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ON THE EVOLUTION OF A HERMITIAN METRIC BY ITS CHERN-RICCI FORM

Valentino Tosatti & Ben Weinkove

Abstract

We consider the evolution of a Hermitian metric on a compact complex manifold by its Chern-Ricci form. This is an evolution equation first studied by M. Gill, and coincides with the Kähler-Ricci flow if the initial metric is Kähler. We find the maximal existence time for the flow in terms of the initial data. We investigate the behavior of the flow on complex surfaces when the initial metric is Gauduchon, on complex manifolds with negative first Chern class, and on some Hopf manifolds. Finally, we discuss a new estimate for the complex Monge-Ampère equation on Hermitian manifolds.

1. Introduction

Let (M, J) be a compact complex manifold of complex dimension n. Let g_0 be a Hermitian metric on M, that is a Riemannian metric g_0 satisfying $g_0(JX, JY) = g_0(X, Y)$ for all vectors X, Y. In local complex coordinates $(z_1, \ldots, z_n)$, the metric g_0 is given by a Hermitian matrix with components $(g_0)_{i\bar{j}}$. Associated to g_0 is a real $(1, 1)$ form $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}} dz_i \wedge d\bar{z}_j$, which we will also often refer to as a Hermitian metric.

Given the success of Hamilton's Ricci flow [**22**] in establishing deep results in the setting of topological, smooth, and Riemannian manifolds (see e.g. [**5**, **23**, **30**]), it is natural to ask whether there is a parabolic flow of metrics on M which starts at g_0, preserves the Hermitian condition, and reveals information about the structure of M as a complex manifold. In the case when g_0 is Kähler (meaning $d\omega_0 = 0$), the Ricci flow does precisely this. It gives a flow of Kähler metrics whose behavior is deeply intertwined with the complex and algebro-geometric properties of M (see [**8**, **9**, **15**, **31**, **32**, **34**, **35**, **36**, **37**, **38**, **40**, **41**, **45**, **50**, **51**, **52**, **56**], for example).

However, if g_0 is not Kähler, then in general the Ricci flow does not preserve the Hermitian condition $g(JX, JY) = g(X, Y)$. Alternative parabolic flows on complex manifolds which do preserve the Hermitian

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property have been proposed by Streets-Tian [**42**, **43**] and also Liu-Yang [**28**].

This paper is concerned with another such flow, first investigated by M. Gill [**19**], which we will call the *Chern-Ricci flow*:

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \tag{1.1}$$

where here $\mathrm{Ric}(\omega)$ is the *Chern-Ricci form* (sometimes called the *first Chern form*) associated to the Hermitian metric g, which in local coordinates is given by

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log\det g. \tag{1.2}$$

In the case when g is Kähler, $\mathrm{Ric}(\omega) = \sqrt{-1}R_{i\overline{j}}dz_i \wedge d\overline{z}_j$, where $R_{i\overline{j}}$ is the usual Ricci curvature of g. Thus if g_0 is Kähler, (1.1) coincides with the Kähler-Ricci flow. In general, $\mathrm{Ric}(\omega)$ does not have a simple relationship with the Ricci curvature of g. The Bott-Chern cohomology class determined by the closed form $\mathrm{Ric}(\omega)$ is denoted by $c_1^{\mathrm{BC}}(M)$. We call this the *first Bott-Chern class* of M. It is independent of the choice of Hermitian metric ω (see Section 2 for more details).

The following result for the Chern-Ricci flow was proved by Gill [**19**]:

Theorem 1.1 (Gill)**.** *If $c_1^{\mathrm{BC}}(M) = 0$ then, for any initial metric ω_0, there exists a solution $\omega(t)$ to the Chern-Ricci flow (1.1) for all time and the metrics $\omega(t)$ converge smoothly as $t \to \infty$ to a Hermitian metric ω_∞ satisfying $\mathrm{Ric}(\omega_\infty) = 0$.*

Moreover, the Hermitian metric ω_∞ is the unique Chern-Ricci flat metric on M of the form $\omega_\infty = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi$ for some function φ. The Chern-Ricci flat metrics were already known to exist [**10**, **54**], and the estimate of [**54**] is used in the proof of Theorem 1.1. If ω_0 is Kähler then Theorem 1.1 is due to Cao [**8**], with ω_∞ being the Ricci-flat metric of Yau [**60**]. In Section 2 below, we discuss the work of Gill [**19**] further, and also explain how the Chern-Ricci flow compares with some other parabolic flows on complex manifolds studied in the literature.

Our first result characterizes the maximal existence time for a solution to the Chern-Ricci flow using information from the initial Hermitian metric ω_0. First observe that the flow equation (1.1) may be rewritten as

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial}\theta(t), \quad \text{with } \theta(t) = \log\frac{\det g(t)}{\det g_0}.$$

Thus, as long as the flow exists, the solution $\omega(t)$ starting at ω_0 must be of the form $\omega(t) = \alpha_t + \sqrt{-1}\partial\overline{\partial}\Theta$, for some function $\Theta = \Theta(t)$, with

$$\alpha_t = \omega_0 - t\mathrm{Ric}(\omega_0). \tag{1.3}$$

Now define a number $T = T(\omega_0)$ with $0 < T \leqslant \infty$ by

$$T = \sup\{t \geqslant 0 \mid \exists\psi \in C^\infty(M) \text{ with } \alpha_t + \sqrt{-1}\partial\overline{\partial}\psi > 0\}. \tag{1.4}$$

By the observation above, a solution to (1.1) cannot exist beyond time T. We prove:

Theorem 1.2. *There exists a unique maximal solution to the Chern-Ricci flow (*1.1*) on* $[0, T)$.

In the special case when ω_0 is Kähler, this is already known by the result of Tian-Zhang [**51**], who extended earlier work of Cao and Tsuji [**8**, **56**, **57**]. In the Kähler case, T depends only on the cohomology class $[\omega_0]$ and can be written

$$T = \sup\{t \geqslant 0 \mid [\omega_0] - tc_1(M) > 0\}.$$

Furthermore, the Nakai-Moishezon criterion, due to Buchdahl [**7**] and Lamari [**25**] for Kähler surfaces and to Demailly-Păun [**11**] for general Kähler manifolds, implies that at time T either the volume of M goes to zero, or the volume of some proper analytic subvariety of M goes to zero (cf. the discussion in [**15**]).

Note that in the general Hermitian case, we can consider the equivalence relation of $(1,1)$ forms on M:

$$\alpha \sim \alpha' \quad \Longleftrightarrow \quad \alpha = \alpha' + \sqrt{-1}\partial\overline{\partial}\psi \quad \text{for some function } \psi \in C^\infty(M).$$

Then T defined by (1.4) depends only on the equivalence class of ω_0.

In the special case when M is a complex surface ($n = 2$) a result of Gauduchon [**16**] is that every Hermitian metric is conformal to a $\partial\overline{\partial}$-closed metric ω_0. If ω_0 is $\partial\overline{\partial}$-closed then so is $\omega(t)$ for $t \in [0, T)$. Moreover, we have a geometric characterization of the maximal existence time T:

Theorem 1.3. *Let M be a compact complex surface, ω_0 a $\partial\overline{\partial}$-closed Hermitian metric. Then T defined by (*1.4*) can be written as*

$$T = \sup\left\{T_0 \geqslant 0 \;\middle|\; \forall t \in [0, T_0], \int_M \alpha_t^2 > 0, \quad \int_D \alpha_t > 0, \right.$$
$$\left. \text{for all } D \text{ irreducible effective divisors with } D^2 < 0\right\},$$

*for α_t given by (*1.3*).*

Note that for $t \in [0, T)$, the quantity $\int_M \alpha_t^2 = \int_M \omega(t)^2$ is the volume of M (with respect to $\omega(t)$) and $\int_D \alpha_t = \int_D \omega(t)$ is the volume of the curve D. Thus we can restate Theorem 1.3 as:

Corollary 1.4. *Let M be a compact complex surface, ω_0 a $\partial\overline{\partial}$-closed Hermitian metric. Then the Chern-Ricci flow (*1.1*) starting at ω_0 exists until either the volume of M goes to zero, or the volume of a curve of negative self-intersection goes to zero.*

As we remarked above, the same result was known to hold for the Kähler-Ricci flow thanks to the Nakai-Moishezon criterion of [**7**, **25**].

Analogues of Theorems 1.2 and 1.3 and Corollary 1.4 were conjectured by Streets-Tian [**44**] for their pluriclosed flow (see Section 2 below).

The Kähler-Ricci flow has a deep connection to the Minimal Model Program in algebraic geometry, as demonstrated by the work of Song-Tian and others [**14**, **26**, **34**, **35**, **36**, **37**, **38**, **39**, **40**, **41**, **50**, **51**, **56**, **63**]. In the case of algebraic surfaces, the minimal model program is relatively simple. Indeed, a *minimal surface* is defined to be a surface with no (-1)-curves (smooth rational curves C with $C^2 = -1$). To find the minimal model, one can just apply a finite number of *blow-downs*, which are algebraic operations contracting the (-1)-curves. It was shown in [**38**] that the Kähler-Ricci flow on an algebraic surface carries out these algebraic operations, contracting (-1)-curves in the sense of Gromov-Hausdorff, and smoothly outside of the curves. Moreover, the same behavior occurs on a non-algebraic Kähler surface [**40**]. In all dimensions, weak solutions to the Kähler-Ricci flow through singularities were constructed in [**36**] and a number of conjectures were made about the metric behavior of the flow (see also [**41**]).

We return now to the case of a complex (non-Kähler) surface. In this case one can also contract the (-1) curves to arrive at a minimal surface. We conjecture that the Chern-Ricci flow on a complex surface starting at a $\partial\overline{\partial}$-closed metric behaves in an analogous way to the Kähler-Ricci flow on a Kähler surface. We prove the following:

Theorem 1.5. *Let M be a compact complex surface with a $\partial\overline{\partial}$-closed Hermitian metric ω_0, and let $[0, T)$ be the maximal existence time of the Chern-Ricci flow starting from ω_0. Then*

(a) *If $T = \infty$, then M is minimal.*
(b) *If $T < \infty$ and $\mathrm{Vol}(M, \omega(t)) \to 0$ as $t \to T^-$, then M is either birational to a ruled surface or it is a surface of class VII (and in this case it cannot be an Inoue surface).*
(c) *If $T < \infty$ and $\mathrm{Vol}(M, \omega(t))$ stays positive as $t \to T^-$, then M contains (-1)-curves.*

Furthermore, if M is minimal, then $T = \infty$ unless M is $\mathbb{CP}^2$, a ruled surface, a Hopf surface, or a surface of class VII with $b_2 > 0$, in which cases (b) holds.

In the case that M is not minimal, and (c) occurs, we expect that the Chern-Ricci flow will contract a finite number of (-1)-curves and can be uniquely continued on the new manifold. Moreover, we conjecture that this process can be repeated until one obtains a minimal surface, or ends up in case (b) above. More details of this conjecture can be found in Section 6.

To provide some evidence for our conjecture, we prove the following theorem. It is an analogue of a result for the Kähler-Ricci flow, whose proof is essentially contained in [**51**] (for a recent exposition, see Chapter

7 of [**40**]), and which was a key starting point for the work [**36**, **37**, **38**, **39**]. We assume that the maximal existence time T is finite and, roughly speaking, that the limiting 'class' of the flow at time T is given by the pull-back of a Hermitian metric on a manifold N via $\pi : M \to N$, where π is a holomorphic map blowing down an exceptional divisor E to a point $p \in N$. We show that the solution to the Chern-Ricci flow will converge smoothly at time T away from E. In this way, one can obtain a Hermitian metric on the new manifold N, at least away from the point p. Our result holds in any dimension:

Theorem 1.6. *Assume that there exists a holomorphic map between compact Hermitian manifolds $\pi : (M, \omega_0) \to (N, \omega_N)$ blowing down the exceptional divisor E on M to a point $p \in N$. In addition, assume that there exists a smooth function ψ on M with*

$$\omega_0 - T\mathrm{Ric}\,(\omega_0) + \sqrt{-1}\partial\overline{\partial}\psi = \pi^*\omega_N, \tag{1.5}$$

with $T < \infty$ given by (1.4).

Then the solution $\omega(t)$ to the Chern-Ricci flow (1.1) starting at ω_0 converges in C^∞ on compact subsets of $M \setminus E$ to a smooth Hermitian metric ω_T on $M \setminus E$.

There are some new obstacles to proving this in the non-Kähler case that we overcome using a parabolic Schwarz Lemma for volume forms, and a second order estimate for the metric which uses a trick of Phong-Sturm [**33**].

In the cases when the flow has a long time solution, it is natural to investigate its behavior at infinity. If the manifold has vanishing first Bott-Chern class, we have already seen by Gill's result (Theorem 1.1) that the flow converges to a Chern-Ricci flat Hermitian metric. We now suppose that the first Chern class $c_1(M)$ is *negative* (note that $c_1^{\mathrm{BC}}(M) < 0$ implies $c_1(M) < 0$). In this case, the manifold M is Kähler and a fundamental result of Aubin [**1**] and Yau [**60**] says that M admits a unique Kähler-Einstein metric ω_{KE} with negative scalar curvature. Cao [**8**] then proved that the Kähler-Ricci flow (appropriately normalized) deforms any Kähler metric in $-c_1(M)$ to ω_{KE}. The same is true for the normalized Kähler-Ricci flow starting at any Kähler metric [**51**, **56**]. Our next result shows that starting at any *Hermitian metric* on the manifold M, the (normalized) Chern-Ricci flow will converge to the Kähler-Einstein metric ω_{KE}.

Theorem 1.7. *Let M be a compact complex manifold with $c_1(M) < 0$ and let ω_0 be a Hermitian metric on M. Then the Chern-Ricci flow (1.1) has a long time solution $\omega(t)$, and as t goes to infinity the rescaled metrics $\omega(t)/t$ converge smoothly to the unique Kähler-Einstein metric ω_{KE} on M.*

In particular we see that the Chern-Ricci flow on these manifolds, after normalization, deforms any Hermitian metric to a Kähler one.

Next we illustrate the Chern-Ricci flow with an explicit example. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$ with $|\alpha_1| = \cdots = |\alpha_n| \neq 1$, we consider the Hopf manifold $M_\alpha = (\mathbb{C}^n \setminus \{0\})/\sim$ where

$$(z_1, \ldots, z_n) \sim (\alpha_1 z_1, \ldots, \alpha_n z_n).$$

This is a non-Kähler complex manifold of complex dimension n. If $n = 2$, it is an example of a class VII surface. We can write down an exact solution to the Chern-Ricci flow on M_α. Consider the metric $\omega_H = \frac{\delta_{ij}}{r^2}\sqrt{-1}dz_i \wedge d\overline{z}_j$. Then we have:

Proposition 1.8. *The metrics $\omega(t) := \omega_H - t\mathrm{Ric}(\omega_H)$ on M_α give a solution of the Chern-Ricci flow on the maximal existence interval $[0, 1/n)$. As $t \to T = 1/n$, the limiting nonnegative $(1,1)$ form ω_T is given by*

$$\omega_T = \frac{\overline{z}_i z_j}{r^4}\sqrt{-1}dz_i \wedge d\overline{z}_j.$$

In the case of the original Hopf surface, which has $\alpha = (2, 2)$ and is an elliptic fiber bundle over $\mathbb{P}^1$ via the map $(z_1, z_2) \mapsto [z_1, z_2]$, the limiting form ω_T is positive definite along the fibers and zero in directions orthogonal to the fibers.

