most one). By the parking garage structure of the limit foliation, for n large each of the surfaces $(\lambda_n M_n) \cap B_{\lambda_n N}(q_n, 1+\eta)$ contains a unique main component Δ_n which is topologically a disk (this is the component that contains q_n). Therefore, $\beta_n \subset \Delta_n$, which implies that β_n is homotopically trivial. This contradiction proves our claim. Note that these arguments also imply that q lies in the closed ball of radius 1 centered at the origin. This claim together with Lemma 4.13 imply that item (A) of the proposition holds. Also observe that we have completed the proof of the first statement of Theorem 1.1 (the $\lambda_n M_n$ converge to a minimal parking garage structure of $\mathbb{R}^3$ with at least two vertical lines in $S(\mathcal{L})$).

Since l_q and l are at distance 1 apart, then there exist connection loops γ_n on $\lambda_n M_n$ of lengths converging to 2 which converge as $n \to \infty$ to a horizontal line segment of length 1 joining l_q and l. These connection loops satisfy the properties in item (B) of the proposition (see [**29**] for details).

To complete the proof of Proposition 4.20, it only remains to demonstrate the last sentence of item 5 of Theorem 1.1 assuming that there exists a bound on the genus of the surfaces $\lambda_n M_n$. This follows from

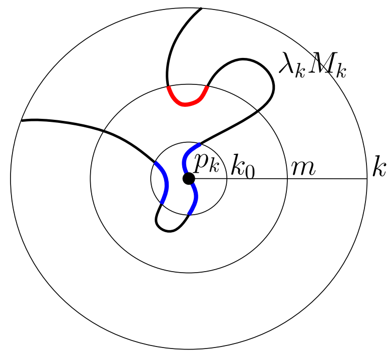

Figure 7. The red portion of $\lambda_k M_k$ cannot enter into the ball of radius k_0, since it does not join to the blue portions within the ball of radius m.

similar arguments as those in the proof of Lemma 3.3, since the surfaces $\lambda_n M_n$ approximate arbitrarily well the behavior of a periodic parking garage surface in $\mathbb{R}^3$. This finishes the proof of Proposition 4.20. q.e.d.

By Proposition 4.20, to finish the proof of Theorem 1.1 it remains to demonstrate that item 6 holds provided that $\mathcal{S} \neq \emptyset$, a hypothesis that will be assumed for the remainder of this section. We will start by stating a property to be used later. Recall that given $R > 0$ and $n \in \mathbb{N}$ sufficiently large, $\Sigma(n, R)$ denotes the closure of the component of $[\lambda_n \overline{B}_M(p_n, \frac{\sqrt{n}}{2\lambda_n})] \cap B_{\lambda_n N}(p_n, R)$ that contains p_n, and that the surface M_k that appears below was defined in equation (12) as a rescaling by $1/\lambda_k = 1/\lambda'_{n(k)}$ of $\Sigma(n(k), k)$, where $n(k)$ was defined in the proof of Lemma 4.4. The purpose of the next result is to show that given a radius k_0, all the components of $\lambda_k M_k$ in an extrinsic ball of that radius centered at the origin (for k sufficiently large depending on k_0) can be joined within an extrinsic ball of a larger radius $m(k_0)$ independent of k, see Figure 7.

Proposition 4.21. *Given $k_0 \in \mathbb{N}$ there exists $m = m(k_0) \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ sufficiently large, we have*

$$(\lambda_k M_k) \cap B_{\lambda_k N}(p_k, k_0) \subset \Sigma(n(k), m). \tag{20}$$

Furthermore, the intrinsic distance in $\lambda_k M_k = \Sigma(n(k), k)$ from any point in $(\lambda_k M_k) \cap B_{\lambda_k N}(p_k, k_0)$ to p_k is not greater than some number independent on k, for all such k.

Proof. To prove (20) we argue by contradiction. Suppose that for some $k_0 \in \mathbb{N}$, there exist sequences $\{m_k\}_k, \{a_k\}_k \subset \mathbb{N}$ so that $m_k \nearrow \infty$, $m_k < a_k$ and for all k,

$$(\lambda_{a_k} M_{a_k}) \cap B_{\lambda_{a_k} N}(p_{a_k}, k_0) \not\subset \Sigma(n(a_k), m_k). \tag{21}$$

Let $\Omega(k)$ be a component of $B_{\lambda_{a_k}N}(p_{a_k}, m_k) - (\lambda_{a_k} M_{a_k})$ that contains $\Sigma(n(a_k), m_k)$ in its boundary $\partial\Omega(k)$, and so that $\partial\Omega(k)$ contains another component $\Delta(k)$ different from $\Sigma(n(a_k), m_k)$, with $\Delta(k) \cap B_{\lambda_{a_k}N}(p_{a_k}, k_0) \neq \emptyset$, which can be done by (21). Observe that the boundary of $\Omega(k)$ is a good barrier for solving Plateau problems in $\Omega(k)$ (here we are using that since $\varepsilon_k \to 0$, then the extrinsic balls $B_N(p_k, \varepsilon_k)$ have mean convex boundaries). Let $\Sigma'(k)$ be a surface of least area in $\Omega(k)$ homologous to $\Sigma(n(a_k), m_k)$, and with $\partial\Sigma'(k) = \partial\Sigma(n(a_k), m_k)$. As for all k the surface $\Sigma'(k)$ intersects $B_{\lambda_{a_k}N}(p_{a_k}, k_0)$, then by uniform curvature estimates for stable minimal surfaces away from their boundaries we conclude that after passing to a subsequence, the $\Sigma'(k)$ converge as $k \to \infty$ to a nonempty (regular) minimal lamination $\mathcal{L}'$ of $\mathbb{R}^3$ all whose leaves are complete, embedded, stable minimal surfaces, and, therefore, all leaves of $\mathcal{L}'$ are planes. To find the desired contradiction, note that the following properties hold.

(H1) The sequence $\{\Sigma'(k)\}_k$ is locally simply connected in $\mathbb{R}^3$ (by uniform curvature estimates for stable minimal surfaces and by the uniform graph lemma, see Colding and Minicozzi [**6**] or Pérez and Ros [**42**]), as well as the sequence $\{\lambda_{a_k} M_{a_k}\}_k$.

(H2) For k large, the surface $\lambda_{a_k} M_{a_k}$ is unstable, and, thus, $\Sigma'(k)$ and $\lambda_{a_k} M_{a_k}$ only intersect along $\partial\Sigma'(k) \subset \partial B_{\lambda_{a_k}N}(p_{a_k}, m_k)$, which diverges to ∞ as $k \to \infty$.

By Property (H2), we deduce that the planes in $\mathcal{L}'$ are either disjoint from $\mathcal{L}$ or they are leaves of $\mathcal{L}$. In fact, Lemma 4.12 implies that the first possibility cannot occur. Also note that as $\Sigma'(k)$ intersects $B_{\lambda_{a_k}N}(p_{a_k}, k_0)$ for all k, then $\mathcal{L}'$ contains a plane Π that intersects the ball $\mathbb{B}(k_0)$. Properties (H1), (H2) insure that we can apply Theorem 7 in [**36**] (or the one-sided curvature estimates by Colding and Minicozzi [**11**]) to obtain uniform local curvature estimates for the surfaces $\lambda_{a_k} M_{a_k}$ in a fixed size neighborhood of Π. As a consequence, Π cannot lie in the collection $\mathcal{P}'$ defined in Lemma 4.11 and, thus, Lemma 4.12 gives a contradiction (hence, (20) is proved).

As for the last sentence in the statement of Proposition 4.21, take $r_{k_0} \in (0, \infty)$ such that $m(k_0) = r_{k_0}\,\delta(r_{k_0})$, where $\delta(r)$ is the function that appears in the first sentence of Theorem 1.1. Applying Proposition 4.2 to $R = R_1 = r_{k_0}$ (recall that equation (7) holds after replacing $\widetilde{\delta}(r)$ by $\delta(r)$, see the paragraph just before Lemma 4.4), for k sufficiently large we have $\Sigma(k, m(k_0)) \subset B_{\widetilde{M}(k)}(p_k, r_{k_0}/2) = B_{\lambda_k M_k}(p_k, r_{k_0}/2)$, from where the last statement of the proposition follows. q.e.d.

Remark 4.22. Proposition 4.21 still holds if the limit object of the surfaces $\lambda_n M_n$ is either a nonsimply connected, properly embedded minimal surface (case 4 of Theorem 1.1) or a minimal parking garage structure (case 5 of the theorem). To see why this generalization holds, note that the arguments in the proof of the last proposition still produce a plane $\Pi \subset \mathbb{R}^3$ which is either disjoint from the limit set of the sequence $\lambda_n M_n$, or it is contained in this limit set. If case 4 of Theorem 1.1 occurs for this sequence, then the halfspace theorem gives a contradiction. If the $\lambda_n M_n$ converge to a minimal parking garage structure $\mathcal{L}$ of $\mathbb{R}^3$, then it is clear that no plane in the complement of $\mathcal{L}$ can exist.

Also note that the constant $m(k_0)$ in Proposition 4.21 can be chosen so that it does not depend on the homogeneously regular manifold or on the surface M to which we apply it.

