TOPOLOGICAL TYPE OF LIMIT LAMINATIONS OF EMBEDDED MINIMAL DISKS

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Abstract

We consider two natural classes of minimal laminations in three-manifolds. Both classes may be thought of as limits—in different senses—of embedded minimal disks. In both cases, we prove that, under a natural geometric assumption on the three-manifold, the leaves of these laminations have genus zero. This answers a question posed by Hoffman and White.

1. Introduction

Let Ω be a fixed Riemannian three-manifold. Suppose that $\Omega_i \subset \Omega$ is an increasing sequence of open sets with $\Omega = \bigcup \Omega_i$ and that Σ_i are minimal surfaces properly embedded in Ω_i. We say that the curvatures of the Σ_i blow up at a point $p \in \Omega$ if there exists a sequence of points $p_i \in \Sigma_i$ converging to p such that $|A_{\Sigma_i}|(p_i)$ becomes arbitrarily large, where $|A_\Sigma|$ denotes the norm of the second fundamental form of Σ. We call such p a blow-up point and observe that the set, K, of blow-up points is closed in Ω. The points of K are precisely the obstruction to the sequence smoothly subconverging. Indeed, up to passing to a subsequence, the $\Sigma_i \backslash K$ converge on compact subsets of $\Omega \backslash K$ to a smooth proper minimal lamination $\mathcal{L}$ in $\Omega \backslash K$—see [**7**, Appendix B]. Recall, a lamination is a foliation that need not fill space. We call the quadruple $(\Omega, K, \mathcal{L}, \mathcal{S})$ a *minimal surface sequence*. A natural question is:

> In a minimal surface sequence, what singular sets, K, and limit laminations, $\mathcal{L}$, can arise?

Work of Anderson [**1**] and White [**23**] answers this question when the total extrinsic curvatures of the surfaces Σ_i—i.e., $\int_{\Sigma_i} |A_{\Sigma_i}|^2$—are uniformly bounded. In this case, K is finite and $\mathcal{L}$ extends smoothly across K. In general, without such a strong assumption on the geometry of the surfaces in the sequence one does not expect such a complete answer.

Remarkably, when the Σ_i are assumed only to be disks, an elegant story also emerges—we call such minimal surface sequences *minimal disk sequences*. In a series of papers [**5**, **6**, **7**, **8**], Colding and Minicozzi

extensively studied these sequences and proved deep structural results about both their singular sets, K, and limit laminations, $\mathcal{L}$. Specifically, they showed that K must be contained in a Lipschitz curve and that for any point $p \in K$ there exists a leaf of $\mathcal{L}$ that extends smoothly across p. When $\Omega = \mathbb{R}^3$, Colding and Minicozzi further showed that either $K = \emptyset$ or $\mathcal{L}$ is a foliation of $\mathbb{R}^3 \backslash K$ by parallel planes and that K consists of a connected Lipschitz curve which meets the leaves of $\mathcal{L}$ transversely. Using this result, Meeks and Rosenberg showed in [**16**] that the helicoid is the unique non-flat properly embedded minimal disk in $\mathbb{R}^3$—see also [**2**]. This uniqueness was then used by Meeks [**15**] to prove that if $\Omega = \mathbb{R}^3$ and $K \neq \emptyset$, then K is a line orthogonal to the leaves of $\mathcal{L}$. This is precisely the limit of a sequence of rescalings of a helicoid.

An example constructed by Colding and Minicozzi in [**4**] illustrates how for general regions such a simple and complete description of the limit lamination does not hold. Specifically, they constructed a sequence of properly embedded minimal disks in the unit ball B_1 of $\mathbb{R}^3$ which has $K = \{0\}$ and whose limit lamination consists of three leaves—two are non-proper and spiralling into the third, which is the punctured unit disk in the x_3-plane. Inspired by this, a plethora of examples have now been constructed which show that the singular set K can consist of any closed subset of a line; see [**9**, **12**, **13**, **11**]. Likewise, Meeks and Weber [**17**] have given examples where K is curved. Strikingly, Hoffman and White [**10**] have also constructed minimal disk sequences in which $K = \emptyset$ and the limit lamination $\mathcal{L}$ has a leaf which is a proper annulus in Ω. In all examples the leaves are either topologically disks or annuli. This motivated Hoffman and White to ask in [**10**]:

> Can a surface of positive genus occur as a leaf of the lami-
> nation $\mathcal{L}$ of a minimal disk sequence? A planar domain with
> more than two holes?

In this paper we show that, under natural geometric assumptions on Ω, the answer to both questions is no. That is, the leaves of $\mathcal{L}$ must be either disks or annuli.

Theorem 1.1. *Let Ω be the interior of a compact oriented three-manifold $N = \overline{\Omega}$ with mean-convex boundary. If Ω contains no closed minimal surfaces and $(\Omega, K, \mathcal{L}, \mathcal{S})$ is a minimal disk sequence, then the leaves of $\mathcal{L}$ are either disks or annuli. Furthermore, if L is a leaf of $\mathcal{L}$ with the property that $\overline{L}$—the closure in Ω of L—is a properly embedded minimal surface, then $\overline{L}$ is either a disk or it is an annulus which is disjoint from K.*

The theorem is proved by realizing the disks in the sequence $\mathcal{S}$ as effective universal covers of the leaves of $\mathcal{L}$. Geometric considerations—specifically the fact that the disks are embedded—strongly restrict these covers and this restricts the topology of the leaves as claimed. Most of

our proof uses relatively elementary topological and geometric properties of embedded minimal disks.

Simple examples show that it is possible for the leaves of $\mathcal{L}$ to be planar domains with more than two holes when the boundary is not assumed to be mean-convex. Nevertheless, the methods of the present paper continue to show that the leaves are genus zero. However, as treating this more general case introduces several technical points, we do not pursue it.

We further remark that it is unclear whether the condition that Ω admits no closed minimal surface is necessary. To better understand this question we consider also the topological type of leaves of a different—though related—class of minimal laminations. Specifically, we say that a quadruple $(\Omega, K, \mathcal{L}, \mathcal{L}_0)$ is a *minimal disk closure* if $\mathcal{L}_0$ is a minimal lamination in Ω all of whose leaves are disks; K is a relatively closed subset of Ω; and $\mathcal{L} = \overline{\mathcal{L}}_0$, the closure of $\mathcal{L}_0$ in $\Omega \backslash K$, is a proper minimal lamination in $\Omega \backslash K$.

Theorem 1.2. *Let Ω be the interior of a compact oriented three-manifold with boundary $N = \overline{\Omega}$ which has mean-convex boundary. If Ω contains no closed minimal surfaces and $(\Omega, K, \mathcal{L}, \mathcal{L}_0)$ is a minimal disk closure, then the leaves of $\mathcal{L}$ are either topologically open subsets of $\mathbb{S}^2$ or open subsets of $\mathbb{RP}^2$.*

Notice that the conclusions of Theorem 1.2 are weaker than those of Theorem 1.1. For instance, we can no longer rule out the existence of one-sided leaves of $\mathcal{L}$. In Appendix A, we give an example to show that this is unavoidable—that is, we construct a minimal disk closure $(\Omega, \emptyset, \mathcal{L}, \mathcal{L}_0)$ where Ω is a solid torus and $\mathcal{L}$ contains a Möbius band as a leaf. We also give an example in Appendix A that shows that Theorem 1.2 is sharp—i.e., we construct a minimal disk closure $(\Omega, \emptyset, \mathcal{L}, \mathcal{L}_0)$ for which one leaf of $\mathcal{L}$ is a torus.

2. Notation

Fix a smooth oriented Riemannian three-manifold (Ω, g). We denote by dist^Ω the distance function on Ω and by $\exp^\Omega$ the exponential map. Hence,

$$\exp_p^\Omega : B_r \to \mathcal{B}_r(p)$$

where B_r is the usual euclidean ball in $\mathbb{R}^3$ centered at the origin and $\mathcal{B}_r(p)$ is the geodesic ball in Ω. A subset $\Sigma \subset \Omega$ is an *embedded smooth*

surface if for each point $p \in \Sigma$ there is a radius $r_p > 0$ and diffeomorphism $\phi_p : \mathcal{B}_{r_p}(p) \to B_1$ so that $\mathbb{D}_1 = \phi_p(\Sigma \cap \mathcal{B}_{r_p}(p))$. Here, $\mathbb{D}_1 = B_1 \cap \{x_3 = 0\} \subset B_1$ is the unit disk. Such Σ is said to be *proper* in Ω if it is a closed subset of Ω, that is, $\overline{\Sigma} = \Sigma$.

For an embedded smooth surface, Σ, we write

$$\exp^\perp \colon N\Sigma \to \Omega$$

for the normal exponential map where here $N\Sigma$ is the normal bundle. If $N\Sigma$ is trivial then we say that Σ is *two-sided*; otherwise we say that Σ is *one-sided*. As Ω is oriented, Σ is two-sided if and only if Σ is orientable. For a subset $U \subset N\Sigma$ set

$$\mathcal{N}_U(\Sigma) := \exp^\perp(U).$$

The set $\mathcal{N}_U(\Sigma)$ is *regular* if there is an open set V with $U \subset V$ so that $\exp^\perp : V \to \mathcal{N}_V(\Sigma)$ is a diffeomorphism. If $\mathcal{N}_U(\Sigma)$ is regular, then the map $\Pi_\Sigma : \mathcal{N}_U(\Sigma) \to \Sigma$ given by nearest point projection is smooth and for any $(q, \mathbf{v}) \in T\mathcal{N}_U(\Sigma)$, there is a natural splitting $\mathbf{v} = \mathbf{v}^\perp + \mathbf{v}^\top$, where $\mathbf{v}^\perp$ is orthogonal to $\mathbf{v}^\top$ and $\mathbf{v}^\perp$ is tangent to the fibers of Π_Σ. We say such $\mathbf{v}$ is *δ-parallel to Σ* if

$$|\mathbf{v}^\perp| \leq \delta|\mathbf{v}| \quad \text{and} \quad \frac{1}{1+\delta}|\mathbf{v}^\top| \leq |d(\Pi_\Sigma)_q(\mathbf{v})| \leq (1+\delta)|\mathbf{v}^\top|.$$

Given $\epsilon > 0$, we set $U_\epsilon = \{(p, \mathbf{v}) \in N\Sigma : |\mathbf{v}| < \epsilon\}$ and define $\mathcal{N}_\epsilon(\Sigma)$, the *$\epsilon$-neighborhood of Σ*, to be $\mathcal{N}_{U_\epsilon}(\Sigma)$. If Σ is an embedded smooth surface and Σ_0 is a pre-compact subset, then there is an $\epsilon > 0$ so that $\mathcal{N}_\epsilon(\Sigma_0)$ is regular.

