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CONVEX VALUATIONS INVARIANT UNDER THE LORENTZ GROUP

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Abstract

We give an explicit classification of translation-invariant, Lorentz-invariant continuous valuations on convex sets. We also classify the Lorentz-invariant even generalized valuations.

1. Introduction

The main result of this paper is to give a complete classification of translation-invariant continuous valuations on convex sets in $\mathbb{R}^n$ invariant under the connected component of the Lorentz group.

Let $\mathcal{K}(\mathbb{R}^n)$ denote the family of convex compact subsets of $\mathbb{R}^n$. A (convex) valuation is a functional $\phi\colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{C}$ which satisfies the following additivity property:

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}(\mathbb{R}^n)$. A valuation is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}(\mathbb{R}^n)$.

Classification results are playing an important role in valuations theory and its applications to integral geometry since the fundamental work of Hadwiger in the 1940's and 1950's. Probably the most famous result in the area is Hadwiger's characterization [**14**] of continuous valuations on convex subsets of a Euclidean space invariant under all isometries, i.e. translations and all orthogonal transformations, as linear combinations of intrinsic volumes (see [**23**] for this notion); the subgroup of orientation-preserving isometries leads to the same list of invariant valuations. In recent years many new classification results have been obtained for various classes of valuations. Klain [**15**] and Schneider [**24**] have classified continuous translation-invariant valuations which are simple, i.e. vanish on convex sets of positive codimension. In [**1**] the first author established the following general results: let G be a compact subgroup of the linear group. The subspace of G-invariant translation-invariant continuous valuations on convex sets is finite dimensional if and only if G acts transitively on the unit sphere; thus for such a group G, one may hope to get a finite classification. The problem of obtaining

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such a classification is being investigated by a few people in recent years. Notice that the cases $G=O(n), SO(n)$ correspond to the Hadwiger theorem. The next interesting case $G=U(n)$ was classified explicitly in geometric terms by the first author [**3**] where also first applications to Hermitian integral geometry were obtained. More thorough and complete further study of $U(n)$-invariant valuations and Hermitian integral geometry was done by Bernig and Fu [**9**] and Fu [**10**]. Several other cases of compact groups acting transitively on the sphere were considered by Bernig [**5**], [**6**], [**7**].

At the same time other classes of valuations were studied under weaker assumptions on continuity but stronger assumptions on the symmetry group, which usually was either $GL_n(\mathbb{R})$ or $SL_n(\mathbb{R})$. Ludwig and Reitzner [**18**] have characterized the affine surface area as the only (up to volume, Euler characteristic, and non-negative multiplicative factor) upper semi-continuous valuation invariant under affine volume preserving transformations. Other results on $SL_n(\mathbb{R})$-invariant valuations were obtained again by Ludwig and Reitzner [**19**]. Quite a few classification results in a different but related direction of convex body valued valuations were obtained in [**16**], [**17**], [**25**], [**26**]; see also references therein.

Let us now discuss in greater detail the main results of the present paper. Let us fix on $\mathbb{R}^n$ the Minkowski metric, i.e. a quadratic form Q of signature $(n-1,1)$. In coordinates it is given by $Q(x)=\sum_{i=1}^{n-1}x_i^2-x_n^2$. Let $O(n-1,1)$ denote the group of all linear transformations of $\mathbb{R}^n$ preserving Q. It is well known that $O(n-1,1)$ has four connected components. Let us denote by $SO^+(n-1,1)$ the connected component of the identity. Throughout the article, we refer to $SO^+(n-1,1)$ as the Lorentz group.

Let $Val(\mathbb{R}^n)$ be the space of all translation-invariant continuous valuations on $\mathcal{K}(\mathbb{R}^n)$. For an integer $0\leq k\leq n$, let us denote by $Val_k(\mathbb{R}^n)$ the subspace of k-homogeneous valuations (a valuation ϕ is called k-homogeneous if $\phi(\lambda K)=\lambda^k\phi(K)$ for any $\lambda\geq 0$ and any convex compact set K). McMullen's decomposition theorem [**20**] says that

$$Val(\mathbb{R}^n)=\oplus_{k=0}^n Val_k(\mathbb{R}^n). \tag{1}$$

$Val_k(\mathbb{R}^n)$ can be decomposed further with respect to parity:

$$Val_k(\mathbb{R}^n)=Val_k^{ev}(\mathbb{R}^n)\oplus Val_k^{odd}(\mathbb{R}^n),$$

where a valuation ϕ is called even (resp. odd) if $\phi(-K)=\phi(K)$ (resp. $\phi(-K)=-\phi(K)$) for any $K\in\mathcal{K}(\mathbb{R}^n)$.

It is easy to see that $Val_0(\mathbb{R}^n)$ is spanned by the Euler characteristic, i.e. valuation which is equal to 1 on any convex compact set. By a theorem of Hadwiger [**14**], $Val_n(\mathbb{R}^n)$ is spanned by the Lebesgue measure.

We denote by $Val(\mathbb{R}^n)^{SO^+(n-1,1)}$ the subspace of $SO^+(n-1,1)$-invariant valuations, and similarly for subspaces of given parity and homogeneity. McMullen's decomposition (1) immediately implies

$$Val(\mathbb{R}^n)^{SO^+(n-1,1)} = \oplus_{k=0}^n \left(Val_k^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)} \oplus Val_k^{odd}(\mathbb{R}^n)^{SO^+(n-1,1)}\right).$$

Our first main result classifies odd $SO^+(n-1,1)$-invariant valuations.