We next continue with the proof of item 6 of Theorem 1.1, provided that $\mathcal{S} \neq \emptyset$. Since the closures of the set of flat leaves is $\mathcal{P}'$ and $\mathcal{P}'$ is a lamination of $\mathbb{R}^3$ with no singularities, then there exists at least one leaf of $\mathcal{L}$ which is not flat, so the first statement of item 6 holds. By Lemmas 4.11 and 4.12, the sublamination $\mathcal{P}$ of flat leaves of $\mathcal{L}$ is nonempty, the closure of every such planar leaf L_1 is a horizontal plane in the family $\mathcal{P}'$ defined in Lemma 4.11, and, hence, by definition of $\mathcal{P}'$ we have that $\overline{L_1}$ intersects $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$. By Lemma 4.13, the distance between any two points in $\overline{L_1} \cap \Delta(\mathcal{L})$ is at least 1. By Lemma 4.17, $\overline{L_1} \cap \Delta(\mathcal{L})$ is either contained in $\mathcal{S}$ or in $S(\mathcal{L})$. So in order to conclude the proof of item 6 of Theorem 1.1 it remains to show the following property.

Proposition 4.23. *Given of leaf L_1 of $\mathcal{P}$, the plane $\overline{L_1}$ intersects $\Delta(\mathcal{L})$ in at least two points.*

We next prove the proposition by contradiction through a series of lemmas. Suppose that $L_1 \in \mathcal{P}$ satisfies that $\overline{L_1} \cap \Delta(\mathcal{L})$ consists of a single point $x \in \Delta(\mathcal{L})$. If $x \in S(\mathcal{L})$, then Lemma 4.17 implies that $\mathcal{L}$ restricts to a foliation of some ε-neighborhood of $\overline{L_1}$. Consider the largest open horizontal slab or halfspace W containing $\overline{L_1}$ so that $\mathcal{L}$ restricts to W as a foliation by planes. As $\mathcal{S} \neq \emptyset$, then $W \neq \mathbb{R}^3$. By Lemma 4.15, we can replace L_1 by a flat leaf in the boundary of W and after this replacement, we have $|I|(\overline{L_1}) = 1$ and $\overline{L_1} \cap \Delta(\mathcal{L}) = \overline{L_1} \cap \mathcal{S}$. Without loss of generality, we may assume that $\overline{L_1}$ is the top boundary plane of W. Arguing as in the discussion of cases (F1) and (F2) in the proof of Proposition 4.18, we can replace $\overline{L_1}$ by a bottom boundary plane P of an open component C of $\mathbb{R}^3 - \cup_{P' \in \mathcal{P}'} P'$ so that the distance from $\overline{L_1}$ to P is less than the number $\mu_0 = \mu_0(\overline{L_1})$ given in Lemma 4.15 (note that P may be equal to $\overline{L_1}$).

Denote by L the nonflat leaf of $\mathcal{L}$ directly above P. After the translation by $-x$, we can assume that $P = \{x_3 = 0\}$ and $x = \vec{0} \in \mathcal{S}$. We next analyze several aspects of the geometry of L in a neighborhood of P in C.

Given a regular value $\mu \in (0, \varepsilon)$ for x_3 restricted to L (this number $\varepsilon > 0$ appears in description (D1)–(D2) above), let

$$L^\mu := L \cap \{(x_1, x_2, x_3) \mid 0 < x_3 \le \mu\},$$

which is a connected surface with boundary for μ sufficiently small by Claim 4.19.

Definition 4.24. Take a sequence of points $q_k \in L^\mu$ converging to $\vec{0}$ and numbers $r_k \in (0, |q_k|/2]$. For each $k \in \mathbb{N}$, consider the function $f_k\colon L^\mu \cap \mathbb{B}(q_k, r_k) \to \mathbb{R}$ given by

$$f_k(x) = \sqrt{|K_L|}(x) \cdot \operatorname{dist}_{\mathbb{R}^3}(x, \partial[L \cap \mathbb{B}(q_k, r_k)]). \tag{22}$$

Let $x_k \in \mathbb{B}(q_k, r_k)$ be a maximum of f_k (note that f_k is continuous and vanishes at $\partial[L \cap B(q_k, r_k)]$). The sequence $\{x_k\}_k$ is called a *blow-up sequence on the scale of curvature* if $f_k(x_k) \to \infty$ as $k \to \infty$.

Lemma 4.25. *Suppose that $\{x_k\}_k \subset L^\mu$ is a blow-up sequence. Then, after passing to a subsequence, the surfaces $L(k) = \sqrt{|K_L|}(x_k)(L^\mu - x_k)$ converge with multiplicity 1 to a vertical helicoid $\mathcal{H} \subset \mathbb{R}^3$ whose axis is the x_3-axis and whose Gaussian curvature is -1 along this axis.*

Proof. First observe that since the surfaces M_n have injectivity radius function larger than or equal to $1/2$, then the ball

$$\mathbb{B}_k = \mathbb{B}\left(x_k, \frac{1}{2}\operatorname{dist}_{\mathbb{R}^3}(x_k, \partial\mathbb{B}(q_k, r_k))\right)$$

intersects L^μ in disks when k is sufficiently large. Also note that the ball $\sqrt{|K_L|}(x_k)(\mathbb{B}_k - x_k)$ is centered at the origin and its radius is $f_k(x_k)/2$, which tends to ∞ as $k \to \infty$ because $\{x_k\}_n$ is a blow-up sequence. Since the second fundamental form of the surfaces $L(k) \cap \mathbb{B}\left(f_k(x_k)/2\right)$ is uniformly bounded, then a subsequence of the $L(k)$ (denoted in the same way) converges to a minimal lamination $\mathcal{L}'$ of $\mathbb{R}^3$ with a leaf L' that is a complete embedded minimal surface that passes through the origin with absolute Gaussian curvature 1 at that point. Standard arguments then show that the multiplicity of the convergence of portions of the $L(k)$ to L' is one. Therefore, a lifting argument of loops on L' implies that L' is simply connected, hence, L' is a helicoid with maximal absolute Gaussian curvature 1 at $\vec{0}$ and L' is the only leaf of $\mathcal{L}'$. The fact that L' is a vertical helicoid with axis the x_3-axis (so $L' = \mathcal{H}$) will follow from the description of the local geometry of L^μ nearby x_n; to see this, note that the blow-up points x_k and the forming helicoids in L^μ on the scale of curvature near x_k for k large imply the existence of pairs of highly sheeted almost-flat multivalued graphs $G^1_{n,k}, G^2_{n,k} \subset M_n$ extrinsically close to x_k for n sufficiently large (recall that portions of the M_n converge to L^μ). These multivalued graphs can be chosen to have any fixed small gradient over the plane perpendicular to the axis of the helicoid L'. For n, k sufficiently large, these multivalued graphs in M_n each contains a two-valued subgraph that extends to an almost-flat two-valued graph on a fixed scale (proportional to the number $\varepsilon > 0$ that appears in description (D1)–(D2) above) and collapse to a punctured disk. Since the punctured (x_1, x_2)-plane $P - \{\vec{0}\}$ is a leaf of the limit minimal lamination $\mathcal{L}$, it then follows that the helicoid L' must be vertical. This completes the proof of the lemma. q.e.d.

The next lemma gives that the same type of limit that appears in Lemma 4.25 at a blow-up sequence on the scale of curvature in L^μ, also appears when using a different notion of blow-up. Namely, when we rescale L^μ around points with heights converging to zero where L^μ is vertical.

Lemma 4.26. *Consider a sequence of points $y_k \in L^\mu$ with $x_3(y_k)$ converging to zero where the tangent planes $T_{y_k}L$ to L^μ are vertical. Then, y_k converge to $\vec{0}$, the numbers $s_k := \sqrt{|K_L|}(y_k)$ diverge to infinity and a subsequence of the surfaces $L'(k) = s_k(L^\mu - y_k)$ converges on compact subsets of $\mathbb{R}^3$ to a vertical helicoid $\mathcal{H}$ containing the x_3-axis and with maximal absolute Gaussian curvature 1 at the origin. Furthermore, the multiplicity of the convergence of the surfaces $L'(k)$ to $\mathcal{H}$ is one.*

Proof. We first show that the points y_k tend to the origin as $k \to \infty$. Arguing by contradiction, suppose after choosing a subsequence that for k large y_k lies outside a ball $\mathbb{B}$ centered at $\vec{0}$. Note that for k large, the injectivity radius function of L is bounded away from zero at the y_k. As these points are arbitrarily close to P, then the Gaussian curvature of L at the y_k blows up (otherwise L could be written locally as graphs over vertical disks in $T_{y_k}L$ of uniform size by the uniform graph lemma, which would contradict that L lies above $\{x_3 = 0\}$) and one obtains a contradiction to the one-sided curvature estimates of Colding–Minicozzi (Corollary 0.4 in [**11**]). Therefore, $y_k \to \vec{0}$. Another consequence of the one-sided curvature estimates is that

(J) There exists $\delta > 0$ such that if $\mu \in (0, \delta)$, then the tangent plane to L^μ at every point in $L^\mu \cap \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \geq \delta^2 x_3^2\}$ makes an angle less than $\pi/4$ with the horizontal.