Given a fixed embedded surface Σ and $\delta \geq 0$, we say that another embedded smooth surface Γ is a *smooth δ-graph* over Σ if there is an $\epsilon > 0$ so that the following holds:

1) $\mathcal{N}_\epsilon(\Sigma)$ is a regular ϵ-neighborhood of Σ;
2) either Γ is a proper subset of $\mathcal{N}_\epsilon(\Sigma)$ or Γ is a proper subset of $\mathcal{N}_\epsilon(\Sigma) \backslash \Sigma$;
3) each $(q, \mathbf{v}) \in T\Gamma$ is δ-parallel to Σ.

We say that a smooth δ-graph over Σ, Γ, is a *smooth δ-cover* of Σ if it is connected and

$$\Pi_\Sigma(\Gamma) = \Sigma.$$

Let $\gamma \colon [0, 1] \to \Sigma$ be a C^1 curve in Σ. We will also denote the image of such γ by γ. We say that a curve $\widetilde{\gamma} \colon [0, 1] \to \mathcal{N}_\delta(\gamma)$ is a *δ-lift* of γ if $\mathcal{N}_\delta(\gamma)$ is regular, $\Pi_\Sigma \circ \widetilde{\gamma} = \gamma$, and for each $t \in [0, 1]$, $(\widetilde{\gamma}(t), \widetilde{\gamma}'(t))$ is δ-parallel to Σ. This definition extends to piece-wise C^1 curves in an obvious manner.

3. Minimal Laminations

We recall some facts about laminations.

Definition 3.1. A subset $\mathcal{L} \subset \Omega$ is a *smooth lamination* if for each $p \in \mathcal{L}$, there is a radius $r_p > 0$, maps $\phi_p, \psi_p : \mathcal{B}_{r_p}(p) \to B_1 \subset \mathbb{R}^3$, and a closed set $0 \in T_p \subset (-1, 1)$ so that:

1) $\phi_p(p) = \psi_p(p) = 0$;
2) ϕ_p is a smooth diffeomorphism and $\mathbb{D}_1 \subset \phi_p(\mathcal{L} \cap \mathcal{B}_{r_p}(p))$;
3) ψ_p is a Lipschitz diffeomorphism and $B_1 \cap \{x_3 = t\}_{t \in T_p} = \psi_p(\mathcal{L} \cap \mathcal{B}_{r_p}(p))$;
4) $\psi_p^{-1}(\mathbb{D}_1) = \phi_p^{-1}(\mathbb{D}_1)$.

We refer to maps ϕ_p satisfying (1) and (2) as *smoothing maps* of $\mathcal{L}$ and to maps ψ_p satisfying (1) and (3) as *straightening maps* of $\mathcal{L}$.

A smooth lamination $\mathcal{L} \subset \Omega$ is *proper* in Ω if it is closed—i.e. $\overline{\mathcal{L}} = \mathcal{L}$. Any embedded smooth surface is a smooth lamination, which is proper if and only if the surface is.

Definition 3.2. Let $\mathcal{L} \subset \Omega$ be a non-empty smooth lamination. A subset $L \subset \mathcal{L}$ is a *leaf of* $\mathcal{L}$ if it is a connected, embedded surface and for any $p \in L$, there is an $r_p > 0$ and a smoothing map ϕ_p so that $\mathbb{D}_1 = \phi_p(L \cap \mathcal{B}_{r_p}(p))$. For each $p \in \mathcal{L}$, let L_p be the unique leaf of $\mathcal{L}$ containing p.

Definition 3.3. A smooth lamination $\mathcal{L}$ is a *minimal lamination* if each leaf is minimal.

A sequence $\{\Omega_i\}$ of open subsets of Ω *exhausts* Ω if $\Omega_i \subset \Omega_{i+1}$ and $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$.

Definition 3.4. Suppose the sequence $\{\Omega_i\}$ exhausts Ω and that $\mathcal{L}_i$ are smooth proper laminations in Ω_i. For any $0 < \alpha < 1$, the $\mathcal{L}_i$ converge in $C_{loc}^{\infty,\alpha}(\Omega)$ to $\mathcal{L}$, a proper smooth lamination in Ω, provided:

1) The sets $\mathcal{L}_i$ converge to $\mathcal{L}$ in pointed Gromov-Hausdorff distance.
2) Smoothing maps of the $\mathcal{L}_i$ converge in C^{∞} to smoothing maps of $\mathcal{L}$. That is, for each $p \in \mathcal{L}$ there is an $r_p > 0$ and an $i_p > 0$ so that: for $i > i_p$, $\mathcal{B}_{r_p}(p) \subset \Omega_i$ and for all $p_i \in \mathcal{B}_{\frac{1}{4}r_p}(p) \cap \mathcal{L}_i$ converging to p, there are $r_{p_i} \geq r_p$ and smoothing maps $\phi_i : \mathcal{B}_{\frac{1}{2}r_p}(p_i) \to B_1$ of the $\mathcal{L}_i$ converging in $C_{loc}^{\infty}(\mathcal{B}_{\frac{1}{4}r_p}(p))$ to a smoothing map $\phi_p : \mathcal{B}_{\frac{1}{4}r_p}(p) \to B_1$ of $\mathcal{L}$.
3) Straightening maps of the $\mathcal{L}_i$ converge in C^{α} to straightening maps of $\mathcal{L}$. That is, for each $p \in \mathcal{L}$ there is an $r_p > 0$ and an $i_p > 0$ so that: for $i > i_p$, $\mathcal{B}_{r_p}(p) \subset \Omega_i$ and for all $p_i \in \mathcal{B}_{\frac{1}{4}r_p}(p) \cap \mathcal{L}_i$ converging to p, there are $r_{p_i} \geq r_p$ and straightening maps $\psi_i : \mathcal{B}_{\frac{1}{2}r_p}(p_i) \to$

B_1 of the $\mathcal{L}_i$ converging in $C^\alpha_{loc}(\mathcal{B}_{\frac{1}{4}r_p}(p))$ to a straightening map $\psi_p : \mathcal{B}_{\frac{1}{4}r_p}(p) \to B_1$ of $\mathcal{L}$.

The following is the natural compactness result for sequences of properly embedded minimal surfaces with uniformly bounded second fundamental form— see [**7**, Appendix B] for a proof.

Theorem 3.5. *Suppose that* $\{\Omega_i\}$ *exhausts* Ω *and that* Σ_i *are properly embedded smooth minimal surfaces in* Ω_i. *If for each compact subset U of* Ω *there is a constant* $C(U) < \infty$ *so that when* $U \subset \Omega_i$

$$\sup_{U \cap \Sigma_i} |A_{\Sigma_i}| \leq C(U),$$

then, for any $0 < \alpha < 1$, *up to passing to a subsequence, the* Σ_i *converge in* $C^{\infty,\alpha}_{loc}(\Omega)$ *to a smooth proper minimal lamination* $\mathcal{L}$ *in* Ω.

REMARK 3.6. While the straightening maps converge in C^α, their Lipschitz norms are uniformly bounded on compact subsets of Ω. This follows from the Harnack inequality and is used in the proof of Theorem 3.5—see (B.3) and (B.5) of [**7**] and also [**21**, Theorem 1.1].

Suppose that $\{\Omega_i\}$ exhausts Ω and that Σ_i are properly embedded smooth minimal surfaces in Ω_i. In light of Theorem 3.5, we define the *regular* points of the sequence $\mathcal{S} = \{\Sigma_i\}$ to be the set of points

$$\mathrm{reg}(\mathcal{S}) := \left\{ p \in \Omega : \exists \rho > 0 \text{ s.t. } \limsup_{i \to \infty} \sup_{B_\rho(p) \cap \Sigma_i} |A_{\Sigma_i}| < \infty \right\}$$

and the *singular* points of $\mathcal{S}$ to be the set

$$\mathrm{sing}(\mathcal{S}) := \left\{ p \in \Omega : \forall \rho > 0, \liminf_{i \to \infty} \sup_{B_\rho(p) \cap \Sigma_i} |A_{\Sigma_i}| = \infty \right\}.$$

Clearly, $\mathrm{reg}(\mathcal{S})$ is an open subset of Ω while $\mathrm{sing}(\mathcal{S})$ is closed in Ω. In general, $\mathrm{sing}(\mathcal{S})$ is a strict subset of $\Omega \backslash \mathrm{reg}(\mathcal{S})$. However, an elementary argument—see [**7**, Lemma I.1.4]—implies that there is a subsequence $\mathcal{S}'$ of $\mathcal{S}$ so that $\Omega = \mathrm{reg}(\mathcal{S}') \cup \mathrm{sing}(\mathcal{S}')$. From now on we consider only sequences for which this decomposition holds.