Theorem 1.1. *For $0 \leq k \leq n$, $k \neq n-1$,*

$$\dim Val_k^{odd}(\mathbb{R}^n)^{SO^+(n-1,1)} = 0.$$

For $k = n-1$,

$$\dim Val_k^{odd}(\mathbb{R}^n)^{SO^+(n-1,1)} = \begin{cases} 1, & n \geq 3 \\ 2, & n = 2 \end{cases}$$

The last space will be described explicitly.

The proof of this result relies on Schneider's imbedding theorem and makes use of Lie group continuous cohomology as one of the tools to show that the Schneider bundle has no non-zero continuous $SO^+(n-1,1)$-invariant sections for $1 \leq k \leq n-2$.

Our second main result classifies even $SO^+(n-1,1)$-invariant valuations. Notice first of all that by the above discussion 0- and n-homogeneous valuations are proportional to the Euler characteristic and Lebesgue measure, respectively. In particular they are even and $SO^+(n-1,1)$-invariant.

For the remaining degrees of homogeneity, namely $1 \leq k \leq n-1$, the classification consists of two parts. First, we define and classify the invariant generalized valuations. Roughly speaking, the generalized valuations are the completion of the space of smooth valuations with respect to a certain weak topology that is defined using the product structure on valuations (see subsection 4.2 for precise definitions). The space of generalized valuations naturally contains the continuous valuations as a dense subspace. We then analyze which of the invariant generalized valuations are in fact continuous. The following two theorems summarize our results:

Theorem 1.2. *For all $1 \leq k \leq n-1$, the space of k-homogeneous, even, $SO^+(n-1,1)$-invariant generalized valuations is 2-dimensional.*

Those spaces will be described explicitly.

Theorem 1.3. *For $1 \leq k \leq n-2$, $\dim Val_k^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)} = 0$. For $k = n-1$, $\dim Val_k^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)} = 2$.*

Again, the last space will be described explicitly.

Let us remark that the generalized Lorentz-invariant odd valuations remain to be classified.

The plan of the classification is as follows: First, we study $SO^+(n-1,1)$-invariant continuous sections of the Klain bundle. For any $1 \leq k \leq n-1$ we get a 2-dimensional space of $SO^+(n-1,1)$-invariant continuous sections. By McMullen's theorem, this finishes the classification of continuous $(n-1)$-homogeneous even valuations. For the remaining $1 \leq k \leq n-2$, it turns out that those sections correspond to generalized valuations, which are not continuous. We construct the corresponding generalized valuations explicitly (section 4), and then proceed to show that they are discontinuous by proving that they cannot be evaluated on the double cone (sections 3, 5). This last part of analysis involves lengthy technical arguments. Another difficulty in comparison to the case of groups transitive on spheres is that $SO^+(n-1,1)$-invariant valuations do not have to be smooth in the sense of [**2**].

Finally, we give some applications of the classification. One is the explicit construction of a continuous section of Klain's bundle that lies in the closure of Klain's imbedding of the continuous valuations, but outside the image of the imbedding. The non-closedness of the image was proved very recently by Parapatits and Wannerer in [**22**] using different methods. Another corollary is the non-extendibility by continuity of the Fourier transform from smooth to continuous valuations, which also was proved in [**22**] using different methods.

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2. Finding the Lorentz-invariant continuous sections of Klain's and Schneider's bundles

Let us introduce the notation used throughout the paper. For a linear space W, $D(W)$ is the 1-dimensional space of densities on W, and $Gr(W,k)$ is the Grassmannian of k-subspaces of W. The signature of a quadratic form Q will be denoted $\mathrm{sign}Q$. If a norm is given on W, $S(W)$ denotes the unit sphere in W. For a vector bundle E over a manifold M, $\Gamma^{\pm\infty}(M,E)$, or sometimes simply $\Gamma^{\pm\infty}(E)$, will denote the space of smooth resp. generalized sections.

In the following, V stands for $\mathbb{R}^n$. Fix two symmetric 2-forms: Euclidean $\langle u,v\rangle = \sum_{j=1}^n u_j v_j$, and Lorentzian $Q(u,v) = \sum_{j=1}^{n-1} u_j v_j - u_n v_n$. Let (e_j) be the standard basis. The unit $n\times n$ matrix is denoted I.

A vector $v \in V$ is called space-like if $Q(v) > 0$, time-like if $Q(v) < 0$, and light-like if $Q(v) = 0$. More generally, a subspace $W \subset V$ is called space-like if $Q|_W$ is positive definite, time-like if $Q|_W$ has signature $(n-1,1)$, and degenerate if $Q|_W$ is degenerate. The collection of all light-like vectors forms the light cone. If no confusion can arise, such as when we consider only the unit vectors S^{n-1} or the projective space $Gr(V,1)$, the collection of elements on which Q vanishes is again called the light cone.

We will make heavy use of the following theorem due to Klain [**15**]:

Theorem. *There is a $GL(n)$-equivariant imbedding of $Val_k^{ev}(V)$ into a $GL(n)$-invariant subspace of the continuous sections of Klain's bundle, which is the line bundle of densities on k-dimensional linear subspaces of $\mathbb{R}^n$, over $Gr(n,k)$.*

A similar result holds for odd continuous valuations; the precise description is given below. To find all Lorentz-invariant valuations, we begin by determining all the invariant sections of those two bundles.