For $k \in \mathbb{N}$ fixed, let $t_k > 0$ be the largest radius such that all points in $L^\mu \cap \mathbb{B}(y_k, t_k)$ have tangent plane making an angle less than $\pi/4$ with $T_{y_k}L$; existence of t_k follows from the fact that L^μ is proper in the slab $\{0 < x_3 \leq \mu\}$. Note that the following properties hold.

(K1) $L^\mu \cap \mathbb{B}(y_k, t_k) \subset \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 < \delta^2 x_3^2\}$ (this follows from (J)),

(K2) $t_k \leq x_3(y_k)$ (otherwise, $\mathbb{B}(y_k, t_k) \cap P$ contains a disk $D \subset P - \{\vec{0}\}$ which is the limit of a sequence of graphs inside $L^\mu \cap \mathbb{B}(y_k, t_k)$ over D; this is clearly impossible by (K1)).

By Property (K2), we have $t_k \to 0$ as $k \to \infty$. Consider the sequence of translated and scaled surfaces

$$L(k) = \frac{2}{t_k}(L^\mu - y_k). \tag{23}$$

We claim that to prove the lemma it suffices to demonstrate that

(N) *Every subsequence of the $L(k)$ has a subsequence that converges with multiplicity one to a vertical helicoid containing the x_3-axis.*

We will prove the lemma, assuming that Property (N) holds. Let $\{L(k_i)\}_i$ be a subsequence of the $L(k)$ that converges with multiplicity one to a vertical helicoid $\mathcal{H}'$ containing the x_3-axis. Then, $\sqrt{|K_{\mathcal{H}'}|}(\vec{0}) \in (0,\infty)$ and

$$\lim_{i\to\infty} \sqrt{|K_{L(k_i)}|}(\vec{0}) = \lim_{i\to\infty} \frac{t_{k_i}}{2}\sqrt{|K_L|}(y_{k_i}).$$

Since $\lim_{i\to\infty} t_{k_i} = 0$, this implies that the numbers $s_{k_i} := \sqrt{|K_L|}(y_{k_i})$ diverge to infinity, and the sequence of surfaces

$$L'(k_i) = s_{k_i}(L^\mu - y_{k_i}) = s_{k_i}\frac{t_{k_i}}{2}L(k_i)$$

converges with multiplicity one to $\mathcal{H} = \sqrt{|K_{\mathcal{H}'}|}(\vec{0})\mathcal{H}'$, and the proposition follows. Thus, it suffices to prove Property (N); there are two cases to consider after choosing a subsequence.

Case (N.1). The sequence $\{L(k)\}_k$ has uniform local bounds of the Gaussian curvature in $\mathbb{R}^3$.

In this case, standard arguments show that, after choosing a subsequence, the $L(k)$ converge to a minimal lamination $\mathcal{L}_\infty$ of $\mathbb{R}^3$. Observe that if L_1 is a nonflat leaf of $\mathcal{L}_\infty$, then the multiplicity of convergence of portions of the $L(k)$ to L_1 is one; in particular, L_1 is simply connected (since the injectivity radius function of the $L(k)$ becomes arbitrarily large at points in any fixed compact set of $\mathbb{R}^3$ as $k \to \infty$). On the other hand, if the multiplicity of convergence of portions of the $L(k)$ to a leaf L_2 of $\mathcal{L}_\infty$ is greater than one, then L_2 is stable, hence, a plane. By the classification of simply connected, complete embedded minimal surfaces in $\mathbb{R}^3$ (Meeks and Rosenberg [**35**] and Colding and Minicozzi [**12**]), we conclude that every leaf of $\mathcal{L}_\infty$ is either a plane or a helicoid. Clearly, if $\mathcal{L}_\infty$ contains a leaf which is a helicoid, then this is the only leaf of $\mathcal{L}_\infty$ and Property (N) is proved in this case. Since the leaf of $\mathcal{L}'$ passing through $\vec{0}$ has a vertical tangent plane at $\vec{0}$ but at some point in the sphere $\partial\mathbb{B}(2)$ there exists a point on a leaf of $\mathcal{L}_\infty$ whose tangent plane makes an angle at least $\pi/4$ (by definition of t_k), then $\mathcal{L}_\infty$ contains a leaf which is not a plane. This finishes Case (N.1).

Case (N.2). There exists $x_\infty \in \mathbb{R}^3$ and points $x_k \in L(k)$ converging to x_∞ such that $|K_{L(k)}|(x_k) \geq k$ for all $k \in \mathbb{N}$.

We will show that this case cannot occur, by dividing it into two subcases after replacing by a subsequence.

(N2.1): Suppose that $\frac{x_3(y_k)}{t_k} \to \infty$.

(N2.2): Suppose that $\frac{x_3(y_k)}{t_k}$ converges to a number D which is greater than or equal to 1 (by Property (K2) above).

In case (N2.1) holds, we consider the sequence of compact embedded minimal surfaces $\{L(k) \cap \mathbb{B}(R_k)\}_k$, where $R_k = \frac{x_3(y_k)}{t_k}$. For k large, every component of $L(k) \cap \mathbb{B}(R_k)$ is a disk with boundary contained in $\partial\mathbb{B}(R_k)$. As the supremum of the norm of the second fundamental form of $L(k) \cap \mathbb{B}(2|x_\infty|)$ tends to ∞ as $k \to \infty$ (by assumption in this case (N.2)), then Theorem 0.1 in Colding and Minicozzi [**11**] and Meeks' regularity theorem [**21**] assure that after choosing a subsequence, the $L(k)$ converge as $k \to \infty$ to a limit parking garage structure with one column. Observe that by equation (23), points of $L(k)\cap\mathbb{B}(1)$ correspond to points of $L^\mu \cap \mathbb{B}(y_k, \frac{t_k}{2})$, and, thus, the tangent plane to $L(k)$ at every point in $L(k) \cap \mathbb{B}(1)$ makes an angle less than $\pi/4$ with $T_{y_k}L$. This property implies that the inner product of the Gauss map of $L(k)$ with the unit normal vector to $T_{y_k}L$ is positive (up to sign) in $L(k) \cap \mathbb{B}(1)$, hence, $L(k) \cap \mathbb{B}(1)$ is stable. Schoen's curvature estimates [**45**] now give that the norm of the second fundamental form of $L(k) \cap \mathbb{B}(1)$ is uniformly bounded. Since the tangent plane Π to $L(k)$ at the origin is vertical, then we conclude that the planes in the limit parking garage structure are parallel to Π. As for the line l given by the singular set of convergence of the $L(k) \cap \mathbb{B}(R_k)$ to the limit parking garage structure, note that by definition of t_k, for large k the tangent plane to $L(k)$ at some point q_k in the sphere $\partial\mathbb{B}(2)$ makes an angle at least $\pi/4$ with Π; this implies that l is the straight line orthogonal to Π that passes through the limit of the q_k (after passing to a subsequence). By similar arguments as those at the end of the proof of Lemma 4.25, we can find pairs of highly sheeted almost-flat almost-vertical multivalued graphs $G^1_{n,k}, G^2_{n,k} \subset M_n$ over portions of Π, and these multivalued graphs in M_n contain two-valued subgraphs that extend to two-valued almost-vertical multivalued graphs on a fixed scale. By the arguments at the end of the proof of Lemma 4.25, these extended two-valued almost-vertical multivalued graphs must be almost-horizontal, which gives a contradiction. This finishes the case (N2.1).

Finally, suppose case (N2.2) occurs. By the application of a diagonal-type argument to the doubly indexed sequence of surfaces $\{M_n - y_k\}_{n,k\in\mathbb{N}}$ where n is chosen to go to infinity sufficiently quickly in terms of k that also goes to infinity, we can produce a sequence

$$\Lambda_{n(k)} = \left\{\frac{2}{t_k}(M_{n(k)} - y_k)\right\}_{k\in\mathbb{N}},$$

such that the following properties hold.

(O1) The injectivity radius function of $\frac{2}{t_k}(M_{n(k)} - y_k)$ can be made arbitrarily large for k large at every point a any fixed ball in $\mathbb{R}^3$ (this follows from Property (Inj) just before the statement of Proposition 4.2 after rescaling by $2/t_k$).

(O2) There exists a (possibly empty) closed set $\mathcal{S}_\infty \subset \mathbb{R}^3$ and a minimal lamination $\mathcal{L}_\infty$ of $\mathbb{R}^3 - \mathcal{S}_\infty$ such that the surfaces $\frac{2}{t_k}(M_{n(k)} - y_k) - \mathcal{S}_\infty$ converge to $\mathcal{L}_\infty$ outside of some singular set of convergence $S(\mathcal{L}_\infty) \subset \mathcal{L}_\infty$, and if we call $\Delta(\mathcal{L}_\infty) = \mathcal{S}_\infty \cup S(\mathcal{L}_\infty)$, then $\Delta(\mathcal{L}_\infty) \neq \emptyset$ (this property holds by similar arguments as those that prove the first part of item 1.1 of Theorem 1.1, which are still valid since we have property (O1)).