If we start with any metric ω_0 which differs from ω_H by $\sqrt{-1}\partial\overline{\partial}\psi$ for some function ψ, then we conjecture that the flow also converges as $t \to T$ to a smooth but degenerate $(1,1)$ form on M_α with properties similar to ω_T. To give evidence for this conjecture, we prove an estimate:

Proposition 1.9. *Let $\omega_0 = \omega_H + \sqrt{-1}\partial\overline{\partial}\psi$ be a Hermitian metric on M_α, and let $\omega(t)$ be the solution of the Chern-Ricci flow (1.1) starting at ω_0 on M_α for $t \in [0, 1/n)$. Then there exists a uniform constant C such that*

$$\omega(t) \leqslant C\omega_H, \quad \text{for } t \in [0, 1/n).$$

In particular, this result shows that we obtain convergence for the flow at the level of potential functions in $C^{1+\beta}$ for any $\beta \in (0, 1)$. For more details see Section 8.

The final result in this paper concerns not the Chern-Ricci flow, but an elliptic equation: the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F\omega^n, \quad \omega' := \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \tag{1.6}$$

on a compact Hermitian manifold (M, ω), where F is a smooth function on M. We give an alternative proof of a result of [**54**] that $\|\varphi\|_{C^0}$ is uniformly bounded (see also [**4**, **12**]). The result makes use of a new second order estimate in this context: $\mathrm{tr}_\omega\omega' \leqslant Ce^{A(\varphi - \inf_M \varphi)}$ which we conjectured to hold in [**53**]. For more details see Section 9. We have included this result here because it follows easily from the argument used in Theorem 1.6 together with the method of [**53**]. The key new ingredient is the trick of Phong-Sturm [**33**] applied to this setting.

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2. Preliminaries and comparison with other flows

In this section, we include for the reader's convenience some background material on local coordinate computations with Hermitian metrics.

Let (M, g) be a compact Hermitian manifold of complex dimension n. We will often compute in complex coordinates $z_1, \ldots, z_n$. In this case, g is determined by the $n \times n$ Hermitian matrix $g_{i\overline{j}} = g(\partial_i, \partial_{\overline{j}})$, where we are writing $\partial_i, \partial_{\overline{j}}$ for $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j}$ respectively. We denote by $g^{\overline{j}i}$ the entries of the inverse matrix of $(g_{i\overline{j}})$.

We define the *Chern connection* ∇ associated to g as follows. Given a vector field $X = X^i \partial_i$ and a $(1,0)$ form $a = a_i dz_i$, we define ∇X and ∇a to be the tensors with components:

$$\nabla_i X^k = \partial_i X^k + \Gamma^k_{ij} X^j, \quad \nabla_i a_j = \partial_i a_j - \Gamma^k_{ij} a_k,$$

where the Christoffel symbols Γ^k_{ij} are given by

$$\Gamma^k_{ij} = g^{\overline{q}k} \partial_i g_{j\overline{q}}.$$

The tensors $\nabla \overline{X}$ and $\nabla \overline{a}$ have components $\nabla_i \overline{X^k} = \partial_i \overline{X^k}$ and $\nabla_i \overline{a_j} = \partial_i \overline{a_j}$. The connection ∇ can be naturally extended to any kind of tensor, and we have $\nabla_k g_{i\overline{j}} = 0$.

We write Δ for the *complex Laplacian* of g, which acts on a function f by

$$\Delta f = g^{\overline{j}i} \partial_i \partial_{\overline{j}} f = g^{\overline{j}i} \nabla_i \nabla_{\overline{j}} f.$$

The *torsion* of g is the tensor T with components

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

The torsion tensor vanishes in the special case that g is Kähler.

We define the *curvature* of g to be the tensor with components

$$R_{k\overline{\ell}i}{}^p = -\partial_{\overline{\ell}} \Gamma^p_{ki}.$$

We will often raise and lower indices using the metric g, writing for example $R_{k\overline{\ell}i\overline{j}} = g_{p\overline{j}} R_{k\overline{\ell}i}{}^p$. Note that $\overline{R_{k\overline{\ell}i\overline{j}}} = R_{\ell\overline{k}j\overline{i}}$. We have the following commutation formulae:

$$[\nabla_k, \nabla_{\overline{\ell}}]X^i = R_{k\overline{\ell}j}{}^i X^j, \quad [\nabla_k, \nabla_{\overline{\ell}}]\overline{X^i} = -R_{k\overline{\ell}}{}^{\overline{i}}{}_{\overline{j}}\,\overline{X^j},$$

$$[\nabla_k, \nabla_{\overline{\ell}}]a_j = -R_{k\overline{\ell}j}{}^i a_i, \quad [\nabla_k, \nabla_{\overline{\ell}}]\overline{a_j} = R_{k\overline{\ell}}{}^{\overline{i}}{}_{\overline{j}}\,\overline{a_i},$$

where we are writing $[\nabla_k, \nabla_{\overline{\ell}}]$ for $\nabla_k\nabla_{\overline{\ell}} - \nabla_{\overline{\ell}}\nabla_k$. We write the *Chern-Ricci curvature* of g as the tensor $R^{\mathrm{C}}_{k\overline{\ell}}$ given by

$$R^{\mathrm{C}}_{k\overline{\ell}} = g^{\overline{j}i} R_{k\overline{\ell}i\overline{j}} = -\partial_k \partial_{\overline{\ell}} \log \det g,$$

so that the Chern-Ricci form is equal to

$$\mathrm{Ric}(\omega) = \sqrt{-1} R^{\mathrm{C}}_{k\overline{\ell}} dz_k \wedge d\overline{z}_\ell.$$

It is a closed real $(1,1)$ form and its cohomology class in the Bott-Chern cohomology group

$$H^{1,1}_{\mathrm{BC}}(M, \mathbb{R}) = \frac{\{\text{closed real } (1,1) \text{ forms}\}}{\{\sqrt{-1}\partial\overline{\partial}\psi, \psi \in C^\infty(M, \mathbb{R})\}}$$

is the first Bott-Chern class of M, and is denoted by $c_1^{\mathrm{BC}}(M)$. It is independent of the choice of Hermitian metric ω. More generally, if Ω is a smooth positive volume form on M we can define locally $\mathrm{Ric}(\Omega) = -\sqrt{-1}\partial\overline{\partial}\log\Omega$, which is a global closed real $(1,1)$ form that represents $c_1^{\mathrm{BC}}(M)$. For notational convenience, we omit the factor of 2π that usually appears in the definition of $c_1^{\mathrm{BC}}(M)$. The downside of this convention is that some factors of 2π will appear later in the cohomological calculations of Section 6.

We end this section by briefly mentioning some related parabolic equations on Hermitian manifolds which have previously been studied in the literature. Streets-Tian [**43**] introduced the flow

$$\frac{\partial}{\partial t} g_{i\overline{j}} = -S_{i\overline{j}} + Q_{i\overline{j}}, \quad g|_{t=0} = g_0, \tag{2.1}$$

where $S_{i\overline{j}}$ is given by taking 'the other trace' of the curvature of the Chern connection:

$$S_{i\overline{j}} = g^{\overline{\ell}k} R_{k\overline{\ell}i\overline{j}},$$

and $Q_{i\overline{j}}$ is a certain quadratic term in the torsion. If the form ω_0 associated to g_0 satisfies $\partial\overline{\partial}\omega_0 = 0$, this equation becomes their *pluriclosed flow*

$$\frac{\partial}{\partial t}\omega = \partial\partial^*\omega + \overline{\partial}\,\overline{\partial}^*\omega - \mathrm{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \tag{2.2}$$

and if g_0 is Kähler, it coincides with the Kähler-Ricci flow. They analyzed (2.2) in detail in [**42**, **44**] and made a number of conjectures about it, two of which are analogues of our Theorems 1.2 and 1.3. They conjecture that their flow can be used to study the topology of class VII$^+$ surfaces. In addition, Streets-Tian considered a family of flows of the form (2.1) with arbitrary quadratic torsion term Q and proved, among other results, a short-time existence theorem [**43**]. The flow (2.1) was extended to the almost complex setting by Vezzoni [**58**]. Liu-Yang [**28**] propose studying the flow (2.1) in the case of $Q = 0$.

In [**19**], Gill introduced the following parabolic complex Monge-Ampère equation on a compact Hermitian manifold $(M, \hat{g})$:

$$\frac{\partial}{\partial t}\varphi = \log \frac{\det(\hat{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)}{\det(\hat{g}_{i\bar{j}})} - F, \quad \hat{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi > 0, \quad \varphi|_{t=0} = 0, \tag{2.3}$$

for a fixed smooth function F on M. He showed that the unique solution to (2.3) exists for all time and, after an appropriate normalization, converges in C^∞ to a smooth function φ_∞ solving the complex Monge-Ampère equation

$$\log \frac{\det(\hat{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi_\infty)}{\det(\hat{g}_{i\bar{j}})} = F + b, \tag{2.4}$$

for a constant b which is uniquely determined. The existence of solutions to the elliptic equation (2.4) on Hermitian manifolds (generalizing Yau's Theorem [**60**]) was already known by the work of Cherrier [**10**] (if $n = 2$) and the authors [**54**] ($n > 2$). See also [**21**, **62**]. In the special case where $\hat{g}$ is Kähler, the flow (2.3) had been considered earlier by Cao [**8**], who proved the analogous results.

In the case when $c_1^{\mathrm{BC}}(M) = 0$, we can find a function F satisfying

$$\partial\overline{\partial} \log \det \hat{g} = \partial\overline{\partial} F,$$

and with this choice, $\omega(t) = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi(t)$ for $\varphi(t)$ solving (2.3) is exactly the Chern-Ricci flow starting at $\hat{\omega}$. In general, the only difference between the Chern-Ricci flow and Gill's flow (2.3) is that for the Chern-Ricci flow we replace the fixed metric $\hat{g}$ by a smoothly varying family of Hermitian metrics $\hat{g}_t$, and replace F by a particular function, which may also depend on t. Many of Gill's estimates carry over easily to the case of the Chern-Ricci flow and we will make extensive use of them here.

A final remark about notation. In the following, C, C' will denote uniform positive constants which may vary from line to line.

3. Evolution of the trace of the metric

In this section we write down a formula for the evolution of the trace of the evolving metric with respect to a fixed Hermitian metric. We will need this calculation in later sections. We carry out the computation here using tensorial quantities, following [**10**], rather than using a particular choice of complex coordinates as in [**19**, **21**, **43**, **53**], for example.

We suppose that we have three Hermitian metrics g, g_0, and $\hat{g}$ such that $g = g(t)$ satisfies the Chern-Ricci flow (1.1), and such that the corresponding forms satisfy

$$\omega = \omega_0 + \eta(t), \tag{3.1}$$

for a closed $(1,1)$ form $\eta(t)$.

We denote by $\hat{\nabla}$, $\hat{T}$, $\hat{\Gamma}$, $\hat{R}$ the Chern connection, torsion, Christoffel symbols, and curvature of $\hat{g}$. Denote by T_0 the torsion tensor of g_0 and by Δ the complex Laplacian associated to $g = g(t)$.

Note that for the purposes of this paper we will in fact only need the case of $\hat{g} = g_0$. However, we included the more general calculation below since we anticipate that it may be useful in the future.

We have:

Proposition 3.1. *The evolution of* $\log \mathrm{tr}_{\hat{g}} g$ *is given by*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \mathrm{tr}_{\hat{g}} g \quad = \quad (I) + (II) + (III) \tag{3.2}$$

where

$$\begin{aligned}
(I) &= \frac{1}{\mathrm{tr}_{\hat{g}} g}\Bigg[- g^{\overline{j}p} g^{\overline{q}i} \hat{g}^{\overline{\ell}k} \hat{\nabla}_k g_{i\overline{j}} \hat{\nabla}_{\overline{\ell}} g_{p\overline{q}} + \frac{1}{\mathrm{tr}_{\hat{g}} g} g^{\overline{\ell}k} \hat{\nabla}_k \mathrm{tr}_{\hat{g}} g \hat{\nabla}_{\overline{\ell}} \mathrm{tr}_{\hat{g}} g \\
&\quad - 2\mathrm{Re}\left(g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \hat{T}^p_{ki} \hat{\nabla}_{\overline{\ell}} g_{p\overline{j}} \right) - g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \hat{T}^p_{ik} \overline{\hat{T}^q_{j\ell}} g_{p\overline{q}} \Bigg] \\
(II) &= \frac{1}{\mathrm{tr}_{\hat{g}} g}\left[g^{\overline{j}i} \hat{g}^{\overline{\ell}k} (\hat{\nabla}_i \overline{\hat{T}^q_{j\ell}} - \hat{R}_{i\overline{\ell}p\overline{j}} \hat{g}^{\overline{q}p}) g_{k\overline{q}} \right] \\
(III) &= -\frac{1}{\mathrm{tr}_{\hat{g}} g}\Bigg[g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \left(\hat{\nabla}_i \left(\overline{(T_0)^p_{j\ell}} (g_0)_{k\overline{p}} \right) + \hat{\nabla}_{\overline{\ell}} \left((T_0)^p_{ik} (g_0)_{p\overline{j}} \right) \right) \\
&\quad - g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \overline{\hat{T}^q_{j\ell}} (T_0)^p_{ik} (g_0)_{p\overline{q}} \Bigg].
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(I) &\leqslant \frac{2}{(\mathrm{tr}_{\hat{g}} g)^2} \mathrm{Re}\left(\hat{g}^{\overline{\ell}i} g^{\overline{q}k} (T_0)^p_{ki} (g_0)_{p\overline{\ell}} \hat{\nabla}_{\overline{q}} \mathrm{tr}_{\hat{g}} g \right), \\
(II) &\leqslant C \mathrm{tr}_g \hat{g},
\end{aligned}$$

for a constant C that depends only on $\hat{g}$. If we are at a point where $\mathrm{tr}_{\hat{g}} g \geqslant 1$, then

$$(III) \leqslant C' \mathrm{tr}_g \hat{g},$$

for C' depending only on g_0 and $\hat{g}$.