We say that $\mathcal{L}$ is the *limit lamination* of $\mathcal{S}$ if for some $\alpha > 0$, $\Sigma_i \to \mathcal{L}$ in $C^{\infty,\alpha}_{loc}(\mathrm{reg}(\mathcal{S}))$. Theorem 3.5 implies that, up to passing to a subsequence, any sequence $\mathcal{S}$ possesses a limit lamination $\mathcal{L}$. Inspired by [**22**], we make the following definition:

Definition 3.7. We say a quadruple $(\Omega, K, \mathcal{L}, \mathcal{S})$ consisting of

1) a Riemannian three-manifold Ω exhausted by $\{\Omega_i\}$;
2) a closed set $K \subset \Omega$;
3) a proper smooth minimal lamination $\mathcal{L}$ in $\Omega \backslash K$; and
4) a sequence $\mathcal{S} = \{\Sigma_i\}$ of properly embedded minimal surfaces Σ_i in Ω_i

is a *minimal surface sequence* if

1) $\text{sing}(\mathcal{S}) = K$; and
2) $\Sigma_i \backslash K$ converge in $C^{\infty,\alpha}_{loc}(\Omega \backslash K)$ to $\mathcal{L}$ for some $0 < \alpha < 1$.

If all the surfaces in $\mathcal{S}$ are disks, then we say this is a *minimal disk sequence.*

The work of Colding and Minicozzi [**5, 6, 8, 7**] implies that if $(\Omega, K, \mathcal{L}, \mathcal{S})$ is a minimal disk sequence, then K and $\mathcal{L}$ have a great deal of structure. We say a leaf L of $\mathcal{L}$ is *regular at* $p \in K$ if $p \in \overline{L}$, the closure of L in Ω, and there is an $r > 0$ so that $\mathcal{B}_r(p) \cap \overline{L}$ is an embedded smooth surface proper in $\mathcal{B}_r(p)$. Then the following holds.

Proposition 3.8. *If* $(\Omega, K, \mathcal{L}, \mathcal{S})$ *is a minimal disk sequence, then there is an embedded one-dimensional Lipschitz curve* $K' \subset \Omega$ *such that* $K \subset K'$ *and*

1) *if* $p \in K$, *then there is a leaf* $L \in \mathcal{L}$ *which is regular at* p;
2) *if* L *is regular at* $p \in K$, *then* $\overline{L}$ *meets* K *transversely—in the strong sense that* $\overline{L}$ *meets* K' *transversely at* p.

White [**22**] has shown that the regularity of K' can be taken to be C^1. Furthermore, Meeks [**14**] has shown that if $K = K'$, then K' can be taken to be $C^{1,1}$.

We will consider also the following related objects.

Definition 3.9. We say a quadruple $(\Omega, K, \mathcal{L}, \mathcal{L}_0)$ consisting of

1) a connected open subset $\Omega \subset N$;
2) a closed set $K \subset \Omega$;
3) a smooth minimal lamination $\mathcal{L}_0$ in $\Omega \backslash K$; and
4) a smooth proper minimal lamination $\mathcal{L}$ in $\Omega \backslash K$

is a *minimal surface closure*, provided

1) for all $p \in K$ and $\rho > 0$, $\sup_{L \in \{\mathcal{L}_0\}} \sup_{B_\rho(p) \cap L} |A_L| = \infty$; and
2) $\overline{\mathcal{L}}_0 = \mathcal{L}$, where here $\overline{\mathcal{L}}_0$ is the closure of $\mathcal{L}_0$ in $\Omega \backslash K$.

If all the leaves of $\mathcal{L}_0$ are disks, then we say this is a *minimal disk closure.*

The limit laminations $\mathcal{L}$ of minimal disk sequences and of minimal disk closures share many properties. Therefore, it is convenient to introduce the following definition.

Definition 3.10. A smooth minimal lamination $\mathcal{L}$ in a Riemannian three-manifold Ω is a *simple minimal lamination in* Ω if there is a relatively closed set $K \subset \Omega$ and either

1) $(\Omega, K, \mathcal{L}, \mathcal{S})$ is a minimal disk sequence for some $\mathcal{S}$; or
2) $(\Omega, K, \mathcal{L}, \mathcal{L}_0)$ is a minimal disk closure for some $\mathcal{L}_0$.

4. Simple Lifts

In order to proceed we will need a technical definition.

Definition 4.1. Let Σ be an embedded surface in a fixed Riemannian three-manifold Ω. The surface Σ has the *simple lift property* if, for any $\delta > 0$, $\gamma : [0,1] \to \Sigma$ a piece-wise C^1 curve, and open pre-compact subset $U \subset \Sigma$ with $\gamma \subset U$, there exist:

   1) a constant $\epsilon = \epsilon(U, \delta) > 0$;
   2) an embedded minimal disk Δ in Ω; and
   3) a δ-lift of γ, $\widehat{\gamma} : [0,1] \to \mathcal{N}_\delta(U)$,

such that

   1) $\widehat{\gamma} \subset \Delta \cap \mathcal{N}_\epsilon(U)$;
   2) $\Delta \cap \mathcal{N}_\epsilon(U)$ is a δ-graph over U;
   3) the connected component of $\Delta \cap \mathcal{N}_\epsilon(U)$ containing $\widehat{\gamma}$ is a δ-cover of U.

Such $\widehat{\gamma}$ is called a *simple δ-lift* of γ into Ω.

If Σ has the simple lift property in Ω and γ is a curve in Σ, then γ has the *embedded lift property* if there is a $\delta_0 > 0$ so that for all $\delta_0 > \delta > 0$, all simple δ-lifts of γ are embedded. Clearly, if γ is an embedded curve, then it has the embedded lift property.

Throughout this paper we study the topology of minimal surfaces with the simple lift property. This is relevant to the study of the topology of a leaf of a simple minimal lamination thanks to the following proposition.

Proposition 4.2. *Leaves of a simple minimal lamination in Ω have the simple lift property.*

Proof. We first consider the case of a minimal disk closure. If L is a leaf of $\mathcal{L}$ which is a disk, then any curve in L is its own simple δ-lift in any pre-compact open set of L containing such curve and there is nothing to prove. If L is not a disk, then $L \subset \mathcal{L} \backslash \mathcal{L}_0$. Hence, for any point $p \in L$ there exists a sequence of points $p_i \in \mathcal{L}_0$ so that $p_i \to p$.

Note first that the definition of smooth lamination—specifically the existence of Lipschitz straightening maps—implies that for each pre-compact open subset U of L there is a constant $C = C(U)$ so that if $1 > C\lambda > 0$, then, for each leaf L' of $\mathcal{L}_0$, $\mathcal{N}_\lambda(U) \cap L'$ is a—possibly empty—$C\lambda$-graph over U. Given a curve $\gamma : [0,1] \to L$, and U some pre-compact open subset of L so that $\gamma \subset U$, let l denote the length of γ and let d denote the diameter of U. For any $\delta > 0$, choose $\epsilon > 0$ such that $C\epsilon < \min\{1, \delta\}$. Let $\mu = \frac{3}{4} e^{-2C(l+d)}$ and pick L_μ to be a leaf of $\mathcal{L}_0$ which satisfies $\mathcal{N}_{\mu\epsilon}(p) \cap L_\mu \neq \emptyset$ where here $p = \gamma(0)$. Let Γ be a component of $L_\mu \cap \mathcal{N}_\epsilon(U)$ which contains a point $q \in \mathcal{N}_{\mu\epsilon}(p) \cap \Gamma$.

The leaf L_μ is, by definition, a disk and we have chosen $\epsilon > 0$ so that $L_\mu \cap \mathcal{N}_\epsilon(U)$ is a δ-graph over U. We claim that Γ is a δ-cover of

U containing a δ-lift of γ. This follows by showing that any curve in U of length at most $2(l + d)$ starting at p has a lift in Γ starting at q. By construction, this lift is necessarily a δ-lift. Indeed, if $\sigma : [0, T] \to U$ is parameterized by arclength and $\widehat{\sigma} : [0, T'] \to \Gamma$ satisfies $\Pi_\Sigma(\widehat{\sigma}(t)) = \sigma(t)$ for some $0 < T' \leq T$, then

$$\left| \frac{d}{dt} \mathrm{dist}^\Omega(\sigma(t), \widehat{\sigma}(t)) \right| \leq C \mathrm{dist}^\Omega(\sigma(t), \widehat{\sigma}(t))$$

and so

$$\mathrm{dist}^\Omega(\sigma(t), \widehat{\sigma}(t)) \leq e^{Ct} \mathrm{dist}^\Omega(p, q) < \epsilon \mu e^{Ct} < \epsilon,$$

where we used that $t \leq T \leq l + d$ to obtain the final inequality. Furthermore, if $t < T$, then the lift $\widehat{\sigma}(t)$ may be extended past t provided $\mathrm{dist}^\Omega(\sigma(t), \widehat{\sigma}(t)) < \epsilon$. This proves that the leaf of a minimal disk closure has the simple lift property as claimed.

In the case of a minimal disk sequence, the argument is identical to the one above except that it uses the Harnack inequality to obtain the bound on the Lipschitz norms of straightening maps. We refer to Remark 3.6 and to [**7**, Appendix B] for the details on how to obtain this bound. q.e.d.

A surface with the simple lift property is one for which, in an effective sense, the universal cover of the surface can be properly embedded as a minimal disk near the surface. For this reason, to understand the topology of the surface, it is important to understand the lifting behavior of closed curves. With this in mind, we give the following definition.

Definition 4.3. Let $\Sigma \subset \Omega$ be an embedded minimal surface with the simple lift property. If $\gamma \colon [0, 1] \to \Sigma$ is a piece-wise C^1 closed curve, then γ has the *open lift property* if there exists a $\delta_0 > 0$ so that, for all $\delta_0 > \delta > 0$, γ does not have a closed simple δ-lift $\widehat{\gamma} : [0, 1] \to \mathcal{N}_\delta(\Sigma)$. Otherwise, γ has the *closed lift property*.

If a closed curve γ has the closed lift property, then there is a sequence $\delta_i \to 0$ so that there are closed simple δ_i-lifts $\widehat{\gamma}_i$ of γ. If it is possible to choose these lifts to be embedded, we say γ has the *embedded closed lift property*.

The next lemma says that if two loops satisfying certain geometric conditions have the open lift property, then their commutator has the closed lift property. Very roughly speaking, it does this by constructing an "effective" homomorphism from the space of loops in the leaf to $\mathbb{Z}$. Indeed, a curve $\widehat{\gamma}$ is a lift of γ if and only if $\widehat{\gamma}^{-1}$ is a lift of γ^{-1}. Thus, γ has the closed lift property if and only if γ^{-1} does.