(O3) Through every point in $\Delta(\mathcal{L}_\infty)$ there passes a plane which contains a planar leaf of $\mathcal{L}_\infty$ and which intersects $\Delta(\mathcal{L}_\infty)$ in exactly one point (two or more points would produce connection loops in the surfaces $\frac{2}{t_k}(M_{n(k)} - y_k)$ for k large, in contradiction with property (O1) above).

(O4) By our hypotheses in Case (N2.2), we deduce that one of the leaves of $\mathcal{L}_\infty$ is contained in the plane $\{x_3 = -2D\}$. In particular, the planes mentioned in property (O3) are horizontal.

(O5) $\mathcal{L}_\infty$ contains a sublamination $\widehat{\mathcal{L}}_\infty$ which is a limit as $k \to \infty$ of the surfaces $L(k)$.

By property (O1), it follows from our previous arguments in this paper that every nonflat leaf Z of $\mathcal{L}_\infty$ is simply connected. Furthermore, the injectivity radius function of such a Z at any point $z \in Z$ is equal to the intrinsic distance from z to boundary of the metric completion $\overline{Z}$ of Z, where the points of this metric completion correspond to certain (singular) points in $\mathcal{S}_\infty$. Observe that such a Z cannot be complete (otherwise, by the discussion in Case (N.1), Z would be a helicoid which is impossible by (O3) or (O4)).

We next show that there exists a nonflat Z_1 of $\mathcal{L}_\infty$ that passes through the origin. Since the norms of the second fundamental forms of the surfaces $\frac{2}{t_k}(M_{n(k)} - y_k)$ are uniformly bounded in the ball of radius 1 centered at the origin, there is a leaf Z_1 of $\mathcal{L}_\infty$ passing through $\vec{0}$ with vertical tangent plane $T_{\vec{0}}Z_1$. Since the flat leaves of $\mathcal{L}$ are horizontal, then Z_1 cannot be flat. By the last paragraph, Z_1 is not complete, hence, there exists $p_0 \in \mathcal{S}_\infty$ in the metric completion of Z_1. By Property (O3), the punctured horizontal plane $\Pi_0 = \{x_3 = x_3(p_0)\} - \{p_0\}$ is a leaf of $\mathcal{L}_\infty$. By the same arguments and the connectedness of Z_1, there is at most one other point p_1 in the metric completion of Z_1, and in this case the plane $\Pi_1 = \{x_3 = x_3(p_1)\} - \{p_1\}$ is a leaf of $\mathcal{L}_\infty$ (if no such p_1 exists, then Z_1 is properly embedded in the upper open halfspace determined by the plane Π_0). Without loss of generality, we can assume that if Π_1 exists, then $x_3(p_1) > x_3(p_0)$.

Given $\delta > 0$, consider the conical region

$$C^+(\delta) = \{(x_1, x_2, x_3) \mid (x_1 - x_1(p_0))^2 + (x_2 - x_2(p_0))^2 < \delta^2 (x_3 - x_3(p_0))^2\},$$

with vertex p_0. Suppose that no p_1 exists. In this case, the injectivity radius function $\mathrm{Inj}_{Z_1}(x)$ at any point $x \in Z_1$ is equal to the intrinsic distance function in Z_1 from x to p_0, which in turn is at least $|x - p_0|$. Therefore, Inj_{Z_1} grows at least linearly with the extrinsic distance in Z_1 to p_0. If p_1 exists, the same property can be proven for $\delta > 0$ sufficiently small with minor modifications, since for such δ, there exists $a = a(\delta) > 0$ such that $C^+(\delta)$ also contains p_1 (if p_1 exists) and

$$\min\{|x - p_0|, |x - p_1|\} \geq a\ |x - p_0|, \tag{24}$$

for all $x \in x_3^{-1}([x_3(p_0), x_3(p_1)]) - C^+(\delta)$. This scale invariant lower bound on Inj_{Z_1} together with the intrinsic version of the one-sided curvature estimate by Colding–Minicozzi (Corollary 0.8 in [**12**]) imply that for $\delta > 0$ sufficiently small, the intersection of Z_1 with $x_3^{-1}([x_3(p_0), x_3(p_1)]) - C^+(\delta)$ consists of two multivalued graphs whose gradient can be made arbitrarily small (in terms of δ). The same scale invariant lower bound on Inj_{Z_1} is sufficient to apply the arguments in page 45 of Colding–Minicozzi [**13**]; especially see the implication that property (D) there implies property (D1). In our current setting, property (D) is the scale invariant lower bound on Inj_{Z_1}, and property (D1) asserts that $Z_1 - C^+(\delta_1)$ consists of a pair of ∞-valued graphs for some $\delta_1 > 0$ small, which can be connected by curves of uniformly bounded length arbitrarily close to p_0. The existence of such ∞-valued graphs over the punctured plane contradicts Corollary 1.2 in [**7**]. This contradiction rules out Case (N2.2), which finishes the proof of Lemma 4.26. q.e.d.

The following corollary is an immediate consequence of Lemma 4.26.

Corollary 4.27. *Given $\varepsilon_1, R > 0$, there exists an $\varepsilon_2 \in (0, \varepsilon_1)$ such that the following holds. Let*

$$\gamma = \{y \in L^\mu \cap \mathbb{B}(\varepsilon_2) \mid T_y L \text{ is vertical }\}.$$

Then for any $y \in \gamma$, there exists a vertical helicoid $\mathcal{H}_y$ with maximal absolute Gaussian curvature 1 at the origin[8] *such that the connected component of $\sqrt{|K_L|}(y)[L^\mu - y] \cap \mathbb{B}(R)$ containing the origin is a normal graph u over its projection $\Omega \subset \mathcal{H}_y$, and*

$$\mathbb{B}(R - 2\varepsilon_1) \cap \mathcal{H}_y \subset \Omega \subset \mathbb{B}(R + 2\varepsilon_1) \cap \mathcal{H}_y, \qquad \|u\|_{C^2(\Omega)} \leq \varepsilon_1.$$

Three immediate consequences of Corollary 4.27 when ε_2 is chosen sufficiently small are:

[8]Observe that for $y_1, y_2 \in \gamma$, the helicoids $\mathcal{H}_{y_1}$, $\mathcal{H}_{y_2}$ coincide up to a rotation around the x_3-axis.

(P1) The set γ in Corollary 4.27 can be parameterized as a connected analytic curve $\gamma(t)$ where $t \in (0, t_0]$ is its positive x_3-coordinate, and $\lim_{t\to 0} \gamma'(t) = (0, 0, 1)$.

(P2) Given $t \in (0, t_0]$, let $f(t) \in \mathbb{R}$ be the angle that the vertical plane $T_{\gamma(t)}L$ makes with the positive x_1-axis, that is, $(-\sin f(t), \cos f(t), 0)$ is the unit normal vector to $T_{\gamma(t)}L$ up to a choice of orientation. Note that $T_{\gamma(t)}L$ rotates infinitely often as $t \searrow 0^+$, and, consequently, the angle function $f(t)$ can be considered to be a smooth function of the height that tends to $+\infty$ as $t \searrow 0^+$ if the forming helicoid $\mathcal{H}_{\gamma(t)}$ is left-handed, or to $-\infty$ if $\mathcal{H}_{\gamma(t)}$ is right-handed. In the sequel we will suppose that this last possibility occurs (after a possible reflection of L^μ in the (x_1, x_3)-plane). Also observe that $f(t)$ is determined up to an additive multiple of 2π, and so, we do not loose generality by assuming that $f(t) < 0$ for each $t \in (0, t_0]$. Since for the right-handed vertical helicoid $\mathcal{H}$ the corresponding angle function $f_{\mathcal{H}}$ is negative linear, then we conclude that after choosing $t_0 > 0$ small enough, $f' > 0$ is bounded away from zero and f''/f' is bounded from above in $(0, t_0]$.

(P3) For any $t \in (0, t_0]$, $L^\mu \cap T_{\gamma(t)}L$ contains a small smooth arc $\alpha_{\gamma(t)}$ passing through $\gamma(t)$ that is a graph over its projection to the horizontal line $x_3^{-1}(t) \cap T_{\gamma(t)}L$. Since for $t > 0$ small the point $\gamma(t)$ is a point of almost-maximal curvature in a certain ball centered at $\gamma(t)$, then the forming double multivalued graph around $\alpha_{\gamma(t)}$ in L^μ extends sideways almost horizontally in $T_{\gamma(t)}L$ by the extension results in Colding and Minicozzi (Theorem II.0.21 in [8], note also that $L^\mu - \gamma$ consists of stable pieces), until exiting the solid vertical cone $\mathcal{C}_{\vec{0}}$ given by description (D2) before Lemma 4.11, whose vertex is the singular point $x = \vec{0} \in \mathcal{S}$. This allows us to extend $\alpha_{\gamma(t)}$ in the vertical plane $T_{\gamma(t)}L$ until it exits $\mathcal{C}_{\vec{0}}$. Once $\alpha_{\gamma(t)}$ exits $\mathcal{C}_{\vec{0}}$, then the almost-horizontal nature of L^μ outside $\mathcal{C}_{\vec{0}}$ for $\mu > 0$ small that comes from curvature estimates, insures that $\alpha_{\gamma(t)}$ can be extended in $L^\mu \cap T_{\gamma(t)}L$ as an almost-horizontal arc, until it possibly intersects the plane $\{x_3 = \mu\}$. The number of intersection points of this extended arc $\alpha_{\gamma(t)}$ with $\{x_3 = \mu\}$ is zero, one or two. If $\alpha_{\gamma(t)}$ never intersects $\{x_3 = \mu\}$ (respectively, if $\alpha_{\gamma(t)}$ intersects $\{x_3 = \mu\}$ exactly once), then $\alpha_{\gamma(t)}$ defines a proper open (respectively, half-open) arc in $\{0 < x_3 \leq \mu\} \cap T_{\gamma(t)}L^\mu$. Otherwise, $\alpha_{\gamma(t)}$ is a compact arc with its two end points at height μ.