Proof. First,

$$\Delta \mathrm{tr}_{\hat{g}} g = g^{\overline{j}i} \hat{\nabla}_i \hat{\nabla}_{\overline{j}} (\hat{g}^{\overline{\ell}k} g_{k\overline{\ell}}) = g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \hat{\nabla}_i \hat{\nabla}_{\overline{j}} g_{k\overline{\ell}}.$$

From the definition of covariant derivative,

$$\hat{\nabla}_{\overline{j}} g_{k\overline{\ell}} = \partial_{\overline{j}} g_{k\overline{\ell}} - \overline{\hat{\Gamma}^p_{j\ell}} g_{k\overline{p}},$$

and skew-symmetrizing in j, ℓ

$$\hat{\nabla}_{\overline{j}} g_{k\overline{\ell}} = \hat{\nabla}_{\overline{\ell}} g_{k\overline{j}} + (\overline{\partial}\omega)_{\overline{j}k\overline{\ell}} - \overline{\hat{T}^p_{j\ell}} g_{k\overline{p}}.$$

But from (3.1), $\overline{\partial}\omega = \overline{\partial}\omega_0$, and we may rewrite this in terms of the torsion of g_0 as $(\overline{\partial}\omega_0)_{\overline{j}k\overline{\ell}} = \overline{(T_0)^p_{j\ell}}(g_0)_{k\overline{p}}$. Thus

$$\hat{\nabla}_i \hat{\nabla}_{\overline{j}} g_{k\overline{\ell}} = \hat{\nabla}_i \hat{\nabla}_{\overline{\ell}} g_{k\overline{j}} + \hat{\nabla}_i \left(\overline{(T_0)^p_{j\ell}}(g_0)_{k\overline{p}} \right) - (\hat{\nabla}_i \overline{\hat{T}^q_{j\ell}}) g_{k\overline{q}} - \overline{\hat{T}^q_{j\ell}} \hat{\nabla}_i g_{k\overline{q}}.$$

Switching covariant derivatives,

$$\hat{\nabla}_i \hat{\nabla}_{\overline{\ell}} g_{k\overline{j}} = \hat{\nabla}_{\overline{\ell}} \hat{\nabla}_i g_{k\overline{j}} - \hat{R}_{i\overline{\ell}k\overline{q}} \hat{g}^{\overline{q}p} g_{p\overline{j}} + \hat{R}_{i\overline{\ell}p\overline{j}} \hat{g}^{\overline{q}p} g_{k\overline{q}}.$$

Arguing as above,

$$\hat{\nabla}_{\overline{\ell}} \hat{\nabla}_i g_{k\overline{j}} = \hat{\nabla}_{\overline{\ell}} \hat{\nabla}_k g_{i\overline{j}} + \hat{\nabla}_{\overline{\ell}} \left((T_0)^p_{ik}(g_0)_{p\overline{j}} \right) - (\hat{\nabla}_{\overline{\ell}} \hat{T}^p_{ik}) g_{p\overline{j}} - \hat{T}^p_{ik} \hat{\nabla}_{\overline{\ell}} g_{p\overline{j}}.$$

Combining all of these we have

$$\begin{aligned} \Delta \mathrm{tr}_{\hat{g}} g = {} & g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \hat{\nabla}_{\overline{\ell}} \hat{\nabla}_k g_{i\overline{j}} + g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \Bigg(\hat{\nabla}_i \left(\overline{(T_0)^p_{j\ell}}(g_0)_{k\overline{p}} \right) \\ & + \hat{\nabla}_{\overline{\ell}} \left((T_0)^p_{ik}(g_0)_{p\overline{j}} \right) - (\hat{\nabla}_i \overline{\hat{T}^q_{j\ell}} - \hat{R}_{i\overline{\ell}p\overline{j}} \hat{g}^{\overline{q}p}) g_{k\overline{q}} \\ & - (\hat{\nabla}_{\overline{\ell}} \hat{T}^p_{ik} + \hat{R}_{i\overline{\ell}k\overline{q}} \hat{g}^{\overline{q}p}) g_{p\overline{j}} - \overline{\hat{T}^q_{j\ell}} \hat{\nabla}_i g_{k\overline{q}} - \hat{T}^p_{ik} \hat{\nabla}_{\overline{\ell}} g_{p\overline{j}} \Bigg). \end{aligned} \tag{3.3}$$

We will make a change to the second to last term using

$$\overline{\hat{T}^q_{j\ell}} \hat{\nabla}_i g_{k\overline{q}} = \overline{\hat{T}^q_{j\ell}} \hat{\nabla}_k g_{i\overline{q}} + \overline{\hat{T}^q_{j\ell}} (T_0)^p_{ik} (g_0)_{p\overline{q}} - \hat{T}^p_{ik} \overline{\hat{T}^q_{j\ell}} g_{p\overline{q}}. \tag{3.4}$$

On the other hand,

$$\frac{\partial}{\partial t} \mathrm{tr}_{\hat{g}} g = \hat{g}^{\overline{\ell}k} \partial_k \partial_{\overline{\ell}} \log \det(g) = g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \partial_k \partial_{\overline{\ell}} g_{i\overline{j}} - g^{\overline{j}p} g^{\overline{q}i} \hat{g}^{\overline{\ell}k} \partial_k g_{i\overline{j}} \partial_{\overline{\ell}} g_{p\overline{q}},$$

and we wish to convert the partial derivatives into covariant ones. For this, we use the relations

$$\partial_k g_{i\overline{j}} = \hat{\nabla}_k g_{i\overline{j}} + \hat{\Gamma}^r_{ki} g_{r\overline{j}}$$

and

$$\begin{aligned} \partial_{\overline{\ell}} \partial_k g_{i\overline{j}} &= \partial_{\overline{\ell}} \hat{\nabla}_k g_{i\overline{j}} + \left(\partial_{\overline{\ell}} \hat{\Gamma}^r_{ki} \right) g_{r\overline{j}} + \hat{\Gamma}^r_{ki} \partial_{\overline{\ell}} g_{r\overline{j}} \\ &= \hat{\nabla}_{\overline{\ell}} \hat{\nabla}_k g_{i\overline{j}} + \overline{\hat{\Gamma}^s_{\ell j}} \hat{\nabla}_k g_{i\overline{s}} - \hat{R}_{k\overline{\ell}i\overline{q}} \hat{g}^{\overline{q}r} g_{r\overline{j}} + \hat{\Gamma}^r_{ki} \hat{\nabla}_{\overline{\ell}} g_{r\overline{j}} + \hat{\Gamma}^r_{ki} \overline{\hat{\Gamma}^s_{\ell j}} g_{r\overline{s}}. \end{aligned}$$

Substituting we get

$$\frac{\partial}{\partial t} \mathrm{tr}_{\hat{g}} g = g^{\overline{j}i} \hat{g}^{\overline{\ell}k} \hat{\nabla}_{\overline{\ell}} \hat{\nabla}_k g_{i\overline{j}} - g^{\overline{j}p} g^{\overline{q}i} \hat{g}^{\overline{\ell}k} \hat{\nabla}_k g_{i\overline{j}} \hat{\nabla}_{\overline{\ell}} g_{p\overline{q}} - \hat{g}^{\overline{\ell}k} \hat{g}^{\overline{j}i} \hat{R}_{k\overline{\ell}i\overline{j}} \tag{3.5}$$

and so, combining (3.3), (3.4), and (3.5),

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\log \mathrm{tr}_{\hat{g}}g \\
&= \frac{1}{\mathrm{tr}_{\hat{g}}g}\Bigg(\Bigg[-g^{\overline{j}p}g^{\overline{q}i}\hat{g}^{\overline{\ell}k}\hat{\nabla}_k g_{i\overline{j}}\hat{\nabla}_{\overline{\ell}}g_{p\overline{q}} + \frac{1}{\mathrm{tr}_{\hat{g}}g}g^{\overline{\ell}k}\hat{\nabla}_k \mathrm{tr}_{\hat{g}}g\hat{\nabla}_{\overline{\ell}}\mathrm{tr}_{\hat{g}}g \\
&\quad - 2\mathrm{Re}\left(g^{\overline{j}i}\hat{g}^{\overline{\ell}k}\hat{T}^p_{ki}\hat{\nabla}_{\overline{\ell}}g_{p\overline{j}}\right) - g^{\overline{j}i}\hat{g}^{\overline{\ell}k}\hat{T}^p_{ik}\overline{\hat{T}^q_{j\ell}}g_{p\overline{q}}\Bigg] \\
&\quad + \left[g^{\overline{j}i}\hat{g}^{\overline{\ell}k}(\hat{\nabla}_i\overline{\hat{T}^q_{j\ell}} - \hat{R}_{i\overline{\ell}p\overline{j}}\hat{g}^{\overline{q}p})g_{k\overline{q}} + \hat{g}^{\overline{\ell}k}(\hat{\nabla}_{\overline{\ell}}\hat{T}^i_{ik} + \hat{R}_{i\overline{\ell}k\overline{q}}\hat{g}^{\overline{q}i} - \hat{R}_{k\overline{\ell}i\overline{q}}\hat{g}^{\overline{q}i})\right] \\
&\quad - \Bigg[g^{\overline{j}i}\hat{g}^{\overline{\ell}k}\left(\hat{\nabla}_i\left(\overline{(T_0)^p_{j\ell}}(g_0)_{k\overline{p}}\right) + \hat{\nabla}_{\overline{\ell}}\left((T_0)^p_{ik}(g_0)_{p\overline{j}}\right)\right) \\
&\quad - g^{\overline{j}i}\hat{g}^{\overline{\ell}k}\overline{\hat{T}^q_{j\ell}}(T_0)^p_{ik}(g_0)_{p\overline{q}}\Bigg]\Bigg).
\end{aligned}$$

Note that in the first set of square brackets above we have used the identity $\hat{T}^p_{ki} = -\hat{T}^p_{ik}$. To obtain (3.2) it remains to show that the expression

$$\hat{g}^{\overline{\ell}k}(\hat{\nabla}_{\overline{\ell}}\hat{T}^i_{ik} + \hat{R}_{i\overline{\ell}k\overline{q}}\hat{g}^{\overline{q}i} - \hat{R}_{k\overline{\ell}i\overline{q}}\hat{g}^{\overline{q}i})$$

in the second set of square brackets vanishes. To see this, note that

$$\hat{R}_{i\overline{\ell}k\overline{q}} - \hat{R}_{k\overline{\ell}i\overline{q}} = -\hat{g}_{j\overline{q}}\partial_{\overline{\ell}}\hat{T}^j_{ik} = -\hat{g}_{j\overline{q}}\hat{\nabla}_{\overline{\ell}}\hat{T}^j_{ik},$$

and so

$$\hat{g}^{\overline{\ell}k}\hat{g}^{\overline{q}i}(\hat{R}_{i\overline{\ell}k\overline{q}} - \hat{R}_{k\overline{\ell}i\overline{q}}) = -\hat{g}^{\overline{\ell}k}\hat{g}^{\overline{q}i}\hat{g}_{j\overline{q}}\hat{\nabla}_{\overline{\ell}}\hat{T}^j_{ik} = -\hat{g}^{\overline{\ell}k}\hat{\nabla}_{\overline{\ell}}\hat{T}^i_{ik},$$

so that

$$\hat{g}^{\overline{\ell}k}(\hat{\nabla}_{\overline{\ell}}\hat{T}^i_{ik} + \hat{R}_{i\overline{\ell}k\overline{q}}\hat{g}^{\overline{q}i} - \hat{R}_{k\overline{\ell}i\overline{q}}\hat{g}^{\overline{q}i}) = 0, \tag{3.6}$$

establishing (3.2).

We now give the estimates on $(I), (II), (III)$. The bounds on (II) and (III) follow immediately from the definitions of these quantities. It remains to prove the bound on (I).

In the special case when g and $\hat{g}$ are Kähler, then it was shown by Aubin [**1**] and Yau [**60**] that $(I) \leqslant 0$. To bound (I) in general we follow Cherrier's generalization of this argument [**10**], as follows. Consider the inequality

$$K = \hat{g}^{\overline{\ell}i}g^{\overline{j}p}g^{\overline{q}k}B_{i\overline{j}k}\overline{B_{\ell\overline{p}q}} \geqslant 0,$$

where

$$B_{i\overline{j}k} = \hat{\nabla}_i g_{k\overline{j}} - g_{i\overline{j}}\frac{\hat{\nabla}_k \mathrm{tr}_{\hat{g}}g}{\mathrm{tr}_{\hat{g}}g} + C_{i\overline{j}k},$$

and $C_{i\bar{j}k}$ will be specified below. Calculate

$$K = \hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} \hat{\nabla}_i g_{k\bar{j}} \hat{\nabla}_{\bar{\ell}} g_{p\bar{q}} + \frac{1}{\mathrm{tr}_{\hat{g}} g} g^{\bar{q}k} \hat{\nabla}_k \mathrm{tr}_{\hat{g}} g \hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g$$
$$- 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{q}k} \hat{\nabla}_i g_{k\bar{\ell}} \frac{\hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g}{\mathrm{tr}_{\hat{g}} g} \right) - 2\mathrm{Re}\left(\hat{g}^{\bar{j}i} g^{\bar{q}k} C_{i\bar{j}k} \frac{\hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g}{\mathrm{tr}_{\hat{g}} g} \right)$$
$$+ 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} C_{i\bar{j}k} \hat{\nabla}_{\bar{\ell}} g_{p\bar{q}} \right) + \hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} C_{i\bar{j}k} \overline{C_{\ell\bar{p}q}}.$$

Using the identity

$$\hat{\nabla}_i g_{k\bar{\ell}} = \hat{\nabla}_k g_{i\bar{\ell}} + (T_0)^p_{ik} (g_0)_{p\bar{\ell}} - \hat{T}^p_{ik} g_{p\bar{\ell}},$$

in the third term we get

$$K = \hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} \hat{\nabla}_i g_{k\bar{j}} \hat{\nabla}_{\bar{\ell}} g_{p\bar{q}} - \frac{1}{\mathrm{tr}_{\hat{g}} g} g^{\bar{q}k} \hat{\nabla}_k \mathrm{tr}_{\hat{g}} g \hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g$$
$$- 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{q}k} (T_0)^p_{ik} (g_0)_{p\bar{\ell}} \frac{\hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g}{\mathrm{tr}_{\hat{g}} g} \right)$$
$$- 2\mathrm{Re}\left(\hat{g}^{\bar{j}i} g^{\bar{q}k} \left[C_{i\bar{j}k} - \hat{T}^p_{ik} g_{p\bar{j}} \right] \frac{\hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g}{\mathrm{tr}_{\hat{g}} g} \right)$$
$$+ 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} C_{i\bar{j}k} \hat{\nabla}_{\bar{\ell}} g_{p\bar{q}} \right) + \hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} C_{i\bar{j}k} \overline{C_{\ell\bar{p}q}},$$

and comparing this expression with (I) we get

$$(I) = \frac{1}{\mathrm{tr}_{\hat{g}} g} \Bigg(- K - 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{q}k} (T_0)^p_{ik} (g_0)_{p\bar{\ell}} \frac{\hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g}{\mathrm{tr}_{\hat{g}} g} \right)$$
$$- 2\mathrm{Re}\left(\hat{g}^{\bar{j}i} g^{\bar{q}k} \left[C_{i\bar{j}k} - \hat{T}^p_{ik} g_{p\bar{j}} \right] \frac{\hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g}{\mathrm{tr}_{\hat{g}} g} \right)$$
$$+ 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} [C_{i\bar{j}k} - \hat{T}^r_{ik} g_{r\bar{j}}] \hat{\nabla}_{\bar{\ell}} g_{p\bar{q}} \right)$$
$$+ \hat{g}^{\bar{\ell}i} g^{\bar{j}p} g^{\bar{q}k} C_{i\bar{j}k} \overline{C_{\ell\bar{p}q}} - g^{\bar{j}i} \hat{g}^{\bar{\ell}k} \hat{T}^p_{ik} \overline{\hat{T}^q_{j\ell}} g_{p\bar{q}} \Bigg),$$

and so the obvious choice to make is

$$C_{i\bar{j}k} = \hat{T}^p_{ik} g_{p\bar{j}},$$

which makes three terms disappear, and gives us

$$(I) = \frac{1}{\mathrm{tr}_{\hat{g}} g} \left(- K - 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{q}k} (T_0)^p_{ik} (g_0)_{p\bar{\ell}} \frac{\hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g}{\mathrm{tr}_{\hat{g}} g} \right) \right). \tag{3.7}$$

Hence

$$(I) \leqslant \frac{2}{(\mathrm{tr}_{\hat{g}} g)^2} \mathrm{Re}\left(\hat{g}^{\bar{\ell}i} g^{\bar{q}k} (T_0)^p_{ki} (g_0)_{p\bar{\ell}} \hat{\nabla}_{\bar{q}} \mathrm{tr}_{\hat{g}} g \right),$$

as required. q.e.d.

4. Maximal existence time for the flow

In this section we give proofs of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. As an aside, note that T can also be defined by

$$T = \sup\{T_0 \geqslant 0 \mid \forall t \in [0, T_0], \exists \psi \in C^\infty(M) \text{ with } \alpha_t + \sqrt{-1}\partial\overline{\partial}\psi > 0\},$$

for α_t given by (1.3).