2.1. Klain's bundle $K^{n,k}$. Let γ_n^k be the tautological vector bundle over $Gr(n,k)$, so that the fiber over $\Lambda \in Gr(\mathbb{R}^n,k)$ is simply Λ; Klain's bundle $K^{n,k}$ is the bundle of densities on its fibers, which is naturally a $GL(n)$-line bundle. The Euclidean structure defines a density in every subspace, i.e. we have a global section $Area \in \Gamma(K^{n,k})$, $Area$ is the only $SO(n)$-invariant continuous section (up to scaling), and it defines a trivialization of the bundle. We will study $SO^+(n-1,1)$-invariant continuous sections of $K^{n,k}$.

Proposition 2.1. *Given a Lorentz-orthogonal family $(v_1,...,v_k)$ s.t. $Q(v_i)=1$ for $i \leq k-1$, and $Q(v_k)=\pm 1$, and denoting $z_j = \langle v_j, e_n\rangle$, one has*

$$Area(v_1,\dots,v_k)^2 = \begin{cases} 1+2\sum_{j=1}^k z_j^2, \ Q(v_k)=1 \\ 2(z_k^2-\sum_{j=1}^{k-1} z_j^2)-1, \ Q(v_k)=-1 \end{cases}$$

Proof. Use the identity

$$Area(v_1,\dots,v_k)^2 = det(\langle v_i,v_j\rangle) = \det(Q(v_i,v_j)+2z_iz_j) = \det(I_\pm + 2zz^T)$$

where $I_\pm$ is a $k\times k$ diagonal matrix with entries $Q(v_1),\dots,Q(v_k)$, and $z=(z_1,\dots,z_k)^T$. The remaining verification is straightforward. Q.E.D.

Proposition 2.2. *Given $T \in SO^+(n-1,1)$, and $\Lambda \in Gr(n,k)$ generic (i.e. Q restricted to Λ is non-degenerate), if $T(\Lambda)=\Lambda$ then $|\det T|_\Lambda| = 1$.*

Proof. Since $Q|_\Lambda$ is non-degenerate, and $T \in GL(\Lambda)$ preserves $Q|_\Lambda$, it follows that $|\det T|_\Lambda|=1$. Q.E.D.

For the following, it will be conveniet to use the following definition: Given a manifold X and a closed subset $Y \subset X$, we say that a net $\{x_\alpha\}_{\alpha\in I} \to Y$ if for any neighborhood $U \supset Y$ there is α_0 s.t. $x_\alpha \in U$ whenever $\alpha > \alpha_0$.

Proposition 2.3. *The space of $G = SO^+(n-1,1)$-invariant continuous sections of $K^{n,k}$ is 2-dimensional.*

Proof. 1. The orbits of the action of G on $Gr(n,k)$ are characterized by the signature of the restriction of Q. The open orbits are $M_+ = \{\Lambda : \mathrm{sign}Q|_\Lambda = (k,0)\}$ and $M_- = \{\Lambda : \mathrm{sign}Q|_\Lambda = (k-1,1)\}$. Together, $M_+ \cup M_-$ are dense in $Gr(n,k)$. The remaining closed orbit is $M_0 = \{\Lambda : signQ|_\Lambda = (k-1,0)\}$, as there are no 2 non-proportional Q-orthogonal vectors on the light cone.

2. Choose some fixed $\Lambda_+ \in M_+$ and $\Lambda_- \in M_-$, and fix arbitrary densities on them. By Proposition 2.2, the stabilizer of $\Lambda_\pm$ would leave the resp. density invariant; thus we may extend those densities to an invariant section $\mu \in \Gamma(M_+ \cup M_-, K^{n,k})^G$. It remains to verify that μ admits a continuous G-invariant extension to all $Gr(n,k)$. Let us show that $\mu(\Lambda) \to 0$ as $\Lambda \to M_0$. For this, it is enough to take a Q-orthonormal basis of Λ, say $v_1, \dots, v_k$, and show that $Area(v_1, \dots, v_k)^2 \to \infty$, since by G-equivariance,

$$\mu(\Lambda) = C \frac{Area(\Lambda)}{Area(v_1, \dots, v_k)} \tag{2}$$

3. First, assume $M_+ \ni \Lambda \to M_0$.

Write $z = (z_1, \dots, z_k)^T \in \mathbb{R}^k$, where $z_j = \langle v_j, e_n \rangle$. Define ϵ by $\langle P_\Lambda e_n, e_n \rangle = \cos(\pi/4 + \epsilon)|P_\Lambda e_n|$, where P_Λ is the (Euclidean) projection onto Λ. We assume $Q(v_j) = 1$ for $1 \le j \le k$. Write $P_\Lambda e_n = \sum \alpha_j v_j$. Then $\langle P_\Lambda e_n - e_n, v_i \rangle = 0$, for all i, i.e. $(I + 2zz^T)(\alpha) = z$. By the Sherman-Morrison [**27**] formula,

$$(I + 2zz^T)^{-1} = I - \frac{2zz^T}{1 + 2z^T z} I.$$

We will denote $A = Area(v_1, \dots, v_k)^2$, $B = z^T z = z_1^2 + \cdots + z_{k-1}^2 + z_k^2$. By Proposition 2.1, $A = 1 + 2B$. Then

$$\alpha = z - \frac{2zz^T z}{1 + 2z^T z} = \frac{1}{A} z.$$

Let us write $\cos^2(\pi/4 + \epsilon) = 1/2 - \delta$. Then

$$\begin{aligned}
\langle P_\Lambda e_n, e_n \rangle^2 &= \cos^2(\pi/4 + \epsilon)|P_\Lambda e_n|^2 \\
&= (1/2 - \delta)\Big(Q(P_\Lambda e_n) + 2\langle P_\Lambda e_n, e_n \rangle^2\Big) \\
\Rightarrow \left(\sum \alpha_j z_j\right)^2 &= (1/2 - \delta)\left(\sum_{j=1}^k \alpha_j^2\right) + (1 - 2\delta)\left(\sum \alpha_j z_j\right)^2 \\
\Rightarrow 2\delta \left(\sum \alpha_j z_j\right)^2 &= (1/2 - \delta)\left(\sum_{j=1}^k \alpha_j^2\right).
\end{aligned}$$