We next show that for $t_0 > 0$ sufficiently small and for all $t \in (0, t_0]$, the number of intersection points of $\alpha_{\gamma(t)}$ with $\{x_3 = \mu\}$ is two. Arguing by contradiction, suppose that for some $t_1 \in (0, t_0]$ small, $\alpha_{\gamma(t_1)}$ is not a compact arc. Then, there exists $t_2 \in (0, t_1)$ such that for all $t \in (0, t_2]$, $\alpha_{\gamma(t)}$ is an open proper arc in $\{0 < x_3 < \mu\} \cap T_{\gamma(t)}L$, which is a graph over the horizontal line $x_3^{-1}(t)\cap T_{\gamma(t)}L$. It follows that the surface

$$\Sigma(t_2)=\bigcup_{t\in(0,t_2]}\alpha_{\gamma(t)} \tag{25}$$

is a proper subdomain of L^μ, $\Sigma(t_2)$ is topologically a disk with connected boundary and when intersected with the domain $x_3^{-1}([0,\mu])\cap\{x_1^2+x_2^2\geq 1\}$, is an ∞-valued graph. This minimal surface $\Sigma(t_2)$ cannot exist by the flux arguments in [7] (specifically see Corollary 1.2 and the paragraph just after this corollary). Thus, we may assume that $\alpha_{\gamma(t)}$ is a compact arc for all $t\in(0,t_2]$ and $t_2>0$ sufficiently small. Observe that $\alpha_{\gamma(t)}$ is transversal to $\{x_3=\mu\}$ at the two end points of $\alpha_{\gamma(t)}$ (because μ was a regular value of x_3 in L), for all $t\in(0,t_2]$.

By the above discussion, for $t_2>0$ sufficiently small, $\Sigma(t_2)$ is a union of the compact arcs $\alpha_{\gamma(t)}$, $t\in(0,t_2]$. Let $\Gamma_1(t),\Gamma_2(t)$ be the end points of $\alpha_{\gamma(t)}$, $t\in(0,t_2]$. Hence, for $i=1,2$, $t\in(0,t_2]\mapsto\Gamma_i(t)$ is an embedded proper arc in $L\cap\{x_3=\mu\}$ that spins infinitely often, and Γ_1,Γ_2 are imbricated (they rotate together). The boundary of $\Sigma(t_2)$ is connected and consists of $\alpha_{\gamma(t_2)}\cup\Gamma_1\cup\Gamma_2$. Consider the piecewise smooth surface $\widehat{\Sigma}(t_2)$ obtained by adding to each $\alpha_{\gamma(t)}$ the two disjoint halflines $l_{t,1},l_{t,2}$ in $x_3^{-1}(\mu)\cap T_{\gamma(t)}L$ that start at $\Gamma_1(t),\Gamma_2(t)$, respectively, for all $t\in(0,t_2]$. Observe that $\Sigma(t_2)$ is a subdomain of $\widehat{\Sigma}(t_2)$, that $\widehat{\Sigma}(t_2)$ fails to be smooth precisely $\Gamma_1\cup\Gamma_2$, and that $\widehat{\Sigma}(t_2)$ fails to be embedded since for certain values $t<t'\in(0,t_2]$, the added halflines $l_{t,1},l_{t',1}$ satisfy $l_{t,1}\subset l_{t',1}$, and, similarly, for the halflines $l_{t,2},l_{t',2}$. Both problems for $\widehat{\Sigma}(t_2)$ can be easily overcome (actually embeddedness is not strictly necessary in what follows) by slightly changing the construction, as we now explain. For each $t\in(0,t_2]$, enlarge slightly $\alpha_{\gamma(t)}$ to a compact arc $\widehat{\alpha}_{\gamma(t)}\subset L\cap T_{\gamma(t)}L$, so that if we call $\widetilde{\Gamma}_1(t),\widetilde{\Gamma}_2(t)\in L\cap x_3^{-1}((\mu,\mu+1])$ to the end points of $\widehat{\alpha}_{\gamma(t)}$, then the following properties hold.

(Q1) For $i=1,2$, the correspondence $\Gamma_i(t)\mapsto\widetilde{\Gamma}_i(t)$ defines a smooth map that goes to zero as $t\searrow 0$. In other words, the curve $t\in(0,t_2]\mapsto\widetilde{\Gamma}_i(t)$ is asymptotic to the planar curve $t\in(0,t_2]\mapsto\Gamma_i(t)$ as $t\searrow 0$.

(Q2) $x_3\circ\widetilde{\Gamma}_1(t)=x_3\circ\widetilde{\Gamma}_2(t)$ is strictly increasing as a function of $t\in(0,t_2]$.

Now add to each $\widehat{\alpha}_{\gamma(t)}$ the two disjoint halflines $\widetilde{l}_{t,1},\widetilde{l}_{t,2}$ in $x_3^{-1}(x_3(\widetilde{\Gamma}_1(t)))\cap T_{\gamma(t)}L$ that start at $\widetilde{\Gamma}_1(t),\widetilde{\Gamma}_2(t)$, respectively, for all $t\in(0,t_2]$. By property (Q2) above, the piecewise smooth surface

$$\widetilde{\Sigma}(t_2)=\bigcup_{t\in(0,t_2]}\left[\widehat{\alpha}_{\gamma(t)}\cup\widetilde{l}_{t,1}\cup\widetilde{l}_{t,2}\right]$$

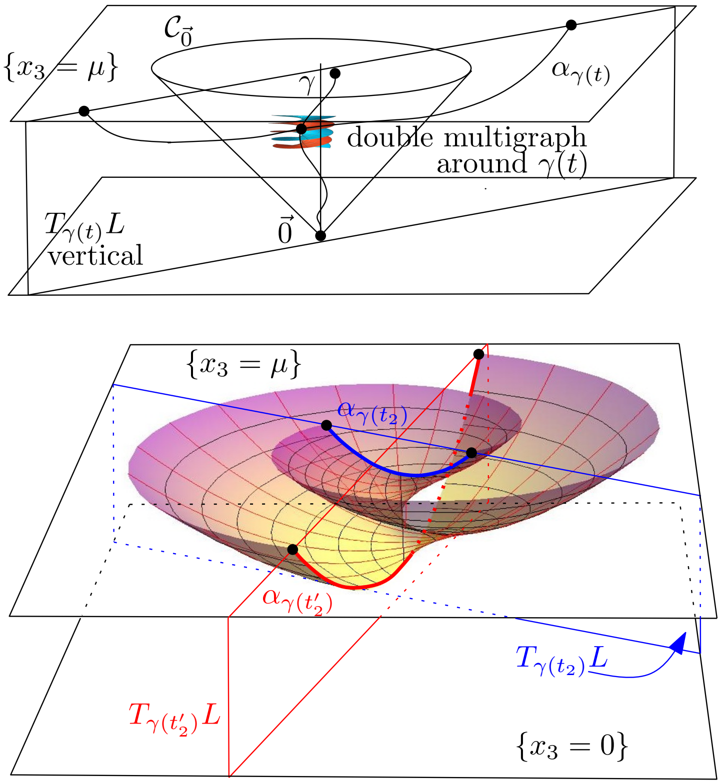

Figure 8. Top: The arc $\alpha_{\gamma(t)} \subset L^\mu \cap T_{\gamma(t)}L$ starts forming in the double multivalued graph around the point $\gamma(t)$, extends sideways until exiting the solid cone $\mathcal{C}_{\vec{0}}$ and eventually intersects $\{x_3 = \mu\}$. Bottom: Schematic representation of a compact portion of the surface $\Sigma(t_2) \subset L^\mu$, foliated by arcs $\alpha_{\gamma(t)}$ in a compact range $t \in [t_2', t_2]$, $0 < t_2' < t_2$.

is embedded and fails to be smooth precisely along $\widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2$. Now smooth $\widetilde{\Sigma}(t_2)$ by rounding off the corners along $\widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2$ in a neighborhood of these curves that is disjoint from $\Sigma(t_2)$, and relabel the resulting smooth embedded surface as $\widetilde{\Sigma}(t_2)$. Furthermore, the above smoothing process can be done so that the tangent spaces to $\widetilde{\Sigma}(t_2)$ form an angle less than $\pi/4$ with the horizontal. Observe that $\Sigma(t_2)$ is a proper subdomain of $\widetilde{\Sigma}(t_2)$, see Figure 8. We denote by $\widetilde{\alpha}_{\gamma(t)} \subset \widetilde{\Sigma}(t_2) \cap T_{\gamma(t)}L$ the smooth proper arc that extends $\alpha_{\gamma(t)}$. Thus, $\widetilde{\Sigma}(t_2)$ is foliated by these arcs $\widetilde{\alpha}_{\gamma(t)}$, $t \in (0, t_2]$.