Proposition 4.4. *Let $L \subset \Omega$ be an embedded minimal surface with the simple lift property and let*

$$\alpha : [0, 1] \to L \text{ and } \beta : [0, 1] \to L$$

be closed piece-wise C^1 curves satisfying the following properties:

    1) both α and β have the open lift property;
    2) $\alpha \cap \beta = \{p_0\}$ where $p_0 = \alpha(0) = \beta(0)$;
    3) there exists a two-sided pre-compact open set $U \subset L$ with $\alpha \cup \beta \subset U$.

Then the curve $\mu = \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$ has the closed lift property. If, in addition, both α and β have the embedded lift property, then either μ has the embedded closed lift property, or one of the following two curves has the embedded closed lift property:

$$\alpha \circ \beta \quad or \quad \beta \circ \alpha^{-1}.$$

Proof. Consider a sequence $\delta_i \to 0$. As Σ has the simple lift property, there exist constants ϵ_i, embedded minimal disks Δ_i, and simple δ_i-lifts of μ, $\widehat{\mu}_i$, so that $\Delta_i \cap \mathcal{N}_{\epsilon_i}(U)$ is a δ_i-graph over U, $\widehat{\mu}_i \subset \Delta_i$ and the connected component Γ_i of $\Delta_i \cap \mathcal{N}_{\epsilon_i}(U)$ containing μ_i is a δ_i-cover of U. By reparameterizing appropriate restrictions of $\widehat{\mu}_i$, we obtain lifts of α, α^{-1}, β, and β^{-1}. We write $\widehat{\mu}_i = \widehat{\alpha}_i \circ \widehat{\beta}_i \circ \widehat{\alpha}_i^{-1} \circ \widehat{\beta}_i^{-1}$ where the $\widehat{\alpha}_i, \widehat{\beta}_i, \widehat{\alpha}_i^{-1}, \widehat{\beta}_i^{-1} \colon [0,1] \to \Gamma_i$ are lifts of the $\alpha, \beta, \alpha^{-1}, \beta^{-1}$.

Setting $p = \mu(0)$, we pick a small simply-connected neighborhood V of p that satisfies $V \subset U$. Because the Δ_i are embedded, there is a natural way to order by height the components of $\Pi_L^{-1}(V) \cap \Delta_i$. We denote these ordered components by $\widehat{V}_i(1), \ldots, \widehat{V}_i(n_i)$. Let $\widehat{p}_i(0) = \widehat{\alpha}_i(0)$, $\widehat{p}_i(1) = \widehat{\alpha}_i(1) = \widehat{\beta}_i(0)$, $\widehat{p}_i(2) = \widehat{\beta}_i(1) = \widehat{\alpha}_i^{-1}(0)$, $\widehat{p}_i(3) = \widehat{\alpha}_i^{-1}(1) = \widehat{\beta}_i^{-1}(0)$, $\widehat{p}_i(4) = \widehat{\beta}_i^{-1}(1)$ and notice that $\widehat{p}_i(j) \in \widehat{V}_i(l(i,j))$ for some function l. Let

$$m_i[\alpha] = l(i,1) - l(i,0),$$
$$m_i[\beta] = l(i,2) - l(i,1),$$
$$m_i[\alpha^{-1}] = l(i,3) - l(i,2), \text{ and}$$
$$m_i[\beta^{-1}] = l(i,4) - l(i,3)$$

represent the (signed) number of sheets between the endpoints of $\widehat{\alpha}_i$, $\widehat{\beta}_i$, $\widehat{\alpha}_i^{-1}$, and $\widehat{\beta}_i^{-1}$. As both $\widehat{\alpha}_i$ and $\widehat{\beta}_i$ are open lifts these numbers are never zero. We now prove that $m_i[\alpha] = -m_i[\alpha^{-1}]$ and $m_i[\beta] = -m_i[\beta^{-1}]$ and, hence, $\widehat{\mu}_i$ is closed. We consider two cases: $m_i[\alpha]m_i[\beta] > 0$ and $m_i[\alpha]m_i[\beta] < 0$.

In the first case we assume, without loss of generality, that $m_i[\alpha], m_i[\beta] > 0$. Using the fact that the Δ_i are embedded and that U is two-sided, we see that there is a disjoint family of "parallel" lifts of α which we denote by $\widehat{\alpha}_i[j]$. The first member of the family is $\widehat{\alpha}_i[0] = \widehat{\alpha}_i$ and the subsequent members of the family are the lifts $\widehat{\alpha}_i[k]$ of α which satisfy $\widehat{\alpha}_i[k](0) = l(i,0) + k$. Namely, the lift $\widehat{\alpha}_i[k]$ starts k sheets above $\widehat{\alpha}_i(0)$. By the embeddedness of Δ_i and the two-sidedness of U, the

signed number of graphs between $\widehat{\alpha}_i[0](t)$ and $\widehat{\alpha}_i[k](t)$ is constant in t. Hence, $\widehat{\alpha}_i[k](1) = l(i, 1) + k$—that is, the lifts have an endpoint k sheets above the endpoint of $\widehat{\alpha}_i$. Clearly, the $\widehat{\alpha}_i[k]$ are well-defined as long as $k \leq m_i[\beta]$. Furthermore, $\widehat{\alpha}_i[m_i[\beta]]$ has an endpoint which is the same as the endpoint of $\widehat{\beta}_i$. Hence, since $\widehat{\alpha}_i[m_i[\beta]]^{-1}$ is a lift of α^{-1} starting at $\widehat{\beta}_i(1)$, the lift $\widehat{\alpha}_i[m_i[\beta]]^{-1}$ must be $\widehat{\alpha}_i^{-1}$. That is, $m_i[\alpha] = -m_i[\alpha^{-1}]$. An identical argument shows that $m_i[\beta] = -m_i[\beta^{-1}]$.

In the second case, we may suppose without loss of generality that $m_i[\alpha] > 0$ and $m_i[\beta] < 0$. We first assume that $m_i[\alpha] + m_i[\beta] + m_i[\alpha^{-1}] \geq 0$ and obtain a contradiction. Under this hypothesis the endpoint of $\widehat{\alpha}_i^{-1}$ is not below that of the initial point of $\widehat{\alpha}_i$. As in the preceding argument, we can construct a family of "parallel" lifts of α^{-1}. The first member of this family is $\widehat{\alpha}_i^{-1}$, the second lift in the family starts just below $\widehat{\alpha}_i^{-1}$, and the last lift in such family, $\widehat{\alpha}_i'$ has endpoint that is the initial point of $\widehat{\alpha}_i$. As before, the number of graphs between the start and endpoints of a lift in this family is constant equal to $m_i[\alpha^{-1}]$. Since $\widehat{\alpha}_i'$, has an endpoint that is the initial point of $\widehat{\alpha}_i$, the lift $(\widehat{\alpha}_i')^{-1}$ must be $\widehat{\alpha}_i$. This implies that $m_i[\alpha] = -m_i[\alpha^{-1}]$. Since $m_i[\beta] < 0$, this leads to a contradiction. Hence, $m_i[\alpha] + m_i[\beta] + m_i[\alpha^{-1}] < 0$. Again one constructs a family of parallel lifts starting from $\widehat{\alpha}_i$ and ending with a curve with initial point the endpoint of $\widehat{\alpha}_i^{-1}$, which again implies that $m_i[\alpha] = -m_i[\alpha^{-1}]$. The same argument shows that in this case $m_i[\beta] = -m_i[\beta^{-1}]$.

Finally, we note that if α and β have the embedded lift property, then, because they meet at only one point, the curves $\widehat{\alpha}_i \circ \widehat{\beta}_i$, $\widehat{\beta}_i \circ \widehat{\alpha}_i^{-1}$, and $\widehat{\alpha}_i^{-1} \circ \widehat{\beta}_i^{-1}$ are all embedded. Hence, the only way that $\widehat{\mu}_i$ can fail to be embedded is if one of the first two is closed.   q.e.d.

5. Main Proof

Rather than prove Theorems 1.1 and 1.2 directly, we prove slightly more general results. To do so, we will restrict the geometry of the three-manifolds Ω we consider.

Assumption 5.1. Let Ω be the interior of a complete, oriented, three-manifold with (possibly empty) boundary $N = \overline{\Omega}$ satisfying the following properties:

1) The boundary of N is mean-convex.
2) There is an exhaustion $\{\Omega_t\}_{t \in [0,1)}$ of Ω so that each Ω_t is pre-compact in N, $\partial \Omega_t$ is mean-convex, and $\{\partial \Omega_t\}_{t \in (0,1)}$ foliates $\Omega \backslash \overline{\Omega}_0$.
3) Ω contains no closed minimal surfaces.

Note that N does not have to be compact. For instance, $\mathbb{H}^3$ satisfies Assumption 5.1. Minimal surfaces in such an Ω automatically satisfy a certain type of uniform isoperimetric inequality. This follows immediately from work of B. White [24].

Lemma 5.2. *If Ω satisfies the conditions of Assumption 5.1, $U \subset \Omega$ is a precompact subset of Ω and Σ is a compact minimal surface with boundary $\partial\Sigma \subset U$, then:*

1) *There is a pre-compact open set U' depending only on U so that $\Sigma \subset U'$.*
2) *There is an increasing function, $\Psi_U : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, depending only on U and satisfying $\Psi_U(0) = 0$ and*

$$|\Sigma| \leq \Psi_U(|\partial\Sigma|).$$

Proof. By Assumption 5.1, there is a pre-compact open subset $U' = \Omega_{t_0}$ with $U \subset \Omega_{t_0}$. As $\Omega\backslash\Omega_{t_0}$ is foliated by mean-convex subsets, the strong maximum principle and the fact that $\partial\Sigma \subset \Omega_{t_0}$ implies that $\Sigma \subset \Omega_{t_0}$. Finally, the existence of the function Ψ_U follows immediately from [**24**, Theorem 2.1] applied to $\overline{U'}$. q.e.d.