Note that $\sum \alpha_j z_j = A^{-1}(z_1^2 + \cdots + z_{k-1}^2 + z_k^2) = \frac{B}{A} = \frac{A-1}{2A} = \frac{1}{2} - \frac{1}{2A}$, and $\sum_{j=1}^k \alpha_j^2 = \frac{B}{A^2} = \frac{A-1}{2A^2}$. Thus

$$\delta\left(1 - \frac{1}{A}\right)^2 = (1/2 - \delta)\frac{1}{A}\left(1 - \frac{1}{A}\right) \Rightarrow \frac{1}{A} = \frac{\delta}{1/2 - \delta}(1 - 1/A)$$
$$\Rightarrow A = \frac{1}{2\delta} = \frac{1}{\sin 2\epsilon}.$$

Thus $Area(v_1, \ldots, v_k) = \frac{1}{|\sin 2\epsilon|^{1/2}} \to \infty$ as $\delta \to 0$, i.e. $\mu(\Lambda) \to 0$ as $M_+ \ni \Lambda \to M_0$.

4. Now assume $M_- \ni \Lambda \to M_0$. Write $z = (z_1, \ldots, z_k)$ and

$$\langle P_\Lambda e_n, e_n\rangle = \cos(\pi/4 - \epsilon)|P_\Lambda e_n|$$

where P_Λ is the orthogonal projection onto Λ. We assume $Q(v_j) = 1$ for $1 \le j \le k-1$, $Q(v_k) = -1$. Write $P_\Lambda e_n = \sum \alpha_j v_j$. Then $\langle P_\Lambda e_n - e_n, v_i\rangle = 0$, for all i, i.e. $(I_- + 2zz^T)(\alpha) = z$. By Sherman-Morrison,

$$(I_- + 2zz^T)^{-1} = I_- - \frac{2I_- zz^T I_-}{1 + 2z^T I_- z}.$$

We will denote $\tilde{z} = I_- z$. Again using Proposition 2.1, we write $B = z^T I_- z = z_1^2 + \cdots + z_{k-1}^2 - z_k^2$, $A = Area(v_1, \ldots, v_k)^2 = -1 - 2B$. Then

$$\alpha = (I_- + 2zz^T)^{-1} z = I_- z - \frac{2}{1 + 2B} I_- zz^T I_- z$$
$$= \tilde{z} - \frac{2B}{1 + 2B} I_- z = \frac{1}{1 + 2B}\tilde{z}.$$

That is,

$$\alpha = -\frac{1}{A}\tilde{z}.$$

Let us write $\cos^2(\pi/4 - \epsilon) = 1/2 + \delta$. Then

$$\langle P_\Lambda e_n, e_n\rangle^2 = \cos^2(\pi/4 - \epsilon)|P_\Lambda e_n|^2$$
$$= (1/2 + \delta)\Big(Q(P_\Lambda e_n) + 2\langle P_\Lambda e_n, e_n\rangle^2\Big)$$
$$\Rightarrow (\sum \alpha_j z_j)^2 = (1/2 + \delta)(\sum_{j=1}^{k-1} \alpha_j^2 - \alpha_k^2) + (1 + 2\delta)(\sum \alpha_j z_j)^2$$
$$\Rightarrow -2\delta(\sum \alpha_j z_j)^2 = (1/2 + \delta)(\sum_{j=1}^{k-1} \alpha_j^2 - \alpha_k^2).$$

Note that $\sum \alpha_j z_j = -A^{-1}(z_1^2 + \cdots + z_{k-1}^2 - z_k^2) = -\frac{B}{A} = \frac{A+1}{2A} = \frac{1}{2} + \frac{1}{2A}$, and $\sum_{j=1}^{k-1} \alpha_j^2 - \alpha_k^2 = \frac{B}{A^2} = -\frac{A+1}{2A^2}$. Thus

$$\delta(1 + \frac{1}{A})^2 = (1/2 + \delta)\frac{1}{A}(1 + \frac{1}{A}) \Rightarrow \frac{1}{A} = \frac{\delta}{1/2 + \delta}(1 + 1/A)$$

$$\Rightarrow A = \frac{1}{2\delta} = \frac{1}{\sin 2\epsilon}.$$

Again $Area(v_1, \dots, v_k) = \frac{1}{|\sin 2\epsilon|^{1/2}} \to \infty$ as $\delta \to 0$, i.e. $\mu(\Lambda) \to 0$ as $M_- \ni \Lambda \to M_0$. Together with step 3, this concludes the proof. Q.E.D.

From the proof it is evident that the space of invariant sections is 2-dimensional, corresponding to the two densities we chose arbitrarily on Λ_+ and Λ_-, and by equation (2) together with McMullen's description [**21**] of $(n-1)$-homogeneous valuations through the Klain imbedding, we get the following.