Now consider the ruled surface

$$R(t_2) = \bigcup_{t \in (0,t_2]} \left(x_3^{-1}(t) \cap T_{\gamma(t)}L \right). \tag{26}$$

The vertical projection $\Pi\colon \widetilde{\Sigma}(t_2) \to R(t_2)$ defined by $\Pi(x,y,z) = (x,y,t)$ if $(x,y,z) \in \widetilde{\alpha}_{\gamma(t)}$, is a quasiconformal diffeomorphism; near γ, this property follows from the fact that both $\widetilde{\Sigma}(t_2), R(t_2)$ can be rescaled around $\gamma(t)$ to produce the same vertical helicoid, and away from γ because the tangent planes to both $\widetilde{\Sigma}(t_2), R(t_2)$ form a small angle with the horizontal for t_2 sufficiently small.

The next lemma shows that the surface $R(t_2)$ is quasiconformally diffeomorphic to a closed halfplane in $\mathbb{C}$. Note that the hypotheses of Lemma 4.28 hold for $R(t_2)$, see items (P1), (P2) above.

Lemma 4.28. *Let $\Gamma\colon (0,1] \to \mathbb{R}^2$ be a smooth curve and $f\colon (0,1] \to (-\infty, 0)$ be a C^2 function. Consider the ruled surface $R \subset \mathbb{R}^3$ parameterized by $X\colon \mathbb{R} \times (0,1] \to \mathbb{R}^3$,*

(27)

$$X(\mu, z) = (\Gamma(z), 0) + (\mu \cos f(z), \mu \sin f(z), z), \qquad (\mu, z) \in \mathbb{R} \times (0,1].$$

If $|\Gamma'|$ is bounded, $\lim_{z\to 0^+} f(z) = -\infty$, f' is bounded away from zero and f''/f' is bounded from above, then R is quasiconformally diffeomorphic to the closed lower half of a vertical helicoid.

Proof. Consider the diffeomorphism $\psi\colon x_3^{-1}((0,1]) \to x_3^{-1}((0,1])$ given by

$$\psi(p,z) = (p - \Gamma(z), z), \qquad (p,z) \in \mathbb{R}^2 \times (0,1].$$

As $|\Gamma'|$ is bounded, then ψ is quasiconformal; this means that there exists $\varepsilon \in (0,1)$ such that given two unitary orthogonal vectors $a, b \in \mathbb{R}^3$, we have

$$\varepsilon \leq \frac{|\psi_*(a)|}{|\psi_*(b)|} \leq \frac{1}{\varepsilon}, \qquad \frac{\langle \psi_*(a), \psi_*(b)\rangle}{|\psi_*(a)|\,|\psi_*(b)|} \in [-1+\varepsilon, 1-\varepsilon],$$

where ψ_* denotes the differential of ψ at any point of $x_3^{-1}((0,1])$. Therefore, after composing with ψ, we may assume in the sequel that $\Gamma(z) = 0$ for all $z \in (0,1]$.

Let $\mathcal{H} = \{(x,y,z) \in \mathbb{R}^3 \mid y = x \tan z\}$ be the standard vertical helicoid. Consider a map $\phi\colon R \to \mathcal{H}$ of the form

$$\phi(X(\mu,z)) = (\widehat{\mu} \cos f(z), \widehat{\mu} \sin f(z), f(z)), \qquad (\mu, z) \in \mathbb{R} \times (0,1],$$

where $\widehat{\mu} = \widehat{\mu}(\mu, z)$ is to be defined later. We will find a choice of $\widehat{\mu}$ for which ϕ is a quasiconformal diffeomorphism from R onto its image; which is the lower half of $\mathcal{H}$ obtained after intersection of $\mathcal{H}$ with $x_3^{-1}((-\infty, f(1)])$. Observe that a global choice of an orthonormal basis for the tangent bundle to R is $\{X_\mu, \frac{1}{|X_z|}X_z\}$. By definition, ϕ is quasiconformal if the following two properties hold.

(R1) $\dfrac{|\phi_*(\frac{1}{|X_z|}X_z)|}{|\phi_*(X_\mu)|}$ is bounded and bounded away from zero (uniformly on M).

(R2) $\dfrac{\langle \phi_*(X_\mu), \phi_*(\frac{1}{|X_z|}X_z)\rangle^2}{|\phi_*(X_\mu)|^2\, |\phi_*(\frac{1}{|X_z|}X_z)|^2} \in [0, 1-\varepsilon]$ uniformly on M, for some $\varepsilon \in (0,1)$.

A direct computation gives

$$\begin{array}{rcl} \phi_*(X_\mu) & = & \widehat{\mu}_\mu(\cos f, \sin f, 0), \\ \phi_*(X_z) & = & \widehat{\mu}_z(\cos f, \sin f, 0) + \widehat{\mu} f'(-\sin f, \cos f, 0) + (0,0,f'(z)), \end{array}$$

where $\widehat{\mu}_\mu = \frac{\partial \widehat{\mu}}{\partial \mu}$, $\widehat{\mu}_z = \frac{\partial \widehat{\mu}}{\partial z}$. Hence,

$$\begin{array}{rcl} |\phi_*(X_\mu)|^2 & = & (\widehat{\mu}_\mu)^2, \\ |\phi_*(X_z)|^2 & = & (\widehat{\mu}_z)^2 + [1+\widehat{\mu}^2](f')^2, \\ \langle \phi_*(X_\mu), \phi_*(X_z)\rangle & = & \widehat{\mu}_\mu \widehat{\mu}_z. \end{array}$$

Thus,

$$\frac{|\phi_*(\frac{1}{|X_z|}X_z)|^2}{|\phi_*(X_\mu)|^2} = \frac{(\widehat{\mu}_z)^2 + [1+\widehat{\mu}^2](f')^2}{[1+\mu^2(f')^2](\widehat{\mu}_\mu)^2}, \tag{28}$$

$$\frac{\langle \phi_*(X_\mu), \phi_*(\frac{1}{|X_z|}X_z)\rangle^2}{|\phi_*(X_\mu)|^2\, |\phi_*(\frac{1}{|X_z|}X_z)|^2} = \frac{(\widehat{\mu}_z)^2}{(\widehat{\mu}_z)^2 + [1+\widehat{\mu}^2](f')^2}. \tag{29}$$

To simplify the last two expressions, we will take $\widehat{\mu}(\mu, z) = \mu f'(z)$ (note that this choice of $\widehat{\mu}$ makes ϕ a diffeomorphism onto its image, as f' does not vanish). The right-hand-side of (28) transforms into

$$E(\mu, z) := 1 + \frac{\mu^2}{1+\mu^2(f')^2}\left(\frac{f''}{f'}\right)^2, \tag{30}$$

which is greater than or equal to 1 and bounded from above under our hypotheses on f' and f''/f', thereby giving (R1). As for the right-hand-side of (29), a direct computation shows that it equals

$$\frac{\mu^2(f'')^2}{(f')^2[1+\mu^2(f')^2] + \mu^2(f'')^2} = 1 - \frac{1}{E(\mu, z)}. \tag{31}$$

As $E(\mu, z)$ is bounded from above, we conclude that the last expression is bounded away from 1 (below 1), and (R2) is also proved. Therefore, ϕ is a quasiconformal diffeomorphism from R onto the lower half of a vertical helicoid, with the choice $\widehat{\mu}(\mu, z) = \mu f'(z)$. q.e.d.

Lemma 4.29. *Let Σ be a simply connected surface which is quasiconformally diffeomorphic to a closed halfplane in $\mathbb{C}$. Then, Σ is conformally diffeomorphic to a closed halfplane.*

Proof. Suppose the lemma fails. Since Σ is simply connected, then Σ can be conformally identified with closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ minus a closed interval $I \subset \partial\mathbb{D}$ that does not reduce to a point. Let $\Sigma^* = (\mathbb{C} \cup \{\infty\}) - I$ be the simply connected Riemann surface obtained after gluing Σ with a copy of itself along $\partial\Sigma$ (by the identity function on $\partial\Sigma$). Thus, by the Riemann mapping theorem, Σ^* is conformally diffeomorphic to the open unit disk $\mathbb{D}$. On the other hand, the quasiconformal version of the Schwarz reflection principle (see e.g., Theorem 3.6 in [**20**]) implies that Σ^* is quasiconformally diffeomorphic to the surface obtained after doubling a closed halfplane along its boundary, which is $\mathbb{C}$. In particular, we deduce that $\mathbb{C}$ is quasiconformally diffeomorphic to $\mathbb{D}$, which is a contradiction (see e.g., Corollary 1 in [**49**]). q.e.d.