Under Assumption 5.1 we have the following gluing property that allows us to "fill in" curves with the embedded closed-lift property.

Lemma 5.3. *Suppose that Ω satisfies Assumption 5.1. If $L \subset \Omega$ is an embedded minimal surface with the simple lift property and $\gamma : [0,1] \to L$ has the embedded closed lift property, then there exists a smooth minimal surface Δ properly embedded in $\Omega\backslash\gamma$ so that:*

1) *Δ has finite area and is contained in a compact subset of Ω;*
2) *$\gamma = \overline{\Delta}\backslash\Delta$ and $\Delta \cup \gamma$ is connected;*
3) *$\Delta \cap L$ is a non-empty open and closed subset of $L\backslash\gamma$; and*
4) *if γ is embedded, then Δ is a disk.*

Proof. By hypothesis, there exists a sequence of closed embedded simple $\frac{1}{n}$-lifts γ_n of γ. Hence, each γ_n bounds a minimal disk Δ'_n inside of the minimal disks Δ_n. Clearly, there is a fixed pre-compact subset U of Ω containing γ together with all of the γ_n. Furthermore, the length of each γ_n is bounded by twice the length of γ and so there is a uniform bound on $|\partial\Delta'_n|$. As a consequence, Lemma 5.2 implies that there is a precompact subset U' of Ω so that the sequence of disks Δ'_n are contained in U' and, moreover, have uniformly bounded area. A result of Schoen and Simon [**19**]—see Theorem B.1 for the statement—then gives uniform curvature bounds for the Δ'_n on compact subsets of $\Omega\backslash\gamma_n$. Hence, up to passing to a subsequence, Theorem 3.5 and the area bounds imply that the Δ'_n converge in $C^\infty_{loc}(\Omega\backslash\gamma)$ to a properly embedded minimal surface $\Delta \subset \Omega\backslash\gamma$ of finite area and bounded curvature on compact sets of $\Omega\backslash\gamma$. As each Δ'_n is contained in U', it follows that $\Delta \subset \overline{U'}$. This proves Item (1). Item (2) follows from the set theoretic convergence of $\bar{\Delta}'_n$.

As the Δ_n are contained in $\frac{1}{n}$ graphs over a fixed neighborhood V of γ, $\Delta\backslash\gamma$ contains a component of $V\backslash\gamma$ and so $\Delta\backslash\gamma$ is non-empty and is contained in L. Indeed, the nature of the convergence implies that $\Delta\backslash\gamma$ is an open and closed subset of $L\backslash\gamma$.

If γ is embedded, then the $\widehat{\gamma}_n$ converge to γ with multiplicity one. Clearly, to prove Item (4) it suffices to show that the Δ'_n also converge to Δ with multiplicity one. This is most conveniently done using the language of varifolds—we refer to [**20**] for details.

Let V_n be the integer multiplicity rectifiable varifold associated to $\overline{\Delta}'_n$. As γ is piecewise smooth, the nature of the convergence of γ_n toward γ implies that $\overline{\Delta}'_n$ has uniformly bounded first variation. Indeed, since Δ_n is minimal, the first variation measure of V_n is supported along γ_n. As the V_n also have uniformly bounded mass, we may apply the compactness theory for integer multiplicity rectifiable varifolds to see that (up to passing to a further subsequence) V_n converges in the sense of varifolds (in Ω) to an integer multiplicity rectifiable varifold V. It follows from the nature of the convergence that $\operatorname{spt} V = \overline{\Delta}$ and, moreover, at any point of Δ the multiplicity of V is a positive integer. Moreover, the first variation measure of V is supported along γ. Notice that as L is a smooth minimal surface and $\operatorname{spt} V \subset L$, the constancy theorem implies that the multiplicity of each component of $\operatorname{spt} V \backslash \gamma$ is constant.

Fix a point $p \in \gamma$ and a small open neighborhood $W \subset L$ about p chosen small enough so that γ divides W into exactly two components W_- and W_+. If both W_- and W_+ meet $\operatorname{spt} V$, then it follows from the strong unique continuation principle for smooth minimal surfaces that $\operatorname{spt} V$ is a closed minimal surface in Ω. This violates Assumption 5.1 and so—up to relabelling—we may assume that $W_- \cap \operatorname{spt} V = \emptyset$. As γ_n converge to γ with multiplicity one, the nature of the convergence of Δ'_n to Δ then immediately implies that the multiplicity of V is one which proves the claim. q.e.d.

Corollary 5.4. *Suppose Ω satisfies Assumption 5.1. If $L \subset \Omega$ is an embedded minimal surface with the simple lift property and γ is a closed embedded curve in L with the closed lift property, then γ is separating.*

Proof. Let Δ be the surface given by Lemma 5.3. If γ is non-separating, then $L \backslash \gamma$ is connected and so, by the previous lemma, $L \backslash \gamma \subset \Delta$. Therefore, by strong unique continuation for smooth minimal surfaces, $\overline{\Delta} = \Delta \cup \gamma = \Delta \cup L$ is a closed minimal surface of finite area in Ω, which contradicts Assumption 5.1. q.e.d.

We are now in a position to prove that surfaces with the simple lift property in regions satisfying Assumption 5.1 must have genus zero.

Proposition 5.5. *Suppose Ω satisfies Assumption 5.1. If $L \subset \Omega$ is an embedded minimal surface with the simple lift property in Ω, then L has genus zero.*

Proof. Arguing by contradiction, suppose that L has genus greater than zero. Then, by the classification of surfaces, there exist two piecewise smooth, non-separating Jordan curves $\alpha, \beta : [0,1] \to L$ and a two-sided pre-compact set $U \subset L$ such that the following holds:

- $\alpha \cap \beta = p = \alpha(0) = \beta(0) = \alpha(1) = \beta(1)$;
- $L \backslash (\alpha \cup \beta)$ is connected;
- $\alpha \cup \beta \subset U$.

By Corollary 5.4 both curves have the open lift property and so α and β satisfy the hypothesis of Proposition 4.4. Hence, there exists a sequence of closed curves ν_n that are closed simple $\frac{1}{n}$-lifts of $\nu = \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$. Proposition 4.4 further tells us that either the curve $\mu = \nu$, the curve $\mu = \alpha \circ \beta$, or the curve $\mu = \beta \circ \alpha^{-1}$ has the embedded closed lift property. In all cases, Lemma 5.3 gives a minimal surface Δ properly embedded in $\Omega \backslash \mu$ and that $[\Delta \backslash \mu] \cap L$ is a non-empty open and closed subset of $L \backslash \mu$. As $L \backslash \mu$ is connected, $\Delta \cap L = L \backslash \mu$. Hence, by the strong unique continuation property of smooth minimal surfaces, $\overline{\Delta} = \Delta \cup L$ is a properly embedded minimal surface in Ω of finite area and we contradict Assumption 5.1.   q.e.d.

Clearly, Theorem 1.2 follows from Propositions 4.2 and 5.5. Indeed, the region Ω of Theorem 1.2 can be seen to satisfy Assumption 5.1 by taking the exhaustion to be Ω itself.

6. Minimal Disk Sequences

In this section we conclude the proof of Theorem 1.1. In particular, we show that in the case of minimal disk sequences, the leaves of the limit lamination are two-sided.

Proposition 6.1. *Suppose Ω satisfies Assumption 5.1. If $(\Omega, K, \mathcal{L}, \mathcal{S})$ is a minimal disk sequence and L is a leaf of $\mathcal{L}$, then*

1) If L is two-sided, then L is either a disk or an annulus.
2) If L is one-sided, then L is a Möbius band.

Proof. We will argue by contradiction. For any three separating Jordan curves γ_1, γ_2, and γ_3 in L with the property that no one of the curves separates the other two, it is the case that $L \backslash (\gamma_1 \cup \gamma_2 \cup \gamma_3)$ has four components L_1, L_2, L_3, L_4. Label the L_i so that $\overline{L}_1 \cap (\gamma_2 \cup \gamma_3) = \emptyset$, $\overline{L}_2 \cap (\gamma_1 \cup \gamma_3) = \emptyset$, $\overline{L}_3 \cap (\gamma_1 \cup \gamma_2) = \emptyset$, and $\gamma_1 \cup \gamma_2 \cup \gamma_3 \subset \overline{L}_4$. By Proposition 5.5, L has genus zero. Hence, if L is neither a disk, an annulus, nor a Möbius band, then the classification of surfaces implies that the γ_i may be chosen so that

- L_1, L_2 and L_3 are not disks;
- L_4 is two-sided.

We claim that for such a choice, γ_1, γ_2, and γ_3 have the open lift and embedded lift property. Indeed, being embedded curves, they clearly

have the embedded lift property. Suppose γ_i had the closed lift property; then applying Lemma 5.3 would give that γ_i is the boundary of a disk Δ and $\Delta \cap L$ is equal to one of the two components of $L \backslash \gamma_i$. By the Colding–Minicozzi one-sided curvature estimate of [8], and the construction of Δ in Lemma 5.3, $\Delta \cap \mathcal{S} = \emptyset$ and therefore one of the components of $L \setminus \gamma_i$ is a disk, contradicting our choice of γ_i.

Let σ be an embedded arc in L_4 which connects $\gamma_1(0)$ to $\gamma_2(0)$. Notice that the classification of surfaces tells us that such σ exists and does not separate L_4. Consider the new closed curve $\gamma_4 = \sigma^{-1} \circ \gamma_2 \circ \sigma$. By an argument analogous to the one described before, this curve must also have the open lift property. In fact, the embeddedness of σ and of γ_2 and the fact that γ_2 has the open-lift property imply that γ_4 has the embedded-lift property.