Fix $T_0 < T$. We will show there exists a solution of (1.1) on $[0, T_0)$. Define reference metrics $\hat{\omega}_t$ for $t \in [0, T_0]$ by

$$\hat{\omega}_t := \alpha_t + \frac{t}{T_0}\sqrt{-1}\partial\overline{\partial} f_{T_0} = \frac{T_0 - t}{T_0}\omega_0 + \frac{t}{T_0}(\alpha_{T_0} + \sqrt{-1}\partial\overline{\partial} f_{T_0}),$$

with f_{T_0} a function satisfying $\alpha_{T_0} + \sqrt{-1}\partial\overline{\partial} f_{T_0} > 0$. Note that these Hermitian metrics vary smoothly on the compact interval $[0, T_0]$ and hence we have estimates on $\hat{\omega}_t$ which are uniform for t in $[0, T_0]$. It is convenient to write $\hat{\omega}_t = \omega_0 + t\chi$ where χ is given by

$$\chi = \frac{1}{T_0}\sqrt{-1}\partial\overline{\partial} f_{T_0} - \mathrm{Ric}(\omega_0).$$

Define a volume form $\Omega = \omega_0^n e^{\frac{f_{T_0}}{T_0}}$, which satisfies $\sqrt{-1}\partial\overline{\partial}\log\Omega = \chi = \frac{\partial}{\partial t}\hat{\omega}_t$. Now consider the parabolic complex Monge-Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log\frac{(\hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}, \quad \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \varphi|_{t=0} = 0. \tag{4.1}$$

If φ solves (4.1) on some time interval, then taking $\sqrt{-1}\partial\overline{\partial}$ of (4.1) shows that $\omega = \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi$ solves (1.1) on the same time interval. Conversely, if ω solves (1.1) on an interval contained in $[0, T_0]$, then we have

$$\frac{\partial}{\partial t}(\omega - \hat{\omega}_t) = -\mathrm{Ric}(\omega) - \chi = \sqrt{-1}\partial\overline{\partial}\left(\log\frac{\omega^n}{\omega_0^n} - \frac{f_{T_0}}{T_0}\right) = \sqrt{-1}\partial\overline{\partial}\log\frac{\omega^n}{\Omega},$$

so if we choose φ to solve

$$\frac{\partial}{\partial t}\varphi = \log\frac{\omega^n}{\Omega}, \quad \varphi|_{t=0} = 0,$$

which is an ODE in t for each fixed point on M, then we have $\frac{\partial}{\partial t}(\omega - \hat{\omega}_t - \sqrt{-1}\partial\overline{\partial}\varphi) = 0$ so that indeed $\omega = \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi$ and φ satisfies (4.1). Therefore, the two flows (1.1) and (4.1) are essentially equivalent.

We know by standard parabolic theory that there exists a unique maximal solution of (4.1) on some time interval $[0, T_{\max})$ with $T_{\max} > 0$. We may as well assume that $T_{\max} \leqslant T_0$. Assume for a contradiction that $T_{\max} < T_0$.

We now prove uniform estimates for φ solving (4.1) up to the maximal time:

Lemma 4.1. *There is a positive constant C_0, independent of $t \in [0, T_{\max})$, such that*

(i) $\|\varphi(t)\|_{C^0} \leqslant C_0$.
(ii) $\|\dot{\varphi}(t)\|_{C^0} \leqslant C_0$.
(iii) $C_0^{-1}\omega_0 \leqslant \omega(t) \leqslant C_0\omega_0$.
(iv) *For each $k = 0, 1, 2, \ldots,$ there exist constants C_k such that*

$$\|\varphi(t)\|_{C^k(\omega_0)} \leqslant C_k.$$

Proof. The proofs of (i) and (ii) follow almost verbatim from the Kähler case [**51**], but we include brief arguments for the reader's convenience. For (i), put $\psi = \varphi - At$ for a constant $A > 0$. Suppose that a maximum of ψ occurs at a point with $t > 0$. Then at this point, by the maximum principle,

$$0 \leqslant \frac{\partial}{\partial t}\psi = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\Omega} - A < 0,$$

if A is chosen sufficiently large, a contradiction. Here we are using the fact that $\hat{\omega}_t$ is a smooth family of metrics on $[0, T_{\max}]$. This gives an upper bound for ψ and hence φ. The lower bound is proved similarly.

For a lower bound for $\dot{\varphi}$, first note that

$$\frac{\partial}{\partial t}\dot{\varphi} = \Delta\dot{\varphi} + \mathrm{tr}_\omega \chi.$$

Put $Q_0 = (T_0 - t)\dot{\varphi} + \varphi + nt$ and compute

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_0 = (T_0 - t)\mathrm{tr}_\omega\chi + n - \Delta\varphi = \mathrm{tr}_\omega(\hat{\omega}_t + (T_0 - t)\chi) = \mathrm{tr}_\omega\hat{\omega}_{T_0} > 0.$$

Hence Q_0 is bounded below by the maximum principle and this gives a lower bound for $\dot{\varphi}$ (since we assume $T_{\max} < T_0$).

For the upper bound of $\dot{\varphi}$, define $Q_1 = t\dot{\varphi} - \varphi - nt$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_1 = t\,\mathrm{tr}_\omega\chi - n + \mathrm{tr}_\omega(\omega - \hat{\omega}_t) = \mathrm{tr}_\omega(t\chi - \hat{\omega}_t) = -\mathrm{tr}_\omega\omega_0 \leqslant 0,$$

and an upper bound for Q_1 and hence $\dot{\varphi}$ follows from the maximum principle.

Note that by (ii) the volume form ω^n is uniformly equivalent to a fixed volume form ω_0^n, say. To prove (iii) then, it suffices to obtain a uniform upper bound of $\mathrm{tr}_{g_0} g$. For this, we could apply the second order estimate of Gill [**19**]. Instead, we give a different proof which uses a trick due to Phong-Sturm [**33**], since we will use it again later in Sections 5 and 9. Choose a constant $\tilde{C}$ so that $\varphi + \tilde{C} \geqslant 1$. Following [**33**], we compute the evolution of

$$Q_2 = \log \mathrm{tr}_{g_0} g - A\varphi + \frac{1}{\varphi + \tilde{C}},$$

for A a large constant to be determined. We wish to show that at a point where Q_2 achieves a maximum, $\mathrm{tr}_{g_0} g$ is uniformly bounded from above. It will then follow from (i) that $\mathrm{tr}_{g_0} g$ is bounded from above on M.

We apply Proposition 3.1 with $\hat{g} = g_0$ to obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \mathrm{tr}_{g_0} g \leqslant \frac{2}{(\mathrm{tr}_{g_0} g)^2} \mathrm{Re}\left(g^{\overline{\ell}k} (T_0)^p_{kp} \partial_{\overline{\ell}} \mathrm{tr}_{g_0} g\right) + C\mathrm{tr}_g g_0, \tag{4.2}$$

assuming we are calculating at a point with $\mathrm{tr}_{g_0} g \geqslant 1$.

To bound the first term on the right hand side of (4.2), we note that at a maximum point of Q_2 we have $\partial_i Q_2 = 0$ and hence

$$\frac{1}{\mathrm{tr}_{g_0} g} \partial_i \mathrm{tr}_{g_0} g - A\partial_i \varphi - \frac{1}{(\varphi + \tilde{C})^2} \partial_i \varphi = 0. \tag{4.3}$$

Then at this maximum point for Q_2,

$$\begin{aligned} &\left|\frac{2}{(\mathrm{tr}_{g_0} g)^2} \mathrm{Re}\left(g^{\overline{\ell}k} (T_0)^p_{kp} \partial_{\overline{\ell}} \mathrm{tr}_{g_0} g\right)\right| \\ &\leqslant \left|\frac{2}{\mathrm{tr}_{g_0} g} \mathrm{Re}\left(\left(A + \frac{1}{(\varphi + \tilde{C})^2}\right) g^{\overline{\ell}k} (T_0)^p_{kp} (\partial_{\overline{\ell}} \varphi)\right)\right| \\ &\leqslant \frac{|\partial \varphi|^2_g}{(\varphi + \tilde{C})^3} + CA^2 (\varphi + \tilde{C})^3 \frac{\mathrm{tr}_g g_0}{(\mathrm{tr}_{g_0} g)^2}, \end{aligned} \tag{4.4}$$

for a uniform constant C. But we may assume that at the maximum of Q_2 we have $(\mathrm{tr}_{g_0} g)^2 \geqslant A^2 (\varphi + \tilde{C})^3$, since otherwise we already have the required bound on $\mathrm{tr}_{g_0} g$.

Thus at the maximum of Q_2, using (4.2), (4.4),

$$\begin{aligned} 0 \leqslant \left(\frac{\partial}{\partial t} - \Delta\right) Q_2 \leqslant & \frac{|\partial \varphi|^2_g}{(\varphi + \tilde{C})^3} + C\mathrm{tr}_g g_0 - \left(A + \frac{1}{(\varphi + \tilde{C})^2}\right) \dot{\varphi} \\ & + \left(A + \frac{1}{(\varphi + \tilde{C})^2}\right) \mathrm{tr}_g (g - \hat{g}_t) - \frac{2}{(\varphi + \tilde{C})^3} |\partial \varphi|^2_g. \end{aligned}$$

Since $\hat{g}_t \geqslant c_0 g_0$ with $t \in [0, T_{\max}]$, for some uniform $c_0 > 0$, we may choose A sufficiently large so that $A\mathrm{tr}_g \hat{g}_t \geqslant (C + 1)\mathrm{tr}_g g_0$. Since $\dot{\varphi}$ is bounded from (ii), we obtain

$$\mathrm{tr}_g g_0 \leqslant C'$$

at the maximum of Q_2, for a uniform constant C'. Hence at the maximum of Q_2,

$$\mathrm{tr}_{g_0} g \leqslant \frac{1}{(n-1)!} (\mathrm{tr}_g g_0)^{n-1} \frac{\det g}{\det g_0} \leqslant C'',$$

where we have applied (ii) again. Hence $\mathrm{tr}_{g_0} g$ is uniformly bounded from above on M, giving (iii).

Part (iv) follows from the higher order estimates of Gill [**19**]. q.e.d.

It is now straightforward to complete the proof of Theorem 1.2. We have uniform estimates for the flow $\varphi(t)$ on $[0, T_{\max})$. Taking limits we have a solution on $[0, T_{\max}]$. Applying the standard parabolic short time existence theory we obtain a solution a little beyond $T_{\max}$, a contradiction. Hence there exists a unique solution of (4.1) on $[0, T_0)$. Taking $\sqrt{-1}\partial\overline{\partial}$ of (4.1) gives us a solution of (1.1) on $[0, T_0)$. Since $T_0 < T$ was chosen arbitrarily, we get a solution of (1.1) on $[0, T)$.

Uniqueness follows from uniqueness of solutions to (4.1). Clearly the flow cannot extend beyond T. q.e.d.

Next we give a proof of Theorem 1.3.

Proof of Theorem 1.3. Note that in general if $\partial\overline{\partial}\omega_0 = 0$ then $\partial\overline{\partial}\omega = 0$ for all later times. If $n = 2$ this means that the flow (1.1) preserves the Gauduchon condition (recall that a Hermitian metric ω is *Gauduchon* if $\partial\overline{\partial}\omega^{n-1} = 0$). The key result that we need is due to Buchdahl [**7**] and it says that if η is a $\partial\overline{\partial}$-closed real $(1,1)$ form and ω_0 a Gauduchon metric on a compact complex surface M such that

$$\int_M \eta^2 > 0, \quad \int_M \eta \wedge \omega_0 > 0, \quad \int_D \eta > 0,$$

for all irreducible effective divisors D on M with $D^2 < 0$, then there exists a smooth real function f such that $\eta + \sqrt{-1}\partial\overline{\partial} f > 0$ is a Gauduchon metric. First of all observe that the following condition

$$\int_M \eta^2 > 0, \quad \int_M \eta \wedge \omega_0 \geqslant 0, \quad \int_D \eta > 0, \tag{4.5}$$

for all D as above is enough to guarantee the same conclusion. Indeed consider the $(1,1)$ forms $\eta_t = \eta + t\omega_0$, and take $t > 0$ large so that η_t is a Gauduchon metric. Then we have

$$\int_M \eta \wedge \eta_t = \int_M \eta^2 + t \int_M \eta \wedge \omega_0 > 0,$$

and so we can apply Buchdahl's result using η_t instead of ω_0. In particular, if (4.5) holds then in fact we have the strict inequality $\int_M \eta \wedge \omega_0 > 0$.

We now apply this discussion to the $(1,1)$ forms $\alpha_t = \omega_0 - t\mathrm{Ric}(\omega_0)$. As we have seen earlier, the evolving metrics $\omega(t)$ are of the form $\omega(t) = \alpha_t + \sqrt{-1}\partial\overline{\partial}\varphi_t$, and so it follows that T is the supremum of all $t \geqslant 0$ such that

$$\int_M \alpha_t^2 > 0, \int_M \alpha_t \wedge \omega_0 \geqslant 0, \int_D \alpha_t > 0, \tag{4.6}$$

for all D as above. Furthermore, if (4.6) holds at some time t then in fact $\int_M \alpha_t \wedge \omega_0 > 0$. Therefore T is also the supremum of all $T_0 \geqslant 0$ such that

$$\int_M \alpha_t^2 > 0, \int_D \alpha_t > 0,$$

hold for all $t \in [0, T_0]$. q.e.d.

5. Estimates away from a divisor

In this section we give the proof of Theorem 1.6. Let $\omega = \omega(t)$ be the solution of (1.1) on the maximal time interval $[0, T)$. We assume in this section that $T < \infty$.

In addition, we make the assumption that there exists a smooth function f_T on M such that

$$\hat{\omega}_T := \alpha_T + \sqrt{-1}\partial\overline{\partial} f_T \geqslant 0, \tag{5.1}$$

where we recall from (1.3) that $\alpha_T = \omega_0 - T\mathrm{Ric}(\omega_0)$. In the Kähler case (5.1) corresponds to the condition that the limiting Kähler class has a nonnegative representative. Define reference metrics

$$\hat{\omega}_t := \frac{1}{T}\left((T-t)\omega_0 + t\hat{\omega}_T\right), \text{ for } t \in [0, T).$$

Then by the same argument as in the beginning of Section 4, we may write $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi$ where $\varphi = \varphi(t)$ solves the parabolic complex Monge-Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}, \quad \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \varphi|_{t=0} = 0, \tag{5.2}$$

for the smooth volume form $\Omega = \omega_0^n e^{\frac{f_T}{T}}$.

We begin with a proposition, which is exactly analogous to a result for the Kähler-Ricci flow [**50**, **51**] (see also the expositions in [**37**, **40**]).

Proposition 5.1. *With the assumptions above, there exists a constant C such that for all $t \in [0, T)$,*

(i) $\|\varphi(t)\|_{C^0} \leqslant C$.

(ii) $\dot{\varphi}(t) \leqslant C$.

Proof. The proof is exactly the same as in the Kähler case. Briefly: the upper bound of φ follows from the same argument as in Lemma 4.1. For the lower bound of φ observe that $\hat{\omega}_t = \frac{(T-t)}{T}\omega_0 + \frac{t}{T}\hat{\omega}_T \geqslant \frac{(T-t)}{T}\omega_0$ and hence $\hat{\omega}_t^n \geqslant c_0(T-t)^n\Omega$ for a uniform $c_0 > 0$. The lower bound of φ follows from applying the maximum principle to the quantity

$$Q = \varphi + n(T-t)(\log(T-t) - 1) - (\log c_0 - 1)t.$$

Indeed if Q achieves its mimimum at some point (x, t) with $t > 0$, then at (x, t) we have $\sqrt{-1}\partial\overline{\partial}\varphi \geqslant 0$ and

$$0 \geqslant \frac{\partial}{\partial t}Q \geqslant \log\frac{\hat{\omega}_t^n}{\Omega} - n\log(T-t) - \log c_0 + 1 \geqslant 1,$$

a contradiction. Hence Q achieves its minimum at time $t = 0$, which gives the lower bound for φ.

The upper bound of $\dot{\varphi}$ follows from the same argument as in Lemma 4.1. q.e.d.

Next we prove a parabolic Schwarz lemma for volume forms for the flow (1.1). It holds in the special case that $\hat{\omega}_T$ is the pull-back of a metric from another Hermitian manifold via a holomorphic map. In the Kähler case this is due to Song-Tian [**34**], and is a parabolic version of Yau's (volume) Schwarz lemma [**61**].