Corollary 2.4. *The space $Val_{n-1}^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)}$ is 2-dimensional, and consists of non-smooth sections. It is spanned by f_S and f_T (standing for space-like and time-like) which are given for any $K \in \mathcal{K}(\mathbb{R}^n)$ by*

$$f_T(K) = \int_{S^{n-1}\cap\{Q\geq 0\}} \sqrt{|\sin 2\epsilon|}d\sigma_K(\omega)$$

and similarly

$$f_S(K) = \int_{S^{n-1}\cap\{Q\leq 0\}} \sqrt{|\sin 2\epsilon|}d\sigma_K(\omega)$$

where ϵ denotes the angle between ω and the light cone, and σ_K is the surface area measure of K.

Remark. Recall that the surface area measure of a smooth convex body K is $\sigma_K = G_*\sigma$ where $G : \partial K \to S^{n-1}$ is the Gauss map, that is, $G(p) \perp T_p\partial K$, and σ is the volume induced on ∂K from the Euclidean space. The definition of surface area measure extends naturally to all convex bodies; see e.g. Schneider [**23**].

2.1.1. A geometric interpretation of $Val_{n-1}^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)}$. The purpose of this subsection is to provide some geometrical intuition into the valuations that we constructed. It will not be used in the rest of the paper.

Let $H^\pm = \{x \in \mathbb{R}^n : Q(x,x) = \pm 1\}$. Both H^+ and H^- inherit a Lorentzian resp. Riemannian metric from $(\mathbb{R}^n, Q)$. Then $H^- \subset (\mathbb{R}^n, Q)$ is the Minkowski model of hyperbolic space, and similarly H^+ is the $(n-2,1)$ de Sitter space. The valuations in $Val_{n-1}^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)}$ can be interpreted as the surface area of K with respect to $H^\pm$ in the following sense:

Define the support functions $h_{H^+}, h_{H^-} : S^{n-1} \to \mathbb{R}$ by setting $h_{H^\pm}(\theta)$ equal to the distance from the origin of the hyperplane P_θ with Euclidean normal equal to θ that is tangent to H^+ (resp. H^-). If no such hyperplane exists, the value of $h_{H^\pm}(\theta)$ is set to 0. Let $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ be the elevation angle on S^{n-1} relative to the space-like coordinate hyperplane $(x_1, \dots, x_{n-1})$, that is, for $\omega = (x_1, \dots, x_n) \in S^{n-1}$, let $x_n = \sin \alpha$. Then these functions are given explicitly by

$$h_{H^+}(\omega) = \begin{cases} \sqrt{|\cos 2\alpha|} & |\alpha| \leq \pi/4 \\ 0 & |\alpha| < \pi/4 \end{cases}$$

$$h_{H^-}(\omega) = \begin{cases} \sqrt{|\cos 2\alpha|} & |\alpha| \geq \pi/4 \\ 0 & |\alpha| < \pi/4 \end{cases}$$

Recall that the mixed volume $V(K_1, \dots, K_n)$ of n convex bodies $K_j \in \mathcal{K}(\mathbb{R}^n)$ is a symmetric functional, given by polarizing the Lebesgue measure with respect to Minkowski sum. If $K_1 = \dots = K_j = K$, we write $V(K[j], K_{j+1}, \dots, K_n)$ for the mixed volume. We may think of f_T informally as a mixed volume:

$$f_T(K) = V(K[n-1], H^+[1]) = \int_{S^{n-1}} h_{H^+}(\omega) d\sigma_K(\omega)$$

and similarly

$$f_S(K) = V(K[n-1], H^-[1]) = \int_{S^{n-1}} h_{H^-}(\omega) d\sigma_K(\omega).$$

Another very similar description is the following. Assume K is smooth. The boundary ∂K inherits from $(\mathbb{R}^n, Q)$ a smooth field of quadratic forms on all tangent spaces, which in turn gives a measure μ_Q on ∂K. Then $f_S(K)$ is the μ_Q-measure of the space-like part of ∂K (that is, where the form is positive definite), and similarly $f_T(K)$ is the μ_Q-measure of the time-like part (that is, the subset of ∂K where the form has mixed signature).

There is also a relation between the $(n-1)$-homogeneous Lorentz-invariant valuations, and the surface area in hyperbolic and de Sitter spaces. More precisely, f_T and f_S correspond to the surface area on H^- and H^+, respectively, in the following sense. For a set $A \subset H^{\pm}$, define $C_A = \{tx : 0 \leq t \leq 1, x \in A\}$ the cone with base A. Denote by $Area_{H^{\mp}}$ the hyperbolic/de Sitter area on $H^{\mp}$. Observe that while C_A is not a convex body, one can nevertheless compute f_S or f_T on C_A at least when A is piecewise geodesic (that is, given by a finite collection of intersections of $H^{\pm}$ with hyperplanes in $\mathbb{R}^n$), simply by applying the explicit formulas of Corollary 2.4.

Proposition 2.5. *1. Let $A \subset H^-$ be piecewise geodesic. Then $Area_{H^-}(\partial A) = f_T(C_A)$.*

2. If $A \subset H^+$ is piecewise geodesic, and we further assume it has spacelike boundary, then $Area_{H^+}(\partial A) = f_S(C_A)$.