We now consider the closed curve $\nu = \gamma_1 \circ \gamma_4 \circ \gamma_1^{-1} \circ \gamma_4^{-1}$. Proposition 4.4 implies that either the curve $\mu = \nu$, the curve $\mu = \gamma_1 \circ \gamma_4$, or the curve $\mu = \gamma_4 \circ \gamma_1^{-1}$ has the embedded closed lift property. In all cases, let Δ be the embedded minimal surface given by Lemma 5.3. The fact that Δ is connected and that $\gamma_1 \cup \gamma_2 \subset \overline{\Delta}$ together imply that $L_4 \subset \Delta$. However, as γ_3 is a Jordan curve disjoint from ν, γ_3 must be the limit of embedded closed curves in Δ_n—that is, it has the embedded closed lift property. This is a contradiction and proves the proposition. q.e.d.

Proposition 6.2. *Let Ω satisfy Assumption 5.1. If $(\Omega, K, \mathcal{L}, \mathcal{S})$ is a minimal disk sequence and L is a leaf of $\mathcal{L}$, then L is two-sided.*

Proof. Suppose that L is a one-sided leaf of $\mathcal{L}$. By Proposition 6.1, L is a Möbius band. As a consequence, there is a closed Jordan curve $\gamma : [0,1] \to L$ that is non-separating and so, by Corollary 5.4, has the open lift property. Let U be an open pre-compact neighborhood of γ and pick $\epsilon > 0$ so that $\overline{\mathcal{N}}_\epsilon(U)$ is a regular neighborhood. As γ is non-separating, U is one-sided and, indeed, the surface $M = \Pi_L^{-1}(\gamma) \cap \overline{\mathcal{N}}_\epsilon(U)$ is a closed Möbius band.

Let Σ_i be the surfaces in $\mathcal{S}$. There are curves, $\widehat{\gamma}_i$, which are components of $\Sigma_i \cap M$ containing δ-lifts of γ for any δ sufficiently small. In particular, the $\widehat{\gamma}_i$ are proper, but not closed, in M. Furthermore, after possibly shrinking ϵ, they are monotone in the sense that $(\widehat{\gamma}_i')^\top \neq 0$ and $\widehat{\gamma}_i$ meets ∂M transversely. Finally, for i large enough, the map $\Pi_L : \widehat{\gamma}_i \to \gamma$ contains a three-fold cover. We claim this yields a contradiction.

To see this, consider $\pi : \widetilde{M} \to M$ the oriented double cover of M. As $\widetilde{M}$ is an annulus and $\widehat{\gamma}_i$ is monotone:

- $\widetilde{M} = \mathbb{S}^1 \times [-1, 1]$ with coordinates (θ, z);
- $M = \widetilde{M}/_\sim$ with $(\theta, z) \sim (\theta + \pi, -z)$;
- $\mathbb{S}^1 \times \{0\} = \pi^{-1}(\gamma)$;
- $\widetilde{\gamma}_i = \pi^{-1}(\widehat{\gamma}_i)$ is a graph over $\mathbb{S}^1$.

As $\widetilde{\gamma}_i$ is a graph, we may parametrize $\widetilde{\gamma}_i(\theta)$ as $(\theta, v_i(\theta))$ for $\theta \in [0, T_i]$ and some continuous function v_i with $|v_i(0)| = |v_i(T_i)| = 1$ and $|v_i(\theta)| < 1$ for $\theta \in (0, T_i)$. Since $\widehat{\gamma}_i$ contains a three-fold cover of γ, $T_i > 3\pi$. The embeddedness of $\widehat{\gamma}_i$ implies that for any $\theta \in [0, T_i - \pi]$, $v_i(\theta + \pi) \neq -v_i(\theta)$ and for any $\theta \in [0, T_i - 2\pi]$, $v_i(\theta + 2\pi) \neq v_i(\theta)$. Without loss of generality, we assume that $v_i(0) = -1$. Consider the continuous function g_i defined for $\theta \in [0, T_i - 2\pi]$ by $g_i(\theta) = v_i(\theta + 2\pi) - v_i(\theta)$. Notice that $g_i(T_i - 2\pi) < 0$ if and only if $v_i(T_i) = -1$. Hence, as $g_i(0) > 0$, the intermediate value theorem implies that $v_i(T_i) = 1$. Finally, consider the continuous function f_i defined for $\theta \in [0, T_i - \pi]$ by $f_i(\theta) = v_i(\theta + \pi) + v_i(\theta)$. Clearly, $f_i(0) < 0$ and $f_i(T_i - \pi) > 0$. Hence the intermediate value theorem contradicts the fact that $f_i(\theta) \neq 0$, completing the proof. q.e.d.

We now finish the proof of Theorem 1.1. For completeness, we recall its statement.

Theorem 1.1. *Let Ω be the interior of a compact oriented three-manifold $N = \overline{\Omega}$ with mean-convex boundary. If Ω contains no closed minimal surfaces and $(\Omega, K, \mathcal{L}, \mathcal{S})$ is a minimal disk sequence, then the leaves of $\mathcal{L}$ are either disks or annuli. Furthermore, if L is a leaf of $\mathcal{L}$ with the property that $\overline{L}$—the closure in Ω of L—is a properly embedded minimal surface, then $\overline{L}$ is either a disk or it is an annulus which is disjoint from K.*

Proof. We first note that Ω satisfies the conditions of Assumption 5.1 by taking the exhaustion to be Ω itself. Moreover, each leaf of $\mathcal{L}$ has the simple lift property by Proposition 4.2. Hence, Propositions 6.1 and 6.2 together imply L is either a disk or an annulus. The remainder of the theorem follows from the deeper result of Colding and Minicozzi that we summarized in Proposition 3.8. Indeed, if L is a leaf of $\mathcal{L}$ with $\overline{L}$ a properly embedded minimal surface, then it is regular at each $p \in \overline{L} \cap K$. Hence, by Proposition 3.8, $\overline{L} \cap K$ is a discrete set of points in $\overline{L}$. As L is either a disk or an annulus, if $\overline{L}$ is an annulus it must be disjoint from K.   q.e.d.

Appendix A. Examples

A.1. One-sided limit leaf. In this section, we construct a simply-connected minimal surface M embedded in a solid torus that is not properly embedded. Moreover, its closure is a lamination in the solid torus consisting of three leaves; the leaf M and two limit leaves. One limit leaf is an annulus while the other is a Möbius band.

Let T be a solid torus obtained by revolving a disk $\mathcal{D}$ in the (x_1, x_3)-plane around the x_3-axis. We take $\mathcal{D}$ small enough so that T is mean-convex and there exists a stable minimal Möbius band M embedded in T whose double cover is also stable and with boundary a simple closed curve in ∂T. For the existence of such a surface, we refer to [**17**]. Since

the double cover of M is stable, a normal neighborhood of M can be foliated by minimal surfaces and, except for M itself, the leaves of this foliation are two-sided annuli. Let Σ denote the outermost leaf of this foliation and let W denote the open region between M and Σ. Let Σ_t, $t \in [0.1]$, be an indexing of the leaves of the foliation with $\Sigma_0 = M$ and $\Sigma_1 = \Sigma$. Let $\widetilde{T}$ be the universal cover of T with the induced metric. We realize $\widetilde{T}$ as

$$\widetilde{T} = \{(x, y, z) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^3$$

in a manner so that for any $\alpha \in \mathbb{R}$ the map

$$G_\alpha \colon \widetilde{T} \to \widetilde{T}, \quad G(x, y, z) = (x, y, z + \alpha)$$

is an isometry and for any $t \in \mathbb{R}$ the set

$$B_t \colon = \{(x, y, z) \in \widetilde{T} \mid z = t\}$$

is a minimal surface that is a lift of a disk obtained by intersecting T with a vertical plane containing the z-axis. The maps $G_{2\pi n}$, $n \in \mathbb{Z}$ are the deck transforms of $\widetilde{T}$. Let $\Pi \colon \widetilde{T} \to T$ denote the natural projection. Given an embedded surface $S \in \widetilde{T}$, if for any $n \in \mathbb{Z}\backslash\{0\}$ it holds that $G_{2\pi n}(S) \cap S = \emptyset$, then $\Pi(S)$ is embedded in T.

The Möbius band M lifts to a strip $\widetilde{M}$ with boundary consisting of two curves in $\partial\widetilde{T}$ and, being a lift, is invariant by the deck transforms $G_{2\pi n}$, $n \in \mathbb{Z}$. The strip $\widetilde{M}$ is two-sided and separates $\widetilde{T}$ into two components. Each leaf Σ_t of the foliation, $t \in (0, 1]$, lifts to two strips $\widetilde{\Sigma}_t^+$ and $\widetilde{\Sigma}_t^-$ on opposite sides of $\widetilde{M}$ and this gives a foliation of a two-sided normal neighborhood of $\widetilde{M}$. We shall denote by $\widetilde{W}$ the region foliated by the leaves $\widetilde{\Sigma}_t^+$ and denote such leaves by $\widetilde{\Sigma}_t$. Given a point $p \in \widetilde{W}$, then $p \in \widetilde{\Sigma}_t$ for a certain $t \in (0, 1)$ and we denote that t by $t(p)$. Note that $\partial\widetilde{W} \cap \partial\widetilde{T}$ consists of two disconnected components, Δ_1 and Δ_2, and let $\alpha_i \subset \Delta_i$, $i = 1, 2$ be analytic curves such that the following holds:

- α_i intersects $\partial\widetilde{\Sigma}_t$, $t \in (0, 1)$, in exactly one point;
- α_i converges to $\partial\widetilde{\Sigma}_1 \cap \Delta_i$ as z goes to infinity and to $\partial\widetilde{M} \cap \Delta_i$ as z goes to minus infinity.

Let $\widetilde{W}_n$ be the region in $\widetilde{W}$ in between the minimal disks $B_{\pm 2\pi n}$. Then, $\widetilde{W} = \bigcup_n \widetilde{W}_n$ and $\partial\widetilde{W}_n$ consists of six surfaces: four minimal surfaces, $\widetilde{\Sigma}_0^n = \widetilde{M} \cap \widetilde{W}_n$, $\widetilde{\Sigma}_1^n = \widetilde{\Sigma}_1 \cap \widetilde{W}_n$, $B_+^n = B_{2\pi n} \cap \widetilde{W}_n$, and $B_-^n = B_{-2\pi n} \cap \widetilde{W}_n$, and two mean convex surfaces $\Delta_1^n = \Delta_1 \cap \widetilde{W}_n$ and $\Delta_2^n = \Delta_2 \cap \widetilde{W}_n$. Since the contact angle between such surfaces is less than π, the boundary of $\widetilde{W}_n$ is mean-convex and a good barrier to solve the Plateau problem.