Proposition 5.2. *Let $\omega = \omega(t)$ solve (1.1) on $[0, T)$ with $T < \infty$. Suppose we have a holomorphic map $\pi : M \to N$ for N a compact Hermitian manifold of the same dimension n, equipped with a Hermitian metric ω_N. Suppose that $\hat{\omega}_T = \pi^*\omega_N$. Then there exists a uniform constant $C > 0$ such that on $M \times [0, T)$,*

$$\omega^n \geqslant \frac{1}{C}(\pi^*\omega_N)^n.$$

Proof. Call

$$u = \frac{(\pi^*\omega_N)^n}{\omega^n}$$

the ratio of the volume forms. At any point where $u > 0$ we can calculate

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log u = \mathrm{tr}_\omega \pi^* \mathrm{Ric}(\omega_N) - \mathrm{tr}_\omega \mathrm{Ric}(\omega) - g^{\bar{j}i}\frac{\partial}{\partial t} g_{i\bar{j}} = \mathrm{tr}_\omega \pi^* \mathrm{Ric}(\omega_N).$$

We apply the maximum principle to

$$Q = \log u - A\varphi - An(T - t)(\log(T - t) - 1),$$

for a constant A to be determined (cf. [**40**, Lemma 7.3]). The maximum of Q is achieved at a point where $u > 0$, so we compute

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) Q &= \mathrm{tr}_\omega \pi^* \mathrm{Ric}(\omega_N) - A\dot{\varphi} + An \log(T - t) + A\mathrm{tr}_\omega(\omega - \hat{\omega}_t) \\ &= -\mathrm{tr}_\omega((A - 1)\hat{\omega}_t - \pi^*\mathrm{Ric}(\omega_N)) - A \log \frac{\omega^n}{\Omega(T - t)^n} \\ &\quad - \mathrm{tr}_\omega \hat{\omega}_t + An.\end{aligned}$$

Choose A sufficiently large so that for all $t \in [0, T]$,

$$(A-1)\hat{\omega}_t - \pi^*\mathrm{Ric}(\omega_N) = \frac{(A - 1)}{T}((T - t)\omega_0 + t\pi^*\omega_N) - \pi^*\mathrm{Ric}(\omega_N) \geqslant \pi^*\omega_N.$$

Note that by the arithmetic-geometric means inequality,

$$\mathrm{tr}_\omega \hat{\omega}_t \geqslant \frac{(T - t)}{T}\mathrm{tr}_\omega \omega_0 \geqslant c\left(\frac{(T - t)^n\Omega}{\omega^n}\right)^{1/n},$$

for a uniform $c > 0$. Then

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) Q &\leqslant -\mathrm{tr}_\omega \pi^*\omega_N + A \log \frac{(T - t)^n\Omega}{\omega^n} - c\left(\frac{(T - t)^n\Omega}{\omega^n}\right)^{1/n} + An \\ &\leqslant -\mathrm{tr}_\omega \pi^*\omega_N + C,\end{aligned}$$

using the fact that the map $x \mapsto A \log x - cx^{1/n}$ is uniformly bounded from above for $x > 0$. Thus at a maximum point of Q we have $\mathrm{tr}_\omega \pi^*\omega_N \leqslant$

C, and applying the arithmetic-geometric means inequality again, u is uniformly bounded from above at this point. Since φ is uniformly bounded by Proposition 5.1, this implies that Q is bounded from above, and hence so is u. q.e.d.

Now assume that we are in the situation of Theorem 1.6. The map $\pi : M \to N$ is a holomorphic map blowing down an exceptional divisor E to a point $p \in N$. More explicitly, a neighborhood of E in M can be identified with

$$\tilde{B} = \{(z, \ell) \in B \times \mathbb{P}^{n-1} \mid z \in \ell\},$$

where B is the open unit ball in $\mathbb{C}^n$ and elements ℓ in $\mathbb{P}^{n-1}$ are identified with lines through the origin in $\mathbb{C}^n$. The map π on $\tilde{B}$ is identified with the projection $(z, \ell) \mapsto z \in B$ and the exceptional divisor $E \subset M$ with the set $\pi^{-1}(0) \subset \tilde{B}$. π is a biholomorphism from $M \setminus E$ to $N \setminus \{p\}$.

By assumption, there exists a function $\psi = f_T$ with

$$\hat{\omega}_T := \alpha_T + \sqrt{-1}\partial\overline{\partial} f_T = \pi^*\omega_N.$$

Now there exists a Hermitian metric h on the fibers of the line bundle $[E]$ associated to the divisor E with the property that for $\varepsilon > 0$ sufficiently small,

$$\pi^*\omega_N - \varepsilon R_h > 0, \quad \text{with } R_h = -\sqrt{-1}\partial\overline{\partial}\log h. \tag{5.3}$$

For a proof of this statement, see [**20**, p. 187]. Although it is stated there in the Kähler case, the same proof carries over with ω_N Hermitian. Fix s a holomorphic section of $[E]$ vanishing along E to order 1.

We have a lemma:

Lemma 5.3. *With these hypotheses, there exists $A > 0$ and C such that*

$$\mathrm{tr}_{g_0}\, g \leqslant \frac{C}{|s|_h^{2A}}.$$

Proof. Define, as in Tsuji's work [**56**], $\tilde{\varphi} = \varphi - \varepsilon_0 \log |s|_h^2$, which is uniformly bounded from below and goes to infinity on E. Here $\varepsilon_0 > 0$ is a small constant that will be specified below. Choose a constant C_0 so that $\tilde{\varphi} + C_0 \geqslant 1$. Following Phong-Sturm [**33**] (and as in Lemma 4.1 above), we compute the evolution of

$$Q = \log \mathrm{tr}_{g_0} g - A\tilde{\varphi} + \frac{1}{\tilde{\varphi} + C_0},$$

for A to be determined (assume at least $A\varepsilon_0 > 1$). Note that the quantity $1/(\tilde{\varphi} + C_0)$ is bounded (in fact it lies between 0 and 1). Moreover, Q tends to negative infinity on E and hence for each fixed time t, the quantity $Q(x, t)$ achieves a maximum at some point in $M \setminus E$.

From Proposition 3.1 we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)\log \mathrm{tr}_{g_0} g \leqslant \frac{2}{(\mathrm{tr}_{g_0} g)^2}\mathrm{Re}\left(g^{\bar{\ell}k}(T_0)^p_{kp}\partial_{\bar{\ell}}\mathrm{tr}_{g_0} g\right)+C\mathrm{tr}_g g_0, \tag{5.4}$$

assuming we are calculating at a point with $\mathrm{tr}_{g_0} g \geqslant 1$. To bound the first term on the right hand side, we note that at a maximum point of Q we have $\partial_i Q = 0$ and hence

$$\frac{1}{\mathrm{tr}_{g_0} g}\partial_i \mathrm{tr}_{g_0} g - A\partial_i\tilde{\varphi} - \frac{1}{(\tilde{\varphi}+C_0)^2}\partial_i\tilde{\varphi}=0.$$

Thus at this maximum point for Q,

$$\begin{aligned}&\left|\frac{2}{(\mathrm{tr}_{g_0} g)^2}\mathrm{Re}\left(g^{\bar{\ell}k}(T_0)^p_{kp}\partial_{\bar{\ell}}\mathrm{tr}_{g_0} g\right)\right|\\ &\leqslant \left|\frac{2}{\mathrm{tr}_{g_0} g}\mathrm{Re}\left(\left(A+\frac{1}{(\tilde{\varphi}+C_0)^2}\right)g^{\bar{\ell}k}(T_0)^p_{kp}(\partial_{\bar{\ell}}\tilde{\varphi})\right)\right|\\ &\leqslant \frac{|\partial\tilde{\varphi}|^2_g}{(\tilde{\varphi}+C_0)^3}+CA^2(\tilde{\varphi}+C_0)^3\frac{\mathrm{tr}_g g_0}{(\mathrm{tr}_{g_0} g)^2},\end{aligned}$$

for a uniform constant C. If at the maximum of Q we have $(\mathrm{tr}_{g_0} g)^2 \leqslant A^2(\tilde{\varphi}+C_0)^3$ then at the same point we have

$$(\mathrm{tr}_{g_0} g)|s|_h^{2A\varepsilon_0} \leqslant A(\varphi-\varepsilon_0\log|s|_h^2+C_0)^{\frac{3}{2}}|s|_h^{2A\varepsilon_0}\leqslant C_A,$$

for a constant C_A depending on A, and so

$$Q=\log((\mathrm{tr}_{g_0} g)|s|_h^{2A\varepsilon_0})-A\varphi+\frac{1}{\tilde{\varphi}+C_0}\leqslant C'_A,$$

and we are done. If on the other hand at the maximum of Q we have $A^2(\tilde{\varphi}+C_0)^3 \leqslant (\mathrm{tr}_{g_0} g)^2$ then

$$\left|\frac{2}{(\mathrm{tr}_{g_0} g)^2}\mathrm{Re}\left(g^{\bar{\ell}k}(T_0)^p_{kp}\nabla_{\bar{\ell}}\mathrm{tr}_{g_0} g\right)\right| \leqslant \frac{|\partial\tilde{\varphi}|^2_g}{(\tilde{\varphi}+C_0)^3}+C\mathrm{tr}_g g_0.$$

Now compute at the maximum of Q, using (5.4),

$$\begin{aligned}0\leqslant\left(\frac{\partial}{\partial t}-\Delta\right)Q \leqslant{}& \frac{|\partial\tilde{\varphi}|^2_g}{(\tilde{\varphi}+C_0)^3}+C\mathrm{tr}_g g_0-\left(A+\frac{1}{(\tilde{\varphi}+C_0)^2}\right)\dot{\varphi}\\ &+\left(A+\frac{1}{(\tilde{\varphi}+C_0)^2}\right)\mathrm{tr}_\omega(\omega-(\hat{\omega}_t-\varepsilon_0 R_h))\\ &-\frac{2}{(\tilde{\varphi}+C_0)^3}|\partial\tilde{\varphi}|^2_g.\end{aligned}\tag{5.5}$$

Since we clearly have $\hat{\omega}_t \geqslant c\hat{\omega}_T$ for some constant $c>0$, we can use (5.3) to get that $\hat{\omega}_t-\varepsilon_0 R_h \geqslant c_0\omega_0$ for some uniform $c_0>0$, provided we choose ε_0 sufficiently small. Hence we may choose A sufficiently large so that

$$\mathrm{tr}_g g_0 \leqslant C\log\frac{\Omega}{\omega^n}+C.$$

Hence at the maximum of Q,

$$\mathrm{tr}_{g_0} g \leqslant \frac{1}{(n-1)!}(\mathrm{tr}_g g_0)^{n-1} \frac{\det g}{\det g_0} \leqslant C \frac{\omega^n}{\Omega}\left(\log \frac{\Omega}{\omega^n}\right)^{n-1} + C \leqslant C',$$

because we know that $\frac{\omega^n}{\Omega} \leqslant C$ and $x \mapsto x|\log x|^{n-1}$ is bounded above for x close to zero. This implies that Q is bounded from above at its maximum, and completes the proof of the lemma. q.e.d.

It is now straightforward to complete the proof of the theorem.

Proof of Theorem 1.6. We apply Proposition 5.2 and Lemma 5.3 to see that for any compact set $K \subset M \setminus E$, there exists a constant $C_K > 0$ such that

$$C_K^{-1}\omega_0 \leqslant \omega(t) \leqslant C_K \omega_0 \quad \text{on } K \times [0, T).$$

Applying the higher order estimates of Gill [**19**], which are local, we obtain uniform C^∞ estimates for $\omega(t)$ on compact subsets of $M \setminus E$. In particular, for every compact set K there exists a constant C'_K such that

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) \leqslant C'_K \omega,$$

which implies that $e^{-C'_K t}\omega(t)$ is decreasing in t as well as being bounded from below. This implies that a limit for $\omega(t)$ exists as $t \to T$, and since we have uniform estimates away from E, we see that $\omega(t)$ converges in C^∞ on compact subsets to a smooth Hermitian metric ω_T on $M \setminus E$. q.e.d.

6. The Chern-Ricci flow on complex surfaces

In this section we give a proof of Theorem 1.5, and we pose some conjectures on the behavior of the Chern-Ricci flow on surfaces, and its relation to the 'minimal model program for complex surfaces'.

First recall that the Kodaira dimension of a compact complex manifold M of dimension n is given by

$$\kappa(M) = \limsup_{\ell \to +\infty} \frac{\log \dim H^0(M, \ell K_M)}{\log \ell} \in \{-\infty, 0, 1, \ldots, n\}.$$

Proof of Theorem 1.5. (a) We need to show that if $T = \infty$ then M is minimal. If M is a non-minimal compact complex surface then we must have $T < \infty$, because if D is any (-1)-curve in M we have that $D \cdot K_M < 0$ and so the volume of D,

$$\int_D \omega(t) = \int_D \omega_0 + 2\pi t D \cdot K_M,$$

becomes zero in finite time.

(b) If the volume goes to zero at time $T < \infty$, we have that $\int_M \alpha_T^2 = 0$, where we recall from (1.3) that α_T is the $\partial\overline{\partial}$-closed $(1,1)$ form $\alpha_T =$

$\omega_0 - T\mathrm{Ric}(\omega_0)$. We claim that in this case the Kodaira dimension $\kappa(M)$ is negative, which by the Kodaira-Enriques classification [**2**, **3**] implies that M is either birational to a ruled surface or of class VII. Indeed, if $\kappa(M) \geqslant 0$ then some power ℓK_M, $\ell \geqslant 1$ of the canonical bundle would be effective. Let E be an effective divisor in $|\ell K_M|$; then E must be nonempty since otherwise ℓK_M would be trivial, and so $c_1^{\mathrm{BC}}(M) = 0$ and by Theorem 1.1 we would have $T = \infty$. We thus conclude that E is nonempty, and let s be a section of ℓK_M defining E and h a smooth metric on the fibers of ℓK_M. Then by the Poincaré-Lelong formula the curvature η of h is a smooth closed real $(1,1)$ form such that

$$2\pi[E] = \eta + \sqrt{-1}\partial\overline{\partial}\log|s|_h^2$$

holds as currents on M (see also [**17**]). In particular for any $\partial\overline{\partial}$-closed smooth $(1,1)$ form γ we have

$$\int_M \gamma \wedge \eta = 2\pi \int_E \gamma.$$

Moreover if γ is a Gauduchon metric we have

$$\int_M \gamma \wedge c_1^{\mathrm{BC}}(M) = -\frac{2\pi}{\ell}\int_E \gamma < 0,$$

because $-\frac{1}{\ell}\eta$ represents $c_1^{\mathrm{BC}}(M)$. Taking $\gamma = \omega(t)$ and letting t approach T we get

$$\int_M \alpha_T \wedge c_1^{\mathrm{BC}}(M) \leqslant 0.$$

We also have

$$\begin{aligned} 0 = \int_M \alpha_T^2 &= -T\int_M \alpha_T \wedge c_1^{\mathrm{BC}}(M) + \int_M \alpha_T \wedge \omega_0 \\ &\geqslant \int_M \alpha_T \wedge \omega_0 \geqslant 0, \end{aligned}$$

which implies that $\int_M \alpha_T \wedge \omega_0 = 0$. Applying [**6**, Lemma 4], we see that $\alpha_T = \sqrt{-1}\partial\overline{\partial} f$ for some smooth function f, which implies that ω_0 is Kähler and that M is Fano, contradicting the assumption that the Kodaira dimension of M is nonnegative. To see that M cannot be an Inoue surface, we apply the observation (below) that on an Inoue surface, the Chern-Ricci flow exists for all time.