Proof. An $(n-2)$-dimensional face F of A lies on $\Lambda \cap H^{\pm}$ for $\Lambda \in Gr(n, n-1)$. By additivity of both sides, it suffices to verify that

$Area_{H^+}(F) = f_S(C_F)$, resp. $Area_{H^-}(F) = f_T(C_F)$. For H^+, by our assumption Λ is space-like, so the statement is simply that the cone measure on the sphere $\Lambda \cap H^+$ coincides with the spherical volume on it. For H^-, Λ is necessarily time-like, and it is again well-known (or easily checked) that the cone measure of the hyperboloid $\Lambda \cap H^-$ coincides with the hyperbolic volume. Q.E.D.

2.2. Schneider's bundle $S^{n,k}$. For every non-oriented subspace $\Omega \subset V$ of dimension $k+1$, consider the bundle of densities on the tautological bundle over the space of k-dimensional cooriented subspaces $\Lambda \subset \Omega$, denoted $\widetilde{K}^{k+1,k}(\Omega)$. For a cooriented pair $\Lambda \subset \Omega$, there is a corresponding pair with reversed coorientation, denoted $\overline{\Lambda} \subset \Omega$. Let $\Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$ denote the space of all global sections $\mu \in \Gamma\big(\widetilde{K}^{k+1,k}(\Omega)\big)$ which are odd w.r.t. coorientation reversal of Λ, i.e. $\mu(\overline{\Lambda}) = -\mu(\Lambda)$.

There is a certain $(k+1)$-dimensional $GL(\Omega)$-invariant linear subspace $L(\Omega) \subset \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$, which we now define. Let $\mathrm{Stab}(\Lambda, \Omega) \subset GL(\Omega)$ denote the linear transformations fixing Λ together with coorientation. Since $\dim \Omega/\Lambda = 1$, the coorientation defines a $\mathrm{Stab}(\Lambda, \Omega)$-equivariant map $S_\Lambda : \Omega/\Lambda \to D^*(\Omega/\Lambda)$. Note that $S_{\bar{\Lambda}} = -S_\Lambda$.

The space $L(\Omega) \simeq D(\Omega) \otimes \Omega$ is defined as follows. By composing the projection $p_\Lambda : \Omega \to \Omega/\Lambda$ with S_Λ, we get a $\mathrm{Stab}(\Lambda, \Omega)$-equivariant map $D^*(\Lambda) \otimes \Omega \to D^*(\Omega)$:

$$Id \otimes (S_\Lambda \circ p_\Lambda) : D^*(\Lambda) \otimes \Omega \to D^*(\Lambda) \otimes D^*(\Omega/\Lambda) = D^*(\Omega).$$

Therefore, we also get a $\mathrm{Stab}(\Lambda, \Omega)$-equivariant map $\mu_\Lambda : D(\Omega) \otimes \Omega \to D(\Lambda)$. Since $S_{\bar{\Lambda}} = -S_\Lambda$, it follows that $\mu_{\bar{\Lambda}}(v) = -\mu_\Lambda(v)$ for all $v \in \Omega$, so in fact $\mu : D(\Omega) \otimes \Omega \to \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$. The image of μ is denoted $L(\Omega)$.

Let $F_\Omega = \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))/L(\Omega)$ be the quotient. Schneider's bundle $S^{n,k}$ has base space $Gr(V, k+1)$, and the fiber over $\Omega \in Gr(V, k+1)$ is F_Ω. The topology can be introduced as follows. Fix an orthonormal basis in V, which yields the identifications $\Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega)) = C_{odd}(S(\Omega))$, $L(\Omega) = \Omega^* = \Omega$. Also, for $\Omega \in Gr(V, k+1)$, the space of cooriented pairs (Ω, Λ) (where $\Lambda \in Gr(\Omega, k)$) is identified with $S(\Omega)$, and $F_\Omega = L(\Omega)^\perp = \Omega^\perp \subset C_{odd}(S(\Omega))$, the orthogonal complement of Ω taken in the $L^2_{odd}(S(\Omega))$ norm. In particular, F_Ω inherits an inner product from $L^2_{odd}(S(\Omega))$. Then the total space is topologized by choosing a system of charts on $Gr(V, k+1)$ and taking locally the product topology. The following theorem is due to Schneider [**24**].

Theorem. *For $1 \le k \le n-1$, there is a $GL(V)$-equivariant imbedding $Val_k^{odd}(V) \to \Gamma(S^{n,k})$.*

Note that for $k = 0$ and $k = n$, there are no odd valuations by Hadwiger's theorem. We will classify the $G = SO^+(n-1, 1)$-invariant continuous sections of $S^{n,k}$, thus proving

Theorem 2.6. *There are no odd G-invariant k-homogeneous valuations for $1 \leq k \leq n-2$. For $k = n-1$ and $n \geq 3$, the space $Val_k^-(\mathbb{R}^n)^{SO^+(n-1,1)}$ is 1-dimensional. The space $Val_1^{odd}(\mathbb{R}^2)^{SO^+(1,1)}$ is 2-dimensional.*

Proof. Let $s \in \Gamma(S^{n,k})^G$ be an invariant section. We assume at first that $k \leq n-2$.