Proof of Proposition 4.23. As the minimal surface $\Sigma(t_2)$ defined in (25) can be considered to be a proper subdomain of the surface $\widetilde{\Sigma}(t_2)$ defined immediately before (26), and $\widetilde{\Sigma}(t_2)$ is conformally diffeomorphic to a closed halfplane (by Lemmas 4.28 and 4.29), then $\Sigma(t_2)$ is a parabolic surface. The restriction of the x_3-coordinate function to $\Sigma(t_2)$ is a bounded harmonic function with boundary values greater than or equal to $m = \min\{x_3(q) \mid q \in \alpha_{\gamma(t_2)}\} > 0$. In particular, the parabolicity of $\Sigma(t_2)$ insures that

$$m = \min_{\partial\Sigma(t_2)} x_3 \leq x_3 \leq \max_{\partial\Sigma(t_2)} x_3,$$

which contradicts that $\Sigma(t_2)$ contains points at height arbitrarily close to zero. Now Proposition 4.23 is proved. q.e.d.

Note that Proposition 4.23 finishes the proof of item 6 of Theorem 1.1 (see the paragraph just after the statement of Proposition 4.23). Therefore, the proof of Theorem 1.1 is complete.

We next prove some additional information about case 6 of Theorem 1.1.

Proposition 4.30. *Suppose that $\mathcal{S} \neq \emptyset$ (hence, item 6 of Theorem 1.1 holds). Then:*

(A) $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$ is a closed set of $\mathbb{R}^3$ which is contained in the union of planes $\bigcup_{L\in\mathcal{P}} \overline{L}$. Furthermore, every plane in $\mathbb{R}^3$ intersects $\mathcal{L}$.

(B) There exists $R_0 > 0$ such that the sequence $\left\{M_n \cap B_M(p_n, \frac{R_0}{\lambda_n})\right\}_n$ does not have bounded genus.

(C) There exist oriented closed geodesics $\gamma_n \subset \lambda_n M_n$ with uniformly bounded lengths which converge to a line segment γ in the closure of some flat leaf in $\mathcal{P}$, which joins two points of $\Delta(\mathcal{L})$, and such that the integrals of $\lambda_n M_n$ along γ_n in the induced exponential $\mathbb{R}^3$-coordinates of $\lambda_n B_N(p_n, \varepsilon_n)$ converge to a horizontal vector orthogonal to γ with length $2\,\mathrm{Length}(\gamma)$.

Proof. $\Delta(\mathcal{L})$ is closed in $\mathbb{R}^3$ since $\mathcal{S}$ is closed in $\mathbb{R}^3$ and $S(\mathcal{L})$ is closed in $\mathbb{R}^3 - \mathcal{S}$. Lemma 4.11 implies that $\Delta(\mathcal{L})$ is contained in $\bigcup_{L\in\mathcal{P}} \overline{L}$. Every plane in $\mathbb{R}^3$ intersects $\mathcal{L}$ by Lemma 4.12, and so item (A) of the proposition holds.

We next prove item (B). Consider a plane $P \in \mathcal{P}'$ such that $P \cap \mathcal{S} \neq \emptyset$. By Proposition 4.23, P intersects $\Delta(\mathcal{L})$ in at least two points. If this intersection consists of exactly two points with opposite orientation numbers, then Proposition 4.18 implies that $\mathcal{L}$ is a foliation of $\mathbb{R}^3$, which contradicts that $\mathcal{S} \neq \emptyset$. Therefore, there exist two points x_1, x_2 in $P \cap \mathcal{S}$ with the same orientation number. In this setting, we can adapt the arguments in the proof of Lemma 3.3 to conclude that $B_{\lambda_n N}(x_1, 2d_n) \cap (\lambda_n M_n)$ has unbounded genus for n large enough, where d_n is the extrinsic distance in $\lambda_n N$ from x_1 to x_2 (note that d_n converges as $n \to \infty$ to $|x_1 - x_2|$). Finally, take $k_0 \in \mathbb{N}$ so that $B_{\lambda_n N}(x_1, 2d_n) \cap (\lambda_n M_n)$ is contained in $B_{\lambda_n N}(p_n, k_0)$, for all $n \in \mathbb{N}$. By Proposition 4.21, each point in $B_{\lambda_n N}(x_1, 2d_n) \cap (\lambda_n M_n)$ is at an intrinsic distance not greater than some fixed number R_0 (depending on k_0) from p_n, for all n sufficiently large. After coming back to the original scale, this implies that the surfaces $M_n \cap B_M(p_n, \frac{R_0}{\lambda_n})$ do not have bounded genus. This finishes the proof of item (B) of the proposition.

Finally, item (C) follows from applying to the plane P that appears in the previous paragraph the arguments in the proof of Lemma 4.13. This concludes the proof of the proposition. q.e.d.

Remark 4.31. The techniques used to prove Theorem 1.1 have other consequences. For example, suppose $\{M_n\}_n$ is a sequence of compact embedded minimal surfaces in $\mathbb{R}^3$ with $\vec{0} \in M_n$ whose boundaries lie in the boundaries of balls $\mathbb{B}(R_n)$, where $R_n \to \infty$. Suppose that there exists some $\varepsilon > 0$ such that for any ball $\mathbb{B}$ in $\mathbb{R}^3$ of radius ε, for n sufficiently large, $M_n \cap \mathbb{B}$ consists of disks, and such that for some fixed compact set C, there exists a $d > 0$ such that for n large, the injectivity radius function of M_n is at most d at some point of $M_n \cap C$. Then the proof of Theorem 1.1 shows that, after replacing by a subsequence, the M_n converge on compact subsets of $\mathbb{R}^3$ to one of the following cases:

(4.29.a) A properly embedded, nonsimply connected minimal surface M_∞ in $\mathbb{R}^3$. In this case, the convergence of the surfaces M_n to M_∞ is smooth of multiplicity one on compact sets of $\mathbb{R}^3$.

(4.29.b) A minimal parking garage structure of $\mathbb{R}^3$ with at least two columns.

(4.29.c) A singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with properties similar to the minimal lamination described in item 6 of Theorem 1.1 and in Proposition 4.30.

Remark 4.32. In [**27**], we will apply Theorem 1.1 under slightly weaker hypotheses for the embedded minimal surface M appearing in it, namely M is not assumed to be complete, but instead we will suppose that M satisfies the following condition.

Suppose M is an embedded minimal surface, not necessarily complete and possibly with boundary, in a homogeneously regular three-manifold N. Observe that the injectivity radius $I_M(p) \in (0, \infty]$ at any interior point $p \in M$ still makes sense, although the exponential map $\exp_p$ is no longer defined in the whole T_pM. We endow M with the structure of a metric space with respect to the intrinsic distance d_M, and let $\overline{M}$ be the metric completion of (M, d_M). In the sequel we will identify M with its isometrically embedded image in $\overline{M}$. Given an interior point $p \in M$, we define $d_M(p, \partial M) > 0$ to be the distance in $\overline{M}$ from p to $\partial M = \overline{M} - \mathrm{Int}(M)$. Consider the continuous function $f\colon \mathrm{Int}(M) \to (0, \infty)$ given by

$$f(p) = \frac{\min\{1, d_M(p, \partial M)\}}{I_M(p)}.$$

Suppose that f is unbounded. Then, the conclusions in Theorem 1.1 hold, i.e., exist points $p_n \in \mathrm{Int}(M)$ and positive numbers $\varepsilon_n = nI_M(p_n) \to 0$ such that items 1,..., 6 of Theorem 1.1 hold.

To prove this version of Theorem 1.1 in the case that either M is incomplete or $\partial M \neq \emptyset$, then one must replace the points $q_n \in M$ with $I_M(q_n) \leq \frac{1}{n}$ that appeared in the first paragraph of Section 4 by points $q_n \in \mathrm{Int}(M)$ such that $f(q_n) \geq n$, and then change the function h_n defined in (1) by the expression

$$h_n(x) = \frac{d_M(x, \partial B_M(q_n, \frac{1}{2}d_M(q_n, \partial M)))}{I_M(x)}, \quad x \in \overline{B}_M(q_n, \tfrac{1}{2}d_M(q_n, \partial M)).$$

From this point on, the above proof of Theorem 1.1 works without changes.

5. Applications

Definition 5.1. Given a complete embedded minimal surface M with injectivity radius zero in a homogeneously regular three-manifold, a *local picture of M on the scale of topology* is one of the blow-up limits that can occur when we apply Theorem 1.1 to M, namely a nonsimply connected properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ as in item 4 of that theorem, a minimal parking garage structure in $\mathbb{R}^3$ with at least two columns as in item 5 or a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ as in item 6, obtained as a limit of M under blow-up around points of almost-minimal injectivity radius.

Similarly, given a complete embedded minimal surface M with unbounded second fundamental form in a homogeneously regular three-manifold, a *local picture of M on the scale of curvature* is a nonflat properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ of bounded Gaussian curvature, obtained as a limit of M under blow-up around points of almost-maximal second fundamental form, in the sense of Theorem 1.1 in [**33**].

An immediate consequence of Theorem 1.1 in the Introduction and of the uniqueness of the helicoid [**35**] is the following statement.