(c) Assume now that $T < \infty$ and that the volume does not collapse at time T, so that $\int_M \alpha_T^2 > 0$. We know from Theorem 1.2 that there is no smooth function f such that $\alpha_T + \sqrt{-1}\partial\overline{\partial} f > 0$ (otherwise we could continue the flow past T). On the other hand, for $\varepsilon > 0$

$$\alpha_T + \varepsilon\omega_0 = (1+\varepsilon)\omega_0 - T\mathrm{Ric}(\omega_0) = (1+\varepsilon)\left(\omega_0 - \frac{T}{1+\varepsilon}\mathrm{Ric}(\omega_0)\right),$$

and since $\frac{T}{1+\varepsilon} < T$ we have $\omega_0 - \frac{T}{1+\varepsilon}\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial} f > 0$ for some function f. Therefore

$$\int_M \omega_0 \wedge (\alpha_T + \varepsilon\omega_0) > 0,$$

and letting $\varepsilon \to 0$ we get $\int_M \alpha_T \wedge \omega_0 \geqslant 0$; therefore

$$\int_M \alpha_T \wedge (\alpha_T + \varepsilon\omega_0) = \int_M \alpha_T^2 + \varepsilon \int_M \alpha_T \wedge \omega_0 > 0.$$

We now apply the main theorem of [**7**], and we see that there is an irreducible effective divisor $D \subset M$ with $D^2 < 0$ such that $\int_D \alpha_T = 0$. Furthermore,

$$2\pi K_M \cdot D = -\int_D \mathrm{Ric}(\omega_0) = \frac{1}{T}\int_D (\alpha_T - \omega_0) = -\frac{1}{T}\int_D \omega_0 < 0, \tag{6.1}$$

and so by the adjunction formula D is a smooth (-1)-curve.

Assume now that M is minimal and consider first the case when M is Kähler. If $\kappa(M) \geqslant 0$ then K_M is nef thanks to [**2**, Corollary III.2.4], while if $\kappa(M) = -\infty$ then by the Kodaira-Enriques classification M is either $\mathbb{CP}^2$ or ruled. If K_M is nef then $\int_M c_1^{\mathrm{BC}}(M)^2 \geqslant 0$ and if $\kappa(M) \geqslant 0$ then some power of K_M is effective, so the argument above implies that $\int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M) < 0$. Thus the volume along the flow is

$$V_t = \int_M \omega(t)^2 = \int_M \omega_0^2 - 2t\int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M) + t^2 \int_M c_1^{\mathrm{BC}}(M)^2,$$

which is always positive, so by Theorem 1.3 we have $T = \infty$. On the other hand, if M is $\mathbb{CP}^2$ or ruled then we must have $T < \infty$, and since case (c) is excluded we must be in case (b). Indeed, if M is $\mathbb{CP}^2$ and E is a line inside it then $E \cdot K_M < 0$ and its volume

$$\int_E \omega(t) = \int_E \omega_0 + 2\pi t E \cdot K_M$$

goes to zero in finite time. The other case is when M is ruled and $E \cong \mathbb{CP}^1$ is a fiber of the ruling; then $E \cdot E = 0$ and by the genus formula $E \cdot K_M = -2$, which again implies that the volume of E goes to zero in finite time.

If on the other hand M is not Kähler, thanks to the Kodaira-Enriques classification [**2**, **3**] we know that minimal non-Kähler compact complex surfaces fall into the following classes:

1) primary and secondary Kodaira surfaces,
2) surfaces of class VII with $b_2(M) = 0$,
3) minimal surfaces of class VII with $b_2(M) > 0$,
4) minimal properly elliptic surfaces,

where a surface of class VII is by definition a compact complex surface with $b_1(M) = 1$ and $\kappa(M) = -\infty$, while a properly elliptic surface is an elliptic surface with $\kappa(M) = 1$. The surfaces in (1), (2), and (4) are completely classified, and while there are many examples of surfaces in (3), a complete classification is still lacking (see e.g. [**2**, **13**, **27**, **29**, **46**, **47**, **49**]). We treat each case separately.

In case (1) the manifold M has torsion canonical bundle (i.e. some power ℓK_M, $\ell \geqslant 1$ is holomorphically trivial). In particular these manifolds have $c_1^{\mathrm{BC}}(M) = 0$, and Theorem 1.1 says that the Chern-Ricci flow starting from any initial Hermitian metric ω_0 has a long time solution $\omega(t)$ (so we are in case (a)) which as t goes to infinity converges smoothly to the unique Hermitian metric of the form $\omega_\infty = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$ with $\mathrm{Ric}(\omega_\infty) = 0$.

In case (2), the manifold M is either an Inoue surface or a Hopf surface [**27**, **46**]. Suppose first that M is an Inoue surface. Then M does not have any curves and by [**48**, Remark 4.2] any Gauduchon metric ω_0 satisfies $\int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M) < 0$. In particular the volume of M along the flow is

$$V_t = \int_M \omega(t)^2 = \int_M \omega_0^2 - 2t \int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M).$$

Since V_t is always positive, Theorem 1.3 implies that the Chern-Ricci flow exists for all positive time, so we are in case (a).

If M is a Hopf surface, it follows from the arguments in [**48**, Remark 4.3] that any Gauduchon metric ω_0 on them satisfies $\int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M) > 0$. Indeed, as we have seen this holds whenever M has a plurianticanonical divisor, and [**48**, Remark 4.3] shows that every primary Hopf surface has an anticanonical divisor. But every Hopf surface is either primary or secondary (i.e. a finite unramified quotient $\tilde{M} \to M$ of a primary one) and an anticanonical divisor on $\tilde{M}$ gives a plurianticanonical divisor on M. In particular the volume of M along the flow is

$$V_t = \int_M \omega(t)^2 = \int_M \omega_0^2 - 2t \int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M),$$

which goes to zero in finite time, and so the Chern-Ricci flow exists for finite time. In fact, since every curve on M is homologous to zero, the flow exists precisely as long as the volume stays positive and then it collapses, so we are in case (b). We will investigate the behavior of the flow on a family of Hopf manifolds in Section 8.

In case (3), if we call $b_2(M) = n > 0$, we have $\int_M c_1^2(M) = -n$ (see e.g. [**47**, p. 494]). It follows that the Chern-Ricci flow starting from any initial Gauduchon metric ω_0 exists only for finite time, because the volume of M along the flow is

$$V_t = \int_M \omega(t)^2 = \int_M \omega_0^2 - 2t \int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M) - 4\pi^2 n t^2,$$

which goes to zero in finite time. Furthermore, since M is minimal and using again Theorem 1.3, we see that we are in case (b). Note that carrying out a space-time rescaling of the flow to have constant volume will still produce a solution that exists only for a finite time (cf. the discussion in [**44**]).

In case (4) we have $\int_M c_1^2(M) = 0$ and by definition some power of the canonical bundle ℓK_M, $\ell \geqslant 1$, is effective. Arguing as before, this implies that

$$\int_M \omega_0 \wedge c_1^{\mathrm{BC}}(M) < 0.$$

Therefore the volume along the Chern-Ricci flow remains positive for all time, and since M is minimal the flow has a long time solution and we are in case (a). q.e.d.

Furthermore, arguing as in [**40**, Proposition 8.4], one can show that in case (b) the volume goes to zero quadratically only on a Fano manifold in a positive multiple of the anticanonical class; otherwise it goes to zero linearly.

We finish this section with some further discussions and conjectures. We begin by considering what happens in case (c), along the lines of [**40**, Theorem 8.3]. First of all note that α_T is a $\partial\overline{\partial}$-closed real $(1,1)$ form with $\int_M \alpha_T^2 > 0$. It follows from [**6**, Lemma 4] that if ψ is another $\partial\overline{\partial}$-closed real $(1,1)$ form with $\int_M \psi \wedge \alpha_T = 0$, then $\int_M \psi^2 \leqslant 0$ with equality if and only if $\psi = \sqrt{-1}\partial\overline{\partial} f$ for some function f. If now D_1, D_2 are irreducible distinct (-1)-curves (so $D_1 \cdot D_2 \geqslant 0$) with $\int_{D_1} \alpha_T = \int_{D_2} \alpha_T = 0$, then as before we can express the divisor $D_1 + D_2$ as

$$2\pi[D_1 + D_2] = \eta + \sqrt{-1}\partial\overline{\partial} \log |s|_h^2,$$

in the sense of currents, where η is a smooth closed form that represents $2\pi c_1(D_1 + D_2)$. Therefore, since $\partial\overline{\partial}\alpha_T = 0$,

$$0 = \int_{D_1} \alpha_T + \int_{D_2} \alpha_T = \frac{1}{2\pi} \int_M \eta \wedge \alpha_T,$$

and so $4\pi^2(D_1 + D_2)^2 = \int_M \eta^2 \leqslant 0$, with equality implying that $\eta = \sqrt{-1}\partial\overline{\partial} f$. But this would give

$$0 = -\int_M \eta \wedge c_1^{\mathrm{BC}}(M) = 2\pi K_M \cdot (D_1 + D_2) < 0,$$

a contradiction. Thus we conclude that $(D_1 + D_2)^2 < 0$, which implies that $D_1 \cdot D_2 = 0$ and D_1, D_2 are disjoint. The set of all these (-1)-curves is finite, $D_1, \ldots, D_k$ say, because they give linearly independent classes in homology. Contracting all of them we get a contraction map $\pi : M \to N$, where N is a compact complex surface which is Kähler if and only if M is.

In light of the behavior of the Kähler-Ricci flow on surfaces, it is natural to ask whether the Chern-Ricci flow contracts, in the sense of [**38**], the (-1)-curves $D_1, \ldots, D_k$ to points $p_1, \ldots, p_k$ on N. First, do the metrics $\omega(t)$ converge smoothly on compact subsets of $M \setminus \bigcup_i D_i$ to a smooth Kähler metric $\omega(T)$ on $M \setminus \bigcup_i D_i$, as in Theorem 1.6? This would hold if we can find β, a $\partial\overline{\partial}$-closed real $(1,1)$ form on N, and f a smooth function on M such that $\pi^*\beta = \alpha_T + \sqrt{-1}\partial\overline{\partial} f$.

Furthermore, does the family $(M, \omega(t))$ converge in the sense of Gromov-Hausdorff to a limiting compact metric space (N, d) as $t \to T^-$? Can we produce a solution of the Chern-Ricci flow $\tilde{\omega}(t)$ on N for $t \in [T, T']$ (with $T' > T$) such that $\tilde{\omega}(T)$ on $N \setminus \{p_1, \ldots, p_k\}$ can be identified with $\omega(T)$ via the blow-down map? Does the family $(N, \tilde{\omega}(t))$ converge in the Gromov-Hausdorff sense to (N, d) as $t \to T^+$?

If this can be carried out, one could continue this process a finite number of times to obtain a solution of the Chern-Ricci flow 'with canonical surgical contractions' [**38**] all the way to the minimal model of M.

Finally, what is the long time behavior of the Chern-Ricci flow on a minimal model M? In the case when M has Kodaira dimension zero, so that it has torsion canonical bundle (and is either a Calabi-Yau surface or a Kodaira surface) the flow always converges to a Chern-Ricci flat metric (which need not be Kähler, even if M is Calabi-Yau) by Gill's Theorem 1.1. Another case (of course there are many more) is when $c_1(M) < 0$. We discuss this case, for any dimension, in the next section.

7. Convergence when $c_1(M) < 0$

In this section we assume that M is a compact Kähler manifold with $c_1(M) < 0$ and we give the proof of Theorem 1.7.

Start by fixing a smooth volume form Ω with $\mathrm{Ric}(\Omega) < 0$, which is possible because $c_1(M) < 0$, and note that $\mathrm{Ric}(\Omega)$ also represents $c_1^{\mathrm{BC}}(M)$. Therefore, for all $t > 0$ the $(1,1)$ form $\omega_0 - t\mathrm{Ric}(\Omega)$ is positive, and by Theorem 1.2 the Chern-Ricci flow (1.1) exists for all time. Call $\tilde{\omega}(s)$ its solution.

We consider now the rescaled metrics $\omega = \frac{\tilde{\omega}}{s+1}$ and a new time parameter $t = \log(s+1)$, so that the new metrics solve

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) - \omega, \quad \omega|_{t=0} = \omega_0, \tag{7.1}$$

for all positive t. First of all we show that (7.1) is equivalent to a parabolic complex Monge-Ampère equation. To see this, call $\hat{\omega} = -\mathrm{Ric}(\Omega) + e^{-t}(\mathrm{Ric}(\Omega) + \omega_0)$, and note that they are Hermitian metrics that satisfy

$$\frac{\partial}{\partial t}\hat{\omega} = -\mathrm{Ric}(\Omega) - \hat{\omega}, \quad \hat{\omega}|_{t=0} = \omega_0, \tag{7.2}$$

and $\hat{\omega}$ converges smoothly to $-\mathrm{Ric}(\Omega)$ as t goes to infinity. It follows that

$$\frac{\partial}{\partial t}(\omega - \hat{\omega}) = -(\omega - \hat{\omega}) + \sqrt{-1}\partial\overline{\partial}\log\frac{\omega^n}{\Omega}.$$

Consider now the solution φ of the equation

$$\frac{\partial}{\partial t}\varphi = \log \frac{\omega^n}{\Omega} - \varphi, \quad \varphi|_{t=0} = 0, \tag{7.3}$$

which exists for all positive time as can be seen by regarding it as an ODE in t for each fixed point on M. We have that

$$\frac{\partial}{\partial t}\left(e^t(\omega - \hat{\omega} - \sqrt{-1}\partial\overline{\partial}\varphi)\right) = 0, \quad (\omega - \hat{\omega} - \sqrt{-1}\partial\overline{\partial}\varphi)|_{t=0} = 0,$$

which implies that $\omega = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi$ holds for $t \geqslant 0$. Then Theorem 1.7 follows directly from:

Theorem 7.1. *As $t \to \infty$ we have that $\varphi \to \varphi_\infty$ smoothly, and $\omega_\infty := -\mathrm{Ric}(\Omega) + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$ equals the unique Kähler-Einstein metric ω_{KE}.*

Proof. First, we derive uniform estimates for φ independent of t. The estimates for $\|\varphi\|_{C^0}$ and $\|\dot{\varphi}\|_{C^0}$ follow from the same arguments as in [**8**, **51**, **56**]. Indeed, a simple maximum principle argument shows that $|\varphi| \leqslant C$ independent of t. Compute

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\dot{\varphi} &= -\dot{\varphi} - \mathrm{tr}_\omega(\mathrm{Ric}(\Omega) + \hat{\omega}) \\ &= -\dot{\varphi} - n + \Delta\varphi - \mathrm{tr}_\omega \mathrm{Ric}(\Omega),\end{aligned}$$

and so

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\varphi + \dot{\varphi}) = -n - \mathrm{tr}_\omega\mathrm{Ric}(\Omega).$$

At the minimum of $\varphi + \dot{\varphi}$, assuming it occurs for $t > 0$, we have $-\mathrm{tr}_\omega\mathrm{Ric}(\Omega) \leqslant n$, and since $\mathrm{Ric}(\Omega) < 0$ the arithmetic-geometric means inequality gives us $\varphi + \dot{\varphi} = \log \frac{\omega^n}{\Omega} \geqslant -C$ at this point and hence everywhere. Since $|\varphi| \leqslant C$, we get $\dot{\varphi} \geqslant -C$. But we also have that $\mathrm{Ric}(\Omega) + \hat{\omega} = e^{-t}(\mathrm{Ric}(\Omega) + \omega_0)$, and so

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\varphi + \dot{\varphi} + nt - e^t\dot{\varphi}) = \mathrm{tr}_\omega\omega_0 > 0,$$

which implies by the maximum principle that, for $t \geqslant 1$, $\dot{\varphi} \leqslant Cte^{-t} \leqslant Ce^{-t/2}$. This estimate is the same as the one in [**51**].