0. Denote $M_+ = \{\Omega \in Gr(V, k+1) : Q|_\Omega > 0\}$, $M_- = \{\Omega \in Gr(V, k+1) : \mathrm{sign}Q|_\Omega = (k,1)\}$, $M_0 = \{\Omega \in Gr(V, k+1) : \mathrm{sign}Q|_\Omega = (k,0)\}$. Those are the orbits of G as it acts on $Gr(V, k+1)$. We will write $\mathrm{Stab}(\Omega) \subset G$ for the stabilizer of Ω, and $\mathrm{Stab}^+(\Omega) = \{\mathrm{T} \in \mathrm{Stab}(\Omega) : \det \mathrm{T}|_\Omega = 1\}$ is the orientation-preserving subgroup of $\mathrm{Stab}(\Omega)$.
1. Observe that s necessarily vanishes on M_+: Fix some $\Omega \in M_+$. Take the Euclidean structure on Ω to be $Q|_\Omega$, which gives a lift $\mu_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))^{\mathrm{Stab}(\Omega)}$ of $s_\Omega \in F_\Omega$ which is $\mathrm{Stab}(\Omega)$-invariant. Since $\mathrm{Stab}(\Omega)$ acts transitively on the oriented Grassmannian $Gr^+(\Omega, k)$ (in fact, it acts transitively even under $\mathrm{Stab}^+(\Omega)$), $\mu_\Omega(\Lambda) = \mu_\Omega(\overline{\Lambda})$ for all Λ, so $\mu_\Omega = 0$ on Ω. Thus $s = 0$ on M_+, and by continuity of s it follows that s vanishes on M_0.
2. Now consider M_-. For any fixed $\Omega \in M_-$ one has a $\mathrm{Stab}(\Omega)$-invariant element $s_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))/L(\Omega)$. Since $H_c^1(\mathrm{Stab}^+(\Omega); \Omega) = 0$ (see Proposition 2.7 below for the computation), we can choose $\mu_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))^{\mathrm{Stab}^+(\Omega)}$ lifting s_Ω. Fix some $g_0 \in \mathrm{Stab}(\Omega)$ which is orientation-reversing. Then by $\mathrm{Stab}(\Omega)$-invariance of s_Ω, we see that $(g_0)_*\mu_\Omega$ also lifts s_Ω to $\Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))^{\mathrm{Stab}^+(\Omega)}$, so by replacing μ_Ω with $\frac{1}{2}(\mu_\Omega + (g_0)_*\mu_\Omega)$ we may assume $\mu_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))^{\mathrm{Stab}(\Omega)}$. In fact, if we fix any $\Omega_0 \in M_-$ and the corresponding $\mu_0 = \mu_{\Omega_0}$, then for any $g \in G$ one can lift s_Ω by setting for $\Omega \in M_-$ $\mu_\Omega = g_*\mu_0$ for any $g \in G$ s.t. $g\Omega_0 = \Omega$. We thus get a G-invariant lift of s to a continuous family of sections $\mu_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))^{\mathrm{Stab}(\Omega)}$ over $\Omega \in M_-$.
3. With $\Omega_0 \in M_-$, we want to inspect an element
$$\mu_0 \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega_0))^{\mathrm{Stab}(\Omega_0)}$$
more closely. The group $\mathrm{Stab}(\Omega_0)$ has the following open orbits as it acts on the cooriented hyperplanes $\Lambda \subset \Omega_0$: Ignoring the coorientation, there are two non-oriented open orbits, consisting of X_+, the Q-positive Λ, and X_-, those Λ with signature $(k-1, 1)$.

   An orientation of $\Lambda \in X_+$ is fixed under $g \in \mathrm{Stab}(\Omega_0) \cap \mathrm{Stab}(\Lambda)$ iff the orientation of Ω_0 is fixed, so coorientation is always preserved. Thus X_+ splits into two orbits X_1 and X_2 when coorientation is accounted for.

On the other hand, X_- constitutes a single orbit including coorientation. There are two cases to consider: when $k = 1$, $\Lambda \in X_-$ is a time-like line and so has its orientation preserved under the action of $g \in \mathrm{Stab}(\Omega_0) \cap \mathrm{Stab}(\Lambda)$, while the orientation of Ω_0 can be preserved or reversed (since $\dim \Omega_0 = k+1 \leq n-1$). If $k \geq 2$, the verification is also straightforward: one can again reverse the orientation of Ω_0 while keeping the orientation of Λ.

We conclude that $\mu_0(\Lambda) = 0$ for all $\Lambda \in X_-$: Indeed, since μ_0 is odd, $\mu_0(\overline{\Lambda}) = -\mu_0(\Lambda)$; but both $\Lambda, \overline{\Lambda}$ lie in the same $\mathrm{Stab}(\Omega_0)$-orbit, so $\mu_0(\Lambda) = 0$.

4. Taking Ω_0 and μ_0 as in step 3, we observe that on any $\Lambda \subset \Omega_0$ which is Q-degenerate, $\mu_0(\Lambda) = 0$ by continuity from X_+. So μ_0 is uniquely defined (since it is odd, and through $\mathrm{Stab}(\Omega_0)$-invariance) by a density $\mu_+ \in D(\Lambda_+)$ for some Q-positive subspace $\Lambda_+ \subset \Omega_0$.

   Note that, as was the case with Klain's bundle, any such μ_+ extends to a continuous $\mu_0 \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega_0))^{\mathrm{Stab}(\Omega_0)}$, and then to a family μ_Ω for $\Omega \in M_-$.