Let $\gamma_n \in \partial W_n$ be a piece-wise smooth simple closed curve constructed in the following way. The curve γ_n is given by the union $\alpha_1^n \cup \beta_+^n \cup \alpha_2^n \cup \beta_-^n$ where $\alpha_i^n = \alpha_i \cap \widetilde{W}_n$. The curve β_+^n is an arc connecting the endpoints, p_i^n of α_i^n in B_+^n. If $t_n = t(p_1^n) = t(p_2^n)$, then we take β_+^n to lie in Σ_{t_n}.

Otherwise, we take β_+^n to intersect each Σ_t in at most one point. We choose the curve β_-^n in an analogous manner in B_-^n. Clearly, by our choices of α_i, for n sufficiently large,

$$\max_{p \in \beta_+^n}\{t(p)\} < \min_{p \in \beta_-^n}\{t(p)\}.$$

This implies that for any n sufficiently large and any $m \in \mathbb{Z}\backslash\{0\}$, then

$$G_{2\pi m}(\gamma_n) \cap \gamma_n = \emptyset.$$

By a result in Meeks and Yau [18], γ_n is the boundary of an embedded, area minimizing disk $D_n \subset \widetilde{W}_n$. Since it is area minimizing and, for n large, $\partial G_{2\pi m}(D_n) \cap \gamma_n = \emptyset$ for any $m \in \mathbb{Z}\backslash\{0\}$, it follows that

$$G_{2\pi m}(D_n) \cap D_n = \emptyset,$$

given that $\Pi(D_n)$ is also embedded. Moreover, since D_n is area minimizing and α_i^n, $i = 1, 2$, are analytic curves, it satisfies curvature estimates up to α_i, $i = 1, 2$ and a standard compactness argument gives that it converges to a complete simply-connected minimal surface D_∞ embedded in $\widetilde{T}$ with boundary α_i, $i = 1, 2$. By construction,

$$G_{2\pi m}(D) \cap D = \emptyset$$

for any $m \in \mathbb{Z}\backslash\{0\}$; therefore, if we let $D = \Pi(D_\infty)$, then D is a complete embedded disk. Clearly it is not properly embedded T. By curvature estimates for stable minimal surfaces, $\overline{D}$, the closure of D in T, is a minimal lamination. We claim that $\overline{D}$ consists of three leaves, D itself and two limit leaves D_1 and D_2. By construction, $\overline{D}$ contains a compact leaf D_1 with boundary $\partial\Sigma$ and a compact leaf D_2 with boundary ∂M. Using the foliation Σ_t and the strong maximum principle, one concludes that $D_1 = \Sigma$ and $D_2 = M$.

A.2. Torus limit leaf. In this section we construct a three-manifold, Ω, and a complete, embedded disk $\Delta \subset \Omega$ whose closure, $\overline{\Delta}$, is a minimal lamination in Ω, one of whose leaves is a minimal torus. More specifically, we take $\Omega = \mathbb{T}^2 \times \mathbb{R}$ together with a certain metric for which Assumption 5.1 does not hold. In Ω we construct an embedded minimal disk Δ that is not properly embedded and so the closure of Δ is a proper minimal lamination in Ω consisting of five leaves; the leaf $L_1 = \Delta$ and four limit leaves. Two of the limit leaves are the tori $L_2 = \mathbb{T}^2 \times \{-1\}$ and $L_3 = \mathbb{T}^2 \times \{1\}$; the other two; L_4 and L_5, are non-proper annuli with $\overline{L}_4 = \overline{L}_5 = L_2 \cup L_3$. The original idea for this construction is due to D. Hoffman; we refer also to [3] for a related construction.

We begin by constructing a metric g on the cylinders

$$C = \mathbb{S}_\theta^1 \times \mathbb{R}_t.$$

Consider the metric

$$g_0 = (2 + \cos \pi t)d\theta^2 + dt^2$$

and the foliation of C by circles, $\mathbb{S}^1[t] = \mathbb{S}^1 \times \{t\} \subset C$. It is an elementary computation to see that these circles all have constant curvature. Moreover, the leaves which are geodesics are $\mathbb{S}^1[i]$ for $i \in \mathbb{Z}$. When $i \in 2\mathbb{Z}$ these geodesics are unstable, while for $i \in 2\mathbb{Z} + 1$ they are stable. Let U be the connected component of $C \backslash \left(\mathbb{S}^1[-1] \cup \mathbb{S}^1[1] \right)$ which contains $(0,0)$. Similarly, we consider the foliation $\alpha_\theta = (\{\theta\} \times \mathbb{R}) \cap U$ of U. It is clear that all the leaves of this foliation are geodesics. Finally, let us denote by T_v the "translation" map $T_v((\theta, t)) = (\theta + v, t)$, which is clearly an isometry, and by R the isometric involution given by $R((\theta, t)) = (-\theta, -t)$.

Standard methods—e.g., a shooting method or a minimization procedure in the universal cover of C—produce an embedded geodesic $\gamma_+ : [0, \infty)_s \to \gamma_+ \subset U$ with $\gamma_+(0) = (0,0)$ and so that the t coordinate of $\gamma_+(s)$ is monotonically increasing in s. Here s is the arclength parameter. It is clear that γ_+ must accumulate at $\mathbb{S}^1[1]$. Let $\gamma = \gamma_+ \cup R(\gamma_+)$. This is a non-proper geodesic in C which accumulates at $\mathbb{S}^1[-1] \cup \mathbb{S}^1[1]$. It follows also from the construction that if $\gamma_v := T_v(\gamma)$, then $\{\gamma_v\}_{v \in \mathbb{R}}$ is a foliation of U. With that in mind, let $\gamma_- = \gamma_{-\pi/2}$ and $\gamma_+ = \gamma_{\pi/2}$ and let V be the component of $U \backslash (\gamma_- \cup \gamma_+)$ which contains $(0,0)$.

We now modify the metric g_0, so that geodesics which pass through $(0,0)$ are unstable. To that end, pick a compactly supported function $\phi \in C_0^\infty(V)$ so that

- $0 \le \phi \le 1$;
- $\mathrm{spt}(\phi) \subset B_{2\epsilon} \subset V \cap \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (-1, 1)$;
- $\phi \circ R = \phi$;
- $\phi = 1$ on B_ϵ.

Here B_r is the geodesic ball (with respect to g_0) about $(0,0)$ of radius r and we choose ϕ so that 2ϵ is smaller than the injectivity radius of g_0 at $(0,0)$. Now fix a point $p \in \mathbb{S}^2$ and let $g_{\mathbb{S}^2}$ be the round metric of curvature one on $\mathbb{S}^2$. We denote by $\mathcal{B}_r$ the geodesic ball of radius r in $\mathbb{S}^2$ about p. As $B_{2\epsilon}$ and $\mathcal{B}_{\frac{7}{8}\pi}$ are disks, there is a smooth diffeomorphism

$$\Psi : B_{2\epsilon} \to \mathcal{B}_{\frac{7}{8}\pi}.$$

Moreover, we may choose this smooth diffeomorphism so that

- $\Psi((0,0)) = p$;
- $\Psi(B_\epsilon) = \mathcal{B}_{\frac{3}{4}\pi}$;
- $\Psi \circ R = \widetilde{R} \circ \Psi$—here $\widetilde{R}$ is the isometry of $\mathbb{S}^2$ given by rotating $180°$ around the line through p and $-p$.

We now set

$$g_1 = (1 - \phi)g_0 + \phi \Psi^* g_{\mathbb{S}^2}.$$

Geodesics of g_1 that pass through $(0,0)$ are, by construction, unstable.

Lemma A.1. *If γ is a geodesic in V for g_1 such that $(0,0) \in \gamma$ and $B_\epsilon \cap \gamma$ is proper in B_ϵ, then γ is unstable.*

Proof. If γ is such a geodesic, then $\Psi(\gamma \cap B_\epsilon)$ is a proper geodesic in $\mathcal{B}_{\frac{3}{4}\pi}$ that contains p. Hence $\Psi(\gamma \cap B_\epsilon)$ has length at least $\frac{3}{2}\pi > \pi$ and so is unstable.   q.e.d.

Note that the curves $\mathbb{S}^1_{\pm 1}$, $\gamma_\pm$, and α_π all remain geodesics for g_1 as $g_1 = g_0$ in a neighborhood of these curves. Furthermore, as $\phi \circ R = \phi$ and $\Psi \circ R = \tilde{R} \circ \Psi$, R is also an isometry of g_1.

Our goal now is to construct the desired disk Δ. To that end, let $\Omega = \mathbb{S}^1_\psi \times C = \mathbb{T}^2 \times \mathbb{R}$ have the product metric

$$g_\Omega = d\psi^2 + g_1.$$

Set $U' = \mathbb{S}^1 \times U$ and $V' = \mathbb{S}^1 \times V$ and note that $\Gamma_\pm = \mathbb{S}^1 \times \gamma_\pm$ are totally geodesic cylinders which accumulate at the totally geodesic tori $\mathbb{T}^2[\pm 1] = \mathbb{S}^1 \times \mathbb{S}^1[\pm 1]$. Likewise, let $A = \mathbb{S}^1 \times \alpha_\pi \subset U'$, which is a totally geodesic annulus. The following "translation" map is an isometry of g_Ω for $v \in \mathbb{R}$

$$T_v((\psi, \theta, t)) = (\psi + v, \theta, t)$$

and the following "reflection" map

$$R'((\psi, \theta, t)) = (-\psi, R(\theta, t)) = (-\psi, -\theta, -t)$$

is an isometric involution.