We feed this into the second order estimate as before. We have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log \mathrm{tr}_{g_0} g \leqslant \frac{2e^{-t}}{(\mathrm{tr}_{g_0} g)^2}\mathrm{Re}\left(g^{\overline{\ell}k}(T_0)^p_{kp}\partial_{\overline{\ell}}\mathrm{tr}_{g_0} g\right) + C\mathrm{tr}_g g_0 - 1,$$

assuming we are calculating at a point with $\mathrm{tr}_{g_0} g \geqslant 1$. Indeed, this follows from the calculations of Proposition 3.1 with $\hat{g} = g_0$, with the minor change that now $g(t)$ evolves by the normalized Chern-Ricci flow. In particular, we now have $d\omega = e^{-t}d\omega_0$ instead of $d\omega = d\omega_0$.

Arguing as in the proof of Lemma 4.1 we get that ω is uniformly equivalent to ω_0 independent of t. Uniform higher order estimates are then provided by Gill's paper [**19**]. In particular, it follows that

$$\left(\frac{\partial}{\partial t} - \Delta\right)(e^t \dot{\varphi}) = -\mathrm{tr}_\omega(\mathrm{Ric}(\Omega) + \omega_0) \geqslant -C,$$

and so the maximum principle implies that $\dot{\varphi} \geqslant -C(1+t)e^{-t} \geqslant -Ce^{-t/2}$. This implies that as t approaches infinity $\dot{\varphi}$ converges uniformly to zero exponentially fast, which implies that φ converges uniformly exponentially fast to a continuous limit function φ_∞. Since we have uniform higher order estimates for φ, it follows that φ_∞ is actually smooth and the convergence of φ to φ_∞ is in the smooth topology. Therefore we can pass to the limit in (7.3) and see that the limiting metric $\omega_\infty = -\mathrm{Ric}(\Omega) + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$ satisfies

$$\log \frac{\omega_\infty^n}{\Omega} = \varphi_\infty,$$

and taking $\sqrt{-1}\partial\overline{\partial}$ of this, we get

$$\mathrm{Ric}(\omega_\infty) = -\omega_\infty,$$

so that ω_∞ is the unique Kähler-Einstein metric on M. q.e.d.

8. Hopf manifolds

In this section we study the Chern-Ricci flow on some Hopf manifolds.

As in the Introduction, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$ with $|\alpha_1| = \cdots = |\alpha_n| \neq 1$, let M_α be the Hopf manifold $M_\alpha = (\mathbb{C}^n \setminus \{0\})/\sim$, where

$$(z_1, \ldots, z_n) \sim (\alpha_1 z_1, \ldots, \alpha_n z_n).$$

We consider the metric

$$\omega_H = \frac{\delta_{ij}}{r^2}\sqrt{-1}dz_i \wedge d\overline{z}_j,$$

where $r^2 = \sum_{j=1}^n |z_j|^2$. If $n = 2$, ω_H is $\partial\overline{\partial}$-closed, but this is false if $n > 2$. We now show that $\omega(t) = \omega_H - t\mathrm{Ric}(\omega_H)$ gives an explicit solution of the Chern-Ricci flow on M_α.

Proof of Proposition 1.8. Observe that $\det(\omega_H) = r^{-2n}$ and

$$\mathrm{Ric}(\omega_H) = n\sqrt{-1}\partial\overline{\partial}\log r^2 = \frac{n}{r^2}\left(\delta_{ij} - \frac{\overline{z}_i z_j}{r^2}\right)\sqrt{-1}dz_i \wedge d\overline{z}_j \geqslant 0.$$

For $t < \frac{1}{n}$ we have the Hermitian metrics

$$\omega(t) = \omega_H - t\mathrm{Ric}(\omega_H) = \frac{1}{r^2}\left((1-nt)\delta_{ij} + nt\frac{\overline{z}_i z_j}{r^2}\right)\sqrt{-1}dz_i \wedge d\overline{z}_j.$$

To compute the determinant of $\omega(t)$, note that the matrix $nt\frac{\overline{z}_i z_j}{r^2}$ has eigenvalue nt with multiplicity 1 and all the other eigenvalues are zero, while the matrix $(1-nt)\delta_{ij}$ has eigenvalue $1-nt$ with multiplicity n and is diagonal in every coordinate system. Choosing a coordinate system that makes $\frac{\overline{z}_i z_j}{r^2}$ diagonal we see that the eigenvalues of $r^2\omega(t)$ are $1-nt$ with multiplicity $n-1$ and 1 with multiplicity 1. Therefore

$$\det(\omega(t)) = \frac{(1-nt)^{n-1}}{r^{2n}},$$

from which it follows that $\mathrm{Ric}(\omega(t)) = \mathrm{Ric}(\omega_H)$, which implies that $\omega(t)$ solves the Chern-Ricci flow on the maximal existence interval $[0, \frac{1}{n})$.

q.e.d.

One can also consider more general Hopf manifolds, such as the Hopf surface M_α with $|\alpha_1| \neq |\alpha_2|$. In this case, Gauduchon and Ornea [**18**] have constructed an explicit Gauduchon metric ω_{GO} (which is also locally conformally Kähler). It would be interesting to see if the solution of the Chern-Ricci flow starting at ω_{GO} can also be written down explicitly.

Next we give the proof of Proposition 1.9.

Proof of Proposition 1.9. Write $\hat{\omega}_t = \omega_H - t\mathrm{Ric}(\omega_H)$. Then we can write $\omega(t)$ as $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi$ for a function $\varphi = \varphi(t)$ solving the parabolic complex Monge-Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log\frac{(\hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}, \quad \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \varphi|_{t=0} = \psi, \tag{8.1}$$

for ψ as in the statement of Proposition 1.9. A solution exists for $t \in [0, 1/n)$. In what follows, we will drop the subscript t and write $\hat{\omega}$ for $\hat{\omega}_t$.

We wish to bound $\mathrm{tr}_{\omega_H}\omega$ from above. First we claim that

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\omega_H}\omega = {} & -g^{\overline{j}i}(\partial_i\partial_{\overline{j}}g_H^{\overline{\ell}k})g_{k\overline{\ell}} + g_H^{\overline{\ell}k}g^{\overline{j}i}(\partial_k\partial_{\overline{\ell}}\hat{g}_{i\overline{j}} - \partial_i\partial_{\overline{j}}\hat{g}_{k\overline{\ell}}) \\ & - 2\mathrm{Re}(g^{\overline{j}i}(\partial_i g_H^{\overline{\ell}k})(\partial_{\overline{j}}g_{k\overline{\ell}})) - g_H^{\overline{\ell}k}g^{\overline{j}p}g^{\overline{q}i}(\partial_k g_{p\overline{q}})(\partial_{\overline{\ell}}g_{i\overline{j}}).\end{aligned} \tag{8.2}$$

To see (8.2), compute

$$\begin{aligned}\Delta\mathrm{tr}_{\omega_H}\omega &= g^{\overline{j}i}\partial_i\partial_{\overline{j}}(g_H^{\overline{\ell}k}g_{k\overline{\ell}}) \\ &= g^{\overline{j}i}(\partial_i\partial_{\overline{j}}g_H^{\overline{\ell}k})g_{k\overline{\ell}} + g^{\overline{j}i}g_H^{\overline{\ell}k}\partial_i\partial_{\overline{j}}g_{k\overline{\ell}} + 2\mathrm{Re}(g^{\overline{j}i}(\partial_i g_H^{\overline{\ell}k})(\partial_{\overline{j}}g_{k\overline{\ell}})),\end{aligned} \tag{8.3}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{tr}_{\omega_H}\omega &= g_H^{\overline{\ell}k}\partial_k\partial_{\overline{\ell}}\log\det g \\ &= -g_H^{\overline{\ell}k}g^{\overline{j}p}g^{\overline{q}i}(\partial_k g_{p\overline{q}})(\partial_{\overline{\ell}}g_{i\overline{j}}) + g_H^{\overline{\ell}k}g^{\overline{j}i}\partial_k\partial_{\overline{\ell}}g_{i\overline{j}}.\end{aligned} \tag{8.4}$$

Then (8.2) follows from combining (8.3) and (8.4) and using the fact that $g_{i\overline{j}} = \hat{g}_{i\overline{j}} + \varphi_{i\overline{j}}$.

For the first term on the right hand side of (8.2), note that

$$g_H^{\overline{\ell}k} = r^2\delta_{k\ell}, \quad \partial_{\overline{j}} g_H^{\overline{\ell}k} = z_j\delta_{k\ell}, \quad \partial_i\partial_{\overline{j}} g_H^{\overline{\ell}k} = \delta_{ij}\delta_{k\ell},$$

so that

$$g^{\overline{j}i}(\partial_i\partial_{\overline{j}} g_H^{\overline{\ell}k})g_{k\overline{\ell}} = \sum_{i,k} g^{\overline{i}i}g_{k\overline{k}} = (\mathrm{tr}_{\omega_H}\omega)(\mathrm{tr}_\omega\omega_H). \tag{8.5}$$

For the second term on the right hand side of (8.2), calculate

$$g_H^{\overline{\ell}k}g^{\overline{j}i}(\partial_k\partial_{\overline{\ell}}\hat{g}_{i\overline{j}} - \partial_i\partial_{\overline{j}}\hat{g}_{k\overline{\ell}}) = \mathrm{tr}_\omega\mathrm{Ric}(\omega_H) - ng^{\overline{j}i}\frac{\overline{z}_iz_j}{r^4} - (n-2)\mathrm{tr}_\omega\omega_H. \tag{8.6}$$

Indeed, to see (8.6) we compute

$$\begin{aligned} \hat{g}_{i\overline{j}} &= \frac{1}{r^2}\left((1-nt)\delta_{ij} + nt\frac{\overline{z}_iz_j}{r^2}\right) \\ \partial_{\overline{\ell}}\hat{g}_{i\overline{j}} &= -\frac{1}{r^4}z_\ell\left((1-nt)\delta_{ij} + \frac{2nt\overline{z}_iz_j}{r^2}\right) + \frac{ntz_j\delta_{i\ell}}{r^4}, \end{aligned} \tag{8.7}$$

and

$$\begin{aligned} \partial_k\partial_{\overline{\ell}}\hat{g}_{i\overline{j}} &= \frac{2}{r^6}\overline{z}_kz_\ell\left((1-nt)\delta_{ij} + \frac{2nt\overline{z}_iz_j}{r^2}\right) \\ &\quad - \frac{1}{r^4}\delta_{k\ell}\left((1-nt)\delta_{ij} + \frac{2nt\overline{z}_iz_j}{r^2}\right) \\ &\quad - \frac{1}{r^4}z_\ell\left(-\frac{2nt\overline{z}_k\,\overline{z}_iz_j}{r^4} + \frac{2nt\overline{z}_i\delta_{jk}}{r^2}\right) - \frac{2nt\overline{z}_kz_j\delta_{i\ell}}{r^6} + \frac{nt\delta_{jk}\delta_{i\ell}}{r^4} \\ &= \frac{6nt\overline{z}_iz_j\overline{z}_kz_\ell}{r^8} + \frac{1}{r^4}(nt\delta_{jk}\delta_{i\ell} - (1-nt)\delta_{k\ell}\delta_{ij}) + \frac{2}{r^6}\overline{z}_kz_\ell\delta_{ij} \\ &\quad - \frac{2nt}{r^6}\left(\delta_{k\ell}\overline{z}_iz_j + \delta_{jk}\overline{z}_iz_\ell + \delta_{i\ell}\overline{z}_kz_j + \delta_{ij}\overline{z}_kz_\ell\right). \end{aligned}$$

Finally, this gives:

$$\partial_k\partial_{\overline{\ell}}\hat{g}_{i\overline{j}} - \partial_i\partial_{\overline{j}}\hat{g}_{k\overline{\ell}} = \frac{2}{r^6}(\overline{z}_kz_\ell\delta_{ij} - \overline{z}_iz_j\delta_{k\ell}),$$

and

$$\begin{aligned} g^{\overline{j}i}g_H^{\overline{\ell}k}(\partial_k\partial_{\overline{\ell}}\hat{g}_{i\overline{j}} - \partial_i\partial_{\overline{j}}\hat{g}_{k\overline{\ell}}) &= \frac{2}{r^2}g^{\overline{j}i}\left(\delta_{ij} - \frac{n\overline{z}_iz_j}{r^2}\right) \\ &= \mathrm{tr}_\omega\mathrm{Ric}(\omega_H) - ng^{\overline{j}i}\frac{\overline{z}_iz_j}{r^4} - (n-2)\mathrm{tr}_\omega\omega_H, \end{aligned}$$

establishing (8.6).

Combining (8.2), (8.5), and (8.6) we obtain

$$\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_{\omega_H}\omega = -(\mathrm{tr}_{\omega_H}\omega)(\mathrm{tr}_{\omega}\omega_H)+\mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H) - ng^{\overline{j}i}\frac{\overline{z}_i z_j}{r^4} - (n-2)\mathrm{tr}_{\omega}\omega_H - 2\mathrm{Re}(g^{\overline{j}i}(\partial_i g_H^{\overline{\ell}k})(\partial_{\overline{j}} g_{k\overline{\ell}})) - g_H^{\overline{\ell}k}g^{\overline{j}p}g^{\overline{q}i}(\partial_k g_{p\overline{q}})(\partial_{\overline{\ell}} g_{i\overline{j}}). \tag{8.8}$$

The troublesome term is the 5th one on the right hand side. We write this term as:

$$\begin{aligned}-2\mathrm{Re}(g^{\overline{j}i}(\partial_i g_H^{\overline{\ell}k})(\partial_{\overline{j}} g_{k\overline{\ell}})) &= -2\mathrm{Re}(g^{\overline{j}i}(\partial_i g_H^{\overline{\ell}k})(\partial_{\overline{\ell}} g_{k\overline{j}})) \\ &\quad - 2\mathrm{Re}(g^{\overline{j}i}(\partial_i g_H^{\overline{\ell}k})(\partial_{\overline{j}}\hat{g}_{k\overline{\ell}} - \partial_{\overline{\ell}}\hat{g}_{k\overline{j}})) \\ &=: A_1 + A_2. \end{aligned}\tag{8.9}$$

For A_2 use (8.7) to compute

$$\partial_{\overline{j}}\hat{g}_{k\overline{\ell}} - \partial_{\overline{\ell}}\hat{g}_{k\overline{j}} = \frac{1}{r^4}(z_\ell\delta_{kj} - z_j\delta_{k\ell}),$$

and so

$$\begin{aligned}A_2 &= -\frac{1}{r^4}2\mathrm{Re}(g^{\overline{j}i}\overline{z}_i\delta_{k\ell}(z_\ell\delta_{kj} - z_j\delta_{k\ell})) \\ &= -\frac{1}{r^4}2\mathrm{Re}\left(g^{\overline{j}i}(\overline{z}_i z_j - n\overline{z}_i z_j)\right) \\ &= 2(n-1)g^{\overline{j}i}\frac{\overline{z}_i z_j}{r^4}.\end{aligned}\tag{8.10}$$

To deal with A_1 we introduce an inner product on tensors of type $\Phi = \Phi_{ij\overline{k}}$. For tensors Ψ and Φ of this type, define

$$\langle \Psi, \Phi\rangle = g_H^{\overline{j}i}g^{\overline{\ell}k}g^{\overline{q}p}\Psi_{ik\overline{q}}\overline{\Phi_{j\ell\overline{p}}}.$$

Then if $\Phi_{ij\overline{k}} = \partial_i g_{j\overline{k}}$ we see that the last term on the right hand side of (8.8) is $-|\Phi|^2$.