Corollary 5.2. *Let M be a complete embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold. If a* *properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ is a local picture of M on the scale of topology (i.e., M_∞ arises as a blow-up limit of M as in item 4 of Theorem 1.1) and M_∞ does not have bounded Gaussian curvature, then every local picture of M_∞ on the scale of curvature is a helicoid.*

5.1. The set of local pictures on the scale of topology. Given $0 < a \leq b$, consider the set $\mathcal{B}_{a,b}$ of all complete embedded minimal surfaces $M \subset \mathbb{R}^3$ with $|K_M| \leq b$ and $|K_M|(p) \geq a$ at some point $p \in \overline{\mathbb{B}}(1)$. Since complete embedded nonflat minimal surfaces in $\mathbb{R}^3$ of bounded absolute Gaussian curvature are proper [**37**] and properly embedded nonflat minimal surfaces in $\mathbb{R}^3$ are connected [**19**], then the surfaces in $\mathcal{B}_{a,b}$ are connected and properly embedded. The topology of uniform C^k-convergence on compact subsets of $\mathbb{R}^3$ is metrizable on the set $\mathcal{B}_{a,b}$ (see Section 5 of [**33**] for a proof of this fact in a slightly different context, for $a = b = 1$). Sequential compactness (hence, compactness) of $\mathcal{B}_{a,b}$ follows immediately from uniform local curvature and area estimates (area estimates come from the existence of a tubular neighborhood of fixed radius, see [**37**]). By the regular neighborhood theorem in [**37**] or [**46**], the surfaces in $\mathcal{B}_{a,b}$ all have cubical area growth, i.e.,

$$R^{-3}\ \mathrm{Area}(M \cap \mathbb{B}(R)) \leq C,$$

for all surfaces $M \in \mathcal{B}_{a,b}$ and for all $R > 1$, where $C = C(b) > 0$ depends on the uniform bound of the curvature.

The next corollary follows directly from the above observations, the Local Picture Theorem on the Scale of Curvature (Theorem 1.1 in [**33**]) and the Local Picture Theorem on the Scale of Topology (Theorem 1.1 in this paper).

Corollary 5.3. *Suppose M is a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold, and suppose M does not have a local picture on the scale of curvature which is a helicoid. Then, there exist positive constants $a \leq b$ depending only on M, such that every local picture of M on the scale of topology lies in $\mathcal{B}_{a,b}$ (in particular, every such local picture of M on the scale of topology arises from item 4 in Theorem 1.1). Furthermore, the set*

$$\mathcal{B}(M) = \{\textit{local pictures of } M \textit{ on the scale of topology}\}$$

is a closed subset of $\mathcal{B}_{a,b}$ (thus, $\mathcal{B}(M)$ is compact), and there is a constant $C = C(M)$ such that every local picture on the scale of topology has area growth at most CR^3.

Remark 5.4. With the notation of Theorem 1.1 in this paper, if M has finite genus or if the sequence $\{\lambda_n M_n\}_n$ has uniformly bounded genus in fixed size intrinsic metric balls, then item 6 of that theorem does not occur, since item (B) of Proposition 4.30 does not occur. This fact will play a crucial role in our forthcoming paper [**25**], when proving a bound on the number of ends for a complete, embedded minimal surface of finite topology in $\mathbb{R}^3$, that only depends on its genus. Also in [**27**], we will apply Theorem 1.1 to give a general structure theorem for singular minimal laminations of $\mathbb{R}^3$ with a countable number of singularities.

5.2. Complete embedded minimal surfaces in $\mathbb{R}^3$ with zero flux. Recall that a nonflat minimal immersion $f\colon M \to \mathbb{R}^3$ has zero flux if the integral of the unit conormal vector around any closed curve on M is zero. By the Weierstrass representation, a nonflat minimal immersion $f\colon M \to \mathbb{R}^3$ has nonzero flux if and only if $f\colon M \to \mathbb{R}^3$ is the unique isometric minimal immersion of M into $\mathbb{R}^3$ up to rigid motions.

The results described in the next corollary to Theorem 1.1 overlap somewhat with the rigidity results for complete embedded constant mean curvature surfaces by Meeks and Tinaglia described in [**41**].

Corollary 5.5. *Let $M \subset \mathbb{R}^3$ be a complete, embedded minimal surface with zero flux. Suppose that M is not a plane or a helicoid. Then, M has infinite genus and one of the following two possibilities hold:*

1) *M is properly embedded in $\mathbb{R}^3$ with positive injectivity radius and one end.*
2) *M has injectivity radius zero and every local picture of M on the scale of topology is a properly embedded minimal surface with one end, infinite genus and zero flux.*

Proof. First suppose that M is properly embedded in $\mathbb{R}^3$. As M has zero flux, then the main result in Choi, Meeks and White [**5**] insures that M has one end. We claim that M has infinite genus. Otherwise, by classification of properly embedded minimal surfaces with finite genus and one end, then M is a helicoid with handles, in particular, M is asymptotic to the helicoid (Bernstein and Breiner [**2**]). We next show that in this case, M has nonzero flux, which contradicts the hypothesis: as M is a helicoid with handles, then after rotation we can assume that M is asymptotic to a vertical helicoid whose axis is the x_3-axis. Consider the intersection Γ_t of M with the horizontal plane $\{x_3 = t\}$. For every $t \in \mathbb{R}$, Γ_t is a proper 1-dimensional analytic set with two ends which are asymptotic to the ends of a straight line; this result can be deduced from the analytic results described in either [**2**] or [**23**]. For $|t|$ large, Γ_t consists of a connected, proper planar arc asymptotic to a straight line. However, since M is not simply connected (because M is not a plane or helicoid, Meeks and Rosenberg [**35**]), there exists a lowest plane $\{x_3 = T\}$ such that Γ_T is not a proper arc. In particular, Γ_T is a limit of proper arcs Γ_t, $t \nearrow T$. By the maximum principle, Γ_T is a connected, 1-dimensional analytic set asymptotic to a straight line. Since Γ_T is not an arc, then $x_3\colon M \to \mathbb{R}$ has a critical point of negative index on Γ_T. As Γ_T only has two ends, then standard topological arguments imply that Γ_T contains a piecewise smooth loop that bounds a horizontal disk on one side of M. By the maximum principle, the unit conormal vector of M along this loop lies in the closed upper (or lower) hemisphere and it is not everywhere horizontal. Thus, the flux of M along this loop is not zero, which is a contradiction. Therefore, M has infinite genus provided that it is proper.

Finally, suppose that M satisfies the hypotheses of Corollary 5.5. If the injectivity radius of M is positive, then M is proper by Theorem 2 in [**36**] and so, Corollary 5.5 holds by the arguments in the last paragraph. Otherwise, the injectivity radius of M is zero and, thus, Theorem 1.1 applies. Note that cases 5 and 6 of Theorem 1.1 cannot occur since in those cases the flux of the approximating surface $\lambda_n M_n$ (with the notation of Theorem 1.1) is not zero by item (B) of Proposition 4.20 and item (C) of Proposition 4.30, but M has zero flux. Hence, every local picture of M on the scale of topology is a properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$. Note that M_∞ has zero flux since M has zero flux. By arguments in the first paragraph of this proof, M_∞ has infinite genus. Finally, observe this last property together with a lifting argument shows that M also has infinite genus in this case. q.e.d.

Remark 5.6. If $M \subset \mathbb{R}^3$ is a complete embedded minimal surface that admits an intrinsic isometry $I\colon M \to M$ which does not extend to an ambient isometry of $\mathbb{R}^3$, then M must have zero flux (because the only isometric minimal immersions from M into $\mathbb{R}^3$ are associated minimal surfaces to M by Calabi [**3**]). Since the associated surfaces to a helicoid which are not congruent to it are not embedded, then such an M cannot admit a local picture on the scale of curvature which is a helicoid. Therefore, either M has positive injectivity radius (so it is properly embedded in $\mathbb{R}^3$ by Theorem 2 in [**36**]) and its Gaussian curvature is bounded (otherwise one could blow-up M on the scale of curvature to produce a limit helicoid by Theorem 1.1 in [**33**], which is a contradiction), or M has injectivity radius zero and Corollaries 5.3 and 5.5 imply that every local picture of M on the scale of topology is a nonsimply connected, properly embedded minimal surface with bounded Gaussian curvature and zero flux. The authors believe that this observation could play an important role in proving the classical conjecture that intrinsic isometries of complete embedded minimal surfaces in $\mathbb{R}^3$ always extend to ambient isometries, and more generally, to prove that a complete embedded minimal surface in $\mathbb{R}^3$ does not admit another noncongruent isometric minimal embedding into $\mathbb{R}^3$.

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Mathematics Department
University of Massachusetts
Amherst, MA 01003
USA
E-mail address: profmeeks@gmail.com

Department of Geometry and Topology
Institute of Mathematics (IEMath-GR)
University of Granada
Granada
Spain
E-mail address: jperez@ugr.es

Department of Geometry and Topology
Institute of Mathematics (IEMath-GR)
University of Granada
Granada
Spain
E-mail address: aros@ugr.es