We denote the universal cover of Ω by $\widehat{\Omega}$. That is,

$$\widehat{\Omega} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

with coordinates $(\widehat{\psi}, \widehat{\theta}, \widehat{t})$. Let $\widehat{\Pi} : \widehat{\Omega} \to \Omega$ be the natural covering map. For subsets $S \subset \Omega$ we will denote lifts of these sets to $\widehat{\Omega}$ by $\widehat{S}$. In particular, the tori $\mathbb{T}^2[\pm 1]$ lift to stable minimal disks $\widehat{\mathbb{T}}^2[\pm 1]$ and the cylinders $\Gamma_\pm$ lift to minimal disks $\widehat{\Gamma}_\pm$ which together bound a region $\widehat{V} := \widehat{V'}$ which contains $(0, 0, 0)$. Likewise, we let $\widehat{A}_i = \mathbb{R} \times \{\pi + 2\pi i\} \times (-1, 1)$ for $i \in \mathbb{Z}$ be lifts of A. We denote by $\widehat{T}_v$ the isometry of $\widehat{\Omega}$ given by $\widehat{T}^1_v(\widehat{\psi}, \widehat{\theta}, \widehat{t}) = (\widehat{\psi} + v, \widehat{\theta}, \widehat{t})$ for $v \in \mathbb{R}$ and let $\widehat{R}$ be the reflection $\widehat{R}(\widehat{\psi}, \widehat{\theta}, \widehat{t}) = (-\widehat{\psi}, -\widehat{\theta}, -\widehat{t})$. Note that $\widehat{T}_{2\pi i}$ for $i \in \mathbb{Z}$ are deck transforms of the cover. Furthermore, $\widehat{R}(\widehat{\Gamma}^+) = \widehat{\Gamma}^-$ and $\widehat{R}(\widehat{A}_i) = \widehat{A}_{-i-1}$. Finally, let us denote by $\widehat{G}_i$, $i \in \mathbb{Z}$, the deck transforms

$$\widehat{G}_i((\psi, \theta, t)) = (\psi, \theta + 2\pi i, t)$$

and note that $\widehat{G}_i(\widehat{V}) \cap \widehat{V} = \emptyset$ for $i \in \mathbb{Z} \backslash \{0\}$.

We now construct an embedded minimal disk $\widehat{\Delta}$ in $\widehat{\Omega}$ so that $\Delta = \widehat{\Pi}(\widehat{\Delta})$. To that end, let $\widehat{\sigma}^+_j$ be the curves $\widehat{\Gamma}_+ \cap \{\widehat{\psi} = j\}$ and $\widehat{\sigma}^-_j$ the curves $\widehat{\Gamma}_- \cap \{\widehat{\psi} = -j\}$. We denote by $\widehat{\sigma}_{j,i}$ the segment of $\widehat{\sigma}_j$ between $\widehat{A}_i$ and $\widehat{A}_{-i-1}$ and likewise for $\widehat{\sigma}^-_{j,i}$. Let $\widehat{\tau}^-_{j,i}$ be a real-analytic curve connecting the endpoint of $\widehat{\sigma}^+_{j,i}$ in $\widehat{A}_{-i-1}$ to the endpoint of $\widehat{\sigma}^-_{j,i}$ which is chosen to

be contained in $\widehat{A}_{-i-1}$ and to have the property that both coordinates $\widehat{t}$ and $\widehat{\psi}$ are strictly monotonic. Set $\widehat{\tau}_{j,i}^{+} = \widehat{R}(\widehat{\tau}_{j,i}^{-})$. One verifies that $\widehat{\tau}_{j,i}^{+}$ connects the other endpoints of $\widehat{\sigma}_{j,i}^{+}$ and $\widehat{\sigma}_{j,i}^{-}$. Hence,

$$\widehat{\delta}_{j,i} := \widehat{\sigma}_{j,i}^{+} \cup \widehat{\sigma}_{j,i}^{-} \cup \tau_{j,i}^{+} \cup \tau_{j,i}^{-}$$

is a closed curve and $\widehat{\delta}_{j,i} = \widehat{R}(\widehat{\delta}_{j,i})$. We note that our choice of curves implies further that $\widehat{T}_v(\widehat{\delta}_{j,i}) \cap \widehat{\delta}_{j,i} = \emptyset$ for $v \neq 0$. Now let $\widehat{\Delta}_{j,i}$ be minimal disks which solve the Plateau problem with boundary $\widehat{\delta}_{j,i}$. By the strong maximum principle $\widehat{T}_v^1(\widehat{\Delta}_{j,i}) \cap \widehat{\Delta}_{j,i} = \emptyset$ for $v \neq 0$. In particular, $\left\{ \widehat{T}_v^1(\widehat{\Delta}_{j,i}) \right\}_{v \in \mathbb{R}}$ is a minimal foliation $\mathcal{D}_{j,i}$ of an open subset, $\widehat{V}_{j,i}$, of $\widehat{V}$ and $\widehat{T}_v$ leaves $\mathcal{D}_{j,i}$ invariant. This together with the strong maximum principle applied to the Jacobi function generated by $\widehat{T}_v$ implies that the leaves of $\mathcal{D}_{j,i}$ are graphs over

$$V_i := \{ (\widehat{\psi}, \widehat{\theta}, \widehat{t}) \in \widehat{V} : \widehat{\psi} = 0, \theta \in (-\pi - 2\pi i, \pi + 2\pi i) \}.$$

As $\widehat{R}$ leaves both $\widehat{\delta}_{j,i}$ and $\widehat{V}_{j,i}$ unchanged, it follows from the strong maximum principle that $\widehat{\Delta}_{j,i} = \widehat{R}(\widehat{\Delta}_{j,i})$. In particular, $(0,0,0) \in \widehat{\Delta}_{j,i}$. By Theorem 3.5, up to passing to a subsequence, the minimal foliations $\mathcal{D}_{j,i}$ converge smoothly on compact subsets of $\widehat{V}$ to a minimal foliation of $\widehat{V}$. This foliation is also invariant under $\widehat{T}_v$. A consequence of this is that if L is a leaf of $\mathcal{D}$, then either L splits as the product $\mathbb{R} \times \eta$ where η is a geodesic in V_∞ or L is a graph over some open subset of V_∞. If the former occurs, then the stability of L implies that η is also a stable geodesic.

Let $\widehat{\Delta}$ be the leaf of $\mathcal{D}$ which contains $(0,0,0)$. As $\widehat{\Delta}$ is the limit of $\widehat{\Delta}_{j,i}$, $\widehat{\Delta}$ is complete. By Lemma A.1, the geodesic in V_∞ through $(0,0,0)$ is unstable and hence $\widehat{\Delta}$ cannot split. In particular, $\widehat{\Delta}$ is a graph over some open subset of V_∞ and $\widehat{T}_v(\widehat{\Delta}) \cap \widehat{\Delta} = \emptyset$. Set $\Delta = \widehat{\Pi}(\widehat{\Delta})$ and note that, as $\widehat{\Delta} \subset \widehat{V}$, $\widehat{G}_i(\widehat{\Delta}) \cap \widehat{\Delta} = \emptyset$ for $i \in \mathbb{Z} \backslash \{0\}$. Hence, Δ is a complete embedded minimal disk in $V \subset \Omega$. Clearly Δ cannot be properly embedded in Ω. Nevertheless, the curvature estimates for stable minimal surfaces imply that $\overline{\Delta}$ is a smooth minimal lamination in Ω. To determine the other leaves, we note first that $\lim_{v \to \infty} \widehat{T}_v(\Delta)$ converges in $\widehat{\Omega}$ to some $\widehat{T}_v$ invariant minimal surface $\widehat{L}_+$—possibly, but not necessarily, $\widehat{\Gamma}_+$. Similarly, $\lim_{v \to -\infty} \widehat{T}_v(\widehat{\Delta})$ converges to a $\widehat{T}_v$ invariant surface $\widehat{L}_- = \widehat{R}(\widehat{L}_+)$. As a consequence, $L_\pm = \widehat{\Pi}(\widehat{L}_\pm)$ are non-proper embedded minimal annuli that are leaves of $\overline{\Delta}$. Finally, by construction $L_\pm$ are contained in $\overline{V}'$, that is, are trapped between Γ_+ and Γ_-. Since the ends of Γ_+ and Γ_- converge to the same side of $\mathbb{T}_1^2$ and $\mathbb{T}_{-1}^2$, one verifies that $\overline{L}_\pm = \mathbb{T}^2[\pm 1]$ and these are the remaining leaves of $\overline{\Delta}$. That is, $\Delta = \widehat{\Pi}(\widehat{\Delta})$ is the desired minimal disk.

Appendix B. Curvature Estimate of Schoen and Simon

For the convenience of the reader we state the curvature estimate for embedded minimal disks with a uniform area bound proved by Schoen and Simon in [**19**].

Theorem B.1. *Fix (Ω, g) a Riemannian three-manifold and let $\mathcal{B}_{2r}(p) \subset N$ satisfy:*

- $\exp_p^\Omega : B_{2r}(0) \to \mathcal{B}_{2r}(p)$ *is a smooth diffeomorphism.*
- *With $g_{ij}dx^i dx^j = (\exp_p^\Omega)^* g$, there is a $0 < \alpha \leq 1$ so*

$$\frac{1}{2}\delta_{ij} < g_{ij} < 2\delta_{ij}, \quad \sup_{B_{2r}} \left(r \left| \frac{\partial g_{jk}}{\partial x^i} \right| + r^2 \left| \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} \right| \right) < 1,$$

and

$$\sup_{(x,y) \in B_{2r} \times B_{2r}} r^{2+\alpha} |x - y|^{-\alpha} \left| \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}(x) - \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}(y) \right| < 1.$$

Given $\mu > 0$, there is a $C = C(\mu) > 0$ so that if $\Sigma \subset \mathcal{B}_{2r}(p)$ is a properly embedded minimal disk in $\mathcal{B}_{2r}(p)$ and $|\Sigma| \leq \mu r^2$, then

$$\sup_{\mathcal{B}_r(p) \cap \Sigma} |A|^2 < Cr^{-2}.$$

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