Now compute

$$\begin{aligned}A_1 &= -2\mathrm{Re}(g^{\overline{j}i}(\partial_i g_H^{\overline{\ell}k})(\partial_{\overline{\ell}} g_{k\overline{j}})) \\ &= -2\mathrm{Re}(g^{\overline{j}i}g_H^{\overline{\ell}p}g^{\overline{v}k}g_{u\overline{v}}(g_H)_{p\overline{q}}(\partial_i g_H^{\overline{q}u})\partial_{\overline{\ell}} g_{k\overline{j}}) \\ &= -2\mathrm{Re}\langle \Psi, \Phi\rangle \\ &\leqslant 2|\Psi||\Phi|,\end{aligned}\tag{8.11}$$

with $\Psi_{ij\overline{k}} = (g_H)_{i\overline{q}}(\partial_j g_H^{\overline{q}u})g_{u\overline{k}}$ and $\Phi_{ij\overline{k}} = \partial_i g_{j\overline{k}}$. But

$$\Psi_{ij\overline{k}} = \frac{1}{r^2}\delta_{iq}\overline{z}_j\delta_{uq}g_{u\overline{k}} = \frac{1}{r^2}\overline{z}_j g_{i\overline{k}},$$

and so

$$|\Psi|^2 = \frac{1}{r^4}g_H^{\overline{j}i}g^{\overline{\ell}k}g^{\overline{q}p}\overline{z}_k g_{i\overline{q}} z_\ell g_{p\overline{j}} = \frac{1}{r^4}g_H^{\overline{j}i}g_{i\overline{j}}g^{\overline{\ell}k}\overline{z}_k z_\ell = (\mathrm{tr}_{\omega_H}\omega)g^{\overline{\ell}k}\frac{\overline{z}_k z_\ell}{r^4}.$$

Then

$$
\begin{aligned}
A_1 &\leqslant |\Psi|^2 + |\Phi|^2 \\
&= (\mathrm{tr}_{\omega_H}\omega) g^{\bar{\ell}k}\frac{\overline{z}_k z_\ell}{r^4} + |\Phi|^2 \\
&= (\mathrm{tr}_{\omega_H}\omega)(\mathrm{tr}_{\omega}\omega_H) - (\mathrm{tr}_{\omega_H}\omega)\frac{1}{r^2}g^{\bar{j}i}\left(\delta_{ij} - \frac{\overline{z}_i z_j}{r^2}\right) + |\Phi|^2. \qquad (8.12)
\end{aligned}
$$

Combining (8.8), (8.9), (8.10), and (8.12), we have

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\omega_H}\omega &\leqslant \mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H) - n g^{\bar{j}i}\frac{\overline{z}_i z_j}{r^4} - (n-2)\mathrm{tr}_{\omega}\omega_H \\
&\quad - (\mathrm{tr}_{\omega_H}\omega)\frac{1}{r^2}g^{\bar{j}i}\left(\delta_{ij} - \frac{\overline{z}_i z_j}{r^2}\right) + 2(n-1)g^{\bar{j}i}\frac{\overline{z}_i z_j}{r^4} \\
&= \mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H) - (n-2)\frac{1}{r^2}g^{\bar{j}i}\left(\delta_{ij} - \frac{\overline{z}_i z_j}{r^2}\right) \\
&\quad - \frac{1}{n}(\mathrm{tr}_{\omega_H}\omega)\mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H) \\
&= \mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H) - \frac{n-2}{n}\mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H) \\
&\quad - \frac{1}{n}(\mathrm{tr}_{\omega_H}\omega)\mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H) \\
&= \left(\frac{2}{n} - \frac{1}{n}\mathrm{tr}_{\omega_H}\omega\right)\mathrm{tr}_{\omega}\mathrm{Ric}(\omega_H).
\end{aligned}
$$

Since $\mathrm{Ric}(\omega_H) \geqslant 0$, we conclude by the maximum principle that $\mathrm{tr}_{\omega_H}\omega$ is uniformly bounded from above. q.e.d.

Finally, we remark that this implies the convergence of the Chern-Ricci flow at the level of potentials. Indeed, recall that $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\overline{\partial}\varphi$ with φ solving (5.2) and $\hat{\omega}_t = \omega_H - t\mathrm{Ric}(\omega_H)$. Moreover, $\hat{\omega}_T$ is nonnegative, so we can apply the argument of Proposition 5.1 to obtain uniform upper bounds for $|\varphi|$ and $\dot{\varphi}$. It follows immediately that as $t \to T$, $\varphi(t)$ converges pointwise to a function $\varphi(T)$ on M_α. On the other hand, by Proposition 1.9 we have uniform bounds for $|\Delta_{g_H}\varphi|$. Thus standard elliptic theory gives a uniform bound for $\|\varphi(t)\|_{C^{1+\beta}}$ for any $\beta \in (0,1)$. It follows that $\varphi \to \varphi(T)$ in $C^{1+\beta}$ for any $\beta \in (0,1)$.

9. The complex Monge-Ampère equation

In this section we prove uniform estimates for solutions of the elliptic complex Monge-Ampère equation.

Let (M^n, ω) be a compact Hermitian manifold, F a smooth function on M, and $\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ a Hermitian metric that satisfies

$$
(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F \omega^n. \qquad (9.1)
$$

We will give an alternative proof of the main result of [**54**] (see also [**4**, **12**] for different proofs):

Theorem 9.1. *There is a constant* C *that depends only on* (M, ω), $\sup_M F$, *and* $\inf_M \Delta F$ *such that*

$$\sup_M \varphi - \inf_M \varphi \leqslant C. \tag{9.2}$$

This is slightly weaker than the result in [**54**], because there is no dependence of C on $\inf_M \Delta F$ there (and in fact the proof of [**54**] can be easily modified to have C depend only on $p > n$ and $\int_M e^{pF}$ rather than $\sup_M F$). The point of our discussion here is to establish Theorem 9.1 via a new second order estimate which we had previously established [**53**] using the maximum principle only in the cases $n = 2$ or (M, ω) balanced.

From now on we will normalize φ by assuming $\sup_M \varphi = 0$. The estimate we wish to prove is:

$$\mathrm{tr}_g g' \leqslant Ce^{A(\varphi - \inf_M \varphi)} \tag{9.3}$$

for uniform constants C, A. The reader may notice that (9.3) has the same form as the second order estimates of Yau and Aubin [**1**, **60**].

Theorem 9.1 then follows from (9.3). Indeed, we can then use the arguments in [**53**] to derive (9.2) from (9.3). The idea is that a second order estimate of the form (9.3), together with the condition $\sqrt{-1}\partial\overline{\partial}\varphi \geqslant -\omega$, implies, via a Moser iteration argument applied to the exponential of φ, a zero order estimate for φ. This method was employed in the Kähler case in [**59**], and a related argument was used in the almost complex setting in [**55**].

Proof of (9.3). Following Phong-Sturm [**33**] we consider the quantity

$$Q = \log \mathrm{tr}_g g' - A\varphi + \frac{1}{\varphi - \inf_M \varphi + 1},$$

for $A \geqslant 1$ to be determined. Note that $0 < \frac{1}{\varphi - \inf_M \varphi + 1} \leqslant 1$. We have

$$\begin{aligned} \Delta' Q &= \Delta' \log \mathrm{tr}_g g' - An + A\mathrm{tr}_{g'} g - \frac{n - \mathrm{tr}_{g'} g}{(\varphi - \inf_M \varphi + 1)^2} \\ &\quad + \frac{2|\partial\varphi|^2_{g'}}{(\varphi - \inf_M \varphi + 1)^3} \\ &\geqslant \Delta' \log \mathrm{tr}_g g' + A\mathrm{tr}_{g'} g + \frac{2|\partial\varphi|^2_{g'}}{(\varphi - \inf_M \varphi + 1)^3} - An - n, \end{aligned} \tag{9.4}$$

writing Δ' for the complex Laplacian associated to g'. To calculate $\Delta' \log \mathrm{tr}_g g'$ we go back to the calculations in Proposition 3.1 where $\hat{g} = g_0$ is now replaced by g and g' takes the role of the evolving metric there. With these substitutions (3.3), (3.4), and (3.6) together read

$$\begin{aligned}\Delta' \mathrm{tr}_g g' &= g'^{\bar{j}i} g^{\bar{\ell}k} \nabla_{\bar{\ell}} \nabla_k g'_{i\bar{j}} + g'^{\bar{j}i} \nabla_i \overline{T^{\ell}_{j\ell}} + g'^{\bar{j}i} g^{\bar{\ell}k} g_{p\bar{j}} \nabla_{\bar{\ell}} T^p_{ik} \\ &- g'^{\bar{j}i} g^{\bar{\ell}k} g'_{k\bar{q}} (\nabla_i \overline{T^q_{j\ell}} - R_{i\bar{\ell}p\bar{j}} g^{\bar{q}p}) \\ &- R - 2\mathrm{Re}\left(g'^{\bar{j}i} g^{\bar{\ell}k} T^p_{ik} \nabla_{\bar{\ell}} g'_{p\bar{j}} \right) \\ &- g'^{\bar{j}i} g^{\bar{\ell}k} T^p_{ik} \overline{T^q_{j\ell}} g_{p\bar{q}} + g'^{\bar{j}i} g^{\bar{\ell}k} T^p_{ik} \overline{T^q_{j\ell}} g'_{p\bar{q}},\end{aligned}$$

where $R = g^{\bar{\ell}k} R^{\mathrm{C}}_{k\bar{\ell}} = g^{\bar{j}i} g^{\bar{\ell}k} R_{k\bar{\ell}i\bar{j}}$ is the Chern scalar curvature of g. On the other hand, by applying $\Delta \log$ to (9.1) we get

$$\Delta F - R = g^{\bar{\ell}k} \partial_k \partial_{\bar{\ell}} \log \det(g') = g'^{\bar{j}i} g^{\bar{\ell}k} \partial_k \partial_{\bar{\ell}} g'_{i\bar{j}} - g'^{\bar{j}p} g'^{\bar{q}i} g^{\bar{\ell}k} \partial_k g'_{i\bar{j}} \partial_{\bar{\ell}} g'_{p\bar{q}},$$

and converting these into covariant derivatives (as in the argument for (3.5) in the proof of Proposition 3.1) we get

$$\Delta F = g'^{\bar{j}i} g^{\bar{\ell}k} \nabla_{\bar{\ell}} \nabla_k g'_{i\bar{j}} - g'^{\bar{j}p} g'^{\bar{q}i} g^{\bar{\ell}k} \nabla_k g'_{i\bar{j}} \nabla_{\bar{\ell}} g'_{p\bar{q}},$$

and so

$$\begin{aligned}\Delta' \log \mathrm{tr}_g g' = \frac{1}{\mathrm{tr}_g g'} \Bigg(&\Bigg[g'^{\bar{j}p} g'^{\bar{q}i} g^{\bar{\ell}k} \nabla_k g'_{i\bar{j}} \nabla_{\bar{\ell}} g'_{p\bar{q}} - \frac{1}{\mathrm{tr}_g g'} g'^{\bar{\ell}k} \nabla_k \mathrm{tr}_g g' \nabla_{\bar{\ell}} \mathrm{tr}_g g' \\ &+ 2\mathrm{Re}\left(g'^{\bar{j}i} g^{\bar{\ell}k} T^p_{ki} \nabla_{\bar{\ell}} g'_{p\bar{j}} \right) + g'^{\bar{j}i} g^{\bar{\ell}k} T^p_{ik} \overline{T^q_{j\ell}} g'_{p\bar{q}} \Bigg] + \Delta F - R \\ &+ g'^{\bar{j}i} \nabla_i \overline{T^{\ell}_{j\ell}} + g'^{\bar{j}i} g^{\bar{\ell}k} g_{p\bar{j}} \nabla_{\bar{\ell}} T^p_{ik} - g'^{\bar{j}i} g^{\bar{\ell}k} g'_{k\bar{q}} (\nabla_i \overline{T^q_{j\ell}} - R_{i\bar{\ell}p\bar{j}} g^{\bar{q}p}) \\ &- g'^{\bar{j}i} g^{\bar{\ell}k} T^p_{ik} \overline{T^q_{j\ell}} g_{p\bar{q}} \Bigg).\end{aligned}$$

The Cauchy-Schwarz argument from (3.7) shows that the quantity inside square brackets equals

$$\Big[\cdots \Big] = K + 2\mathrm{Re}\left(g'^{\bar{q}k} T^i_{ik} \frac{\nabla_{\bar{q}} \mathrm{tr}_g g'}{\mathrm{tr}_g g'} \right),$$

and $K \geqslant 0$ is the same quantity as in (3.7). Putting these together we have

$$\Delta' \log \mathrm{tr}_g g' \geqslant \frac{2}{(\mathrm{tr}_g g')^2} \mathrm{Re}\left(g'^{\bar{q}k} T^i_{ik} \nabla_{\bar{q}} \mathrm{tr}_g g' \right) - C \mathrm{tr}_{g'} g - C, \tag{9.5}$$

where we used the fact that $\mathrm{tr}_g g' \geqslant C^{-1}$, which is a simple consequence of (9.1) and the arithmetic-geometric means inequality. Suppose that Q achieves its maximum at a point $x \in M$. Then at x we have $\partial_i Q = 0$ and hence

$$\frac{1}{\mathrm{tr}_g g'} \partial_i \mathrm{tr}_g g' - A \partial_i \varphi - \frac{1}{(\varphi - \inf_M \varphi + 1)^2} \partial_i \varphi = 0,$$

and therefore

$$\begin{aligned}&\left|\frac{2}{(\mathrm{tr}_g g')^2}\mathrm{Re}\left(g'^{\overline{q}k}T^i_{ik}\nabla_{\overline{q}}\mathrm{tr}_g g'\right)\right|\\&=\left|\frac{2}{\mathrm{tr}_g g'}\mathrm{Re}\left(\left(A+\frac{1}{(\varphi-\inf_M\varphi+1)^2}\right)g'^{\overline{q}k}T^i_{ik}\nabla_{\overline{q}}\varphi\right)\right|\\&\leqslant\frac{|\partial\varphi|^2_{g'}}{(\varphi-\inf_M\varphi+1)^3}+CA^2(\varphi-\inf_M\varphi+1)^3\frac{\mathrm{tr}_{g'}g}{(\mathrm{tr}_g g')^2}.\end{aligned}$$

If at x we have $(\mathrm{tr}_g g')^2(x)\leqslant A^2(\varphi(x)-\inf_M\varphi+1)^3$, then we also have (using that $\frac{3}{2}\log t\leqslant t$ for $t>0$)

$$\begin{aligned}Q\leqslant Q(x)&\leqslant\frac{3}{2}\log(\varphi(x)-\inf_M\varphi+1)+\log A-A\varphi(x)+1\\&\leqslant-(A-1)\varphi(x)-\inf_M\varphi+\log A+2\leqslant-A\inf_M\varphi+\log A+2,\end{aligned}$$

and so in this case (9.3) follows immediately.

Otherwise, we have $(\mathrm{tr}_g g')^2(x)\geqslant A^2(\varphi(x)-\inf_M\varphi+1)^3$, and so at x,

$$\left|\frac{2}{(\mathrm{tr}_g g')^2}\mathrm{Re}\left(g'^{\overline{q}k}T^i_{ik}\nabla_{\overline{q}}\mathrm{tr}_g g'\right)\right|\leqslant\frac{|\partial\varphi|^2_{g'}}{(\varphi-\inf_M\varphi+1)^3}+C\mathrm{tr}_{g'}g.\tag{9.6}$$

Combining (9.4), (9.5), and (9.6) we get, at x,

$$0\geqslant\Delta'Q\geqslant A\mathrm{tr}_{g'}g-C\mathrm{tr}_{g'}g-An-n-C\geqslant\mathrm{tr}_{g'}g-C,$$

if A is chosen sufficiently large. From this (9.3) follows easily. q.e.d.

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Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
Current address:
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston, IL 60208

Mathematics Department
University of California San Diego
9500 Gilman Drive #0112
La Jolla, CA 92093
Current address:
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston, IL 60208