5. Let us show that μ_Ω has a limit μ_∞ in $\Gamma_{odd}\left(\widetilde{K}^{k+1,k}(\Omega_\infty)\right)$ as $\Omega \to \Omega_\infty \in M_0$. Assume for simplicity that some orientation is fixed on Ω_∞. For every Q-positive oriented k-subspace $\Lambda \subset V$, choose $\Omega_\Lambda = \Lambda \oplus \langle e_n \rangle$ with the natural orientation, and $\mu(\Lambda) = \mu_{\Omega_\Lambda}(\Lambda) \in D(\Lambda)$. The family μ_Ω is thus equivalent to a G-invariant collection $\mu(\Lambda)$ of densities on all Q-positive k-dimensional oriented subspaces Λ, s.t. $\mu(\overline{\Lambda}) = -\mu(\Lambda)$. Then for $M_- \ni (\Omega_t, \Lambda_t) \to (\Omega_\infty, \Lambda_\infty)$, either $\mu(\Lambda_t) \to \mu(\Lambda_\infty)$ when Λ_∞ is Q-positive by continuity of μ_Ω, or $\mu(\Lambda_t) \to 0 \in D(\Lambda_\infty)$. Thus μ_∞ is well-defined. The limit of $[\mu_\Omega]$ in $\Gamma_{odd}\left(\widetilde{K}^{k+1,k}(\Omega_\infty)\right)/L(\Omega_\infty)$ is therefore $[\mu_\infty]$, and it must vanish as $\Omega \to \Omega_\infty \in M_0$, by continuity of s and since s vanishes on M_+. Therefore, μ_∞ is a linear section that vanishes on all Q-degenerate k-subspaces. This is equivalent to a linear functional on $\mathbb{R}^{k+1}$ that vanishes on the light cone. So $\mu_\infty = 0$, implying $\mu_\Omega = 0$.

We conclude that when $k \leq n-2$, there are no G-invariant sections of Schneider's bundle. It follows there are no non-trivial continuous, odd, k-homogeneous G-invariant valuations.

Now assume $k = n-1$. Again, since $H^1_c(G; V) = 0$, we may lift s to an invariant section $\mu \in \Gamma_{odd}(\widetilde{K}^{n,n-1}(V))^G$.

If $n \geq 3$, as in step 3 above, μ must vanish on mixed-signature subspaces; and μ is determined by its value μ_+ on one positive subspace. Unlike the case $k \leq n-2$, there are no other restrictions: any μ_+ extends to a global section μ, as was the case with Klain's bundle.

If $n = 2$, as in step 2 above, μ is determined by two independent densities $\mu_+(\Lambda_+)$ and $\mu_-(\Lambda_-)$; and any two such densities give a continuous μ_Ω as with Klain's bundle.

For $k = n-1$, Schneider's imbedding is really just the McMullen characterization of odd $(n-1)$-homogeneous valuations, i.e. the imbedding is an isomorphism, concluding the classification of $(n-1)$-homogeneous invariant valuations. Q.E.D.

2.2.1. Computation of the continuous Lie group cohomology. The main result of this section was explained to us by José Miguel Figueroa-O'Farrill. For the relevant definitions, see [**11**]. We need to compute the continuous cohomology of $G = SO^+(n-1,1)$ with coefficients in the standard representation $V = \mathbb{R}^n$. We will show

Proposition 2.7. *The first continuous group cohomology $H^1_c(G;V)$ vanishes.*

Proof. We assume $n \geq 3$, the case of $n = 2$ being trivial. Consider $SO(n-1) \subset G$—the maximal compact subgroup. By the Hochschild-Mostow Theorem,

$$H^1_c(G;V) = H^1(\mathfrak{so}(n-1,1), \mathfrak{so}(n-1); V).$$

We will write $\mathfrak{g} = \mathfrak{so}(n-1,1)$ and $\mathfrak{h} = \mathfrak{so}(n-1)$. Under the action of $\mathfrak{h}$, $V = W \oplus T$ where $W = \mathbb{R}^{n-1}$ is the standard representation of $SO(n-1)$ (corresponding to the space coordinate hyperplane), and $T = \mathbb{R}$ is the trivial representation (corresponding to the time axis of V). Also, the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = \mathfrak{h} \oplus W$ where the inclusion $i : W \hookrightarrow \mathfrak{g}$ is given by

$$v \mapsto \begin{pmatrix} 0_{(n-1)\times(n-1)} & v_{(n-1)\times 1} \\ v^T_{1\times(n-1)} & 0 \end{pmatrix}$$

Note also that $[\mathfrak{h}, W] = W$. Now

$$C^0(\mathfrak{g},\mathfrak{h};V) = \{v \in V : \mathfrak{h}v = 0\} = T = \mathbb{R}$$

while

$$\begin{aligned} &C^1(\mathfrak{g},\mathfrak{h};V) \\ &\quad = \{f \in \mathrm{Hom}(\mathfrak{g},V) : f(\mathfrak{h}) = 0, f([h,g]) = hf(g)\,\forall g \in \mathfrak{g}, h \in \mathfrak{h}\} \\ &\qquad = \{f \in \mathrm{Hom}(W,V) : f([h,w]) = hf(w)\,\forall w \in W, h \in \mathfrak{h}\} \\ &\qquad\quad = \{f \in \mathrm{Hom}(W,W) : f([h,w]) = hf(w)\,\forall w \in W, h \in \mathfrak{h}\} \end{aligned}$$

that is, $C^1(\mathfrak{g},\mathfrak{h};V) = \mathrm{Hom}(W,W)^{\mathfrak{h}}$. This space consists of scalar operators when $\dim W \geq 3 \iff n \geq 4$, and of complex-linear operators when $n = 3$ and $W = \mathbb{R}^2 = \mathbb{C}$. The differential map $d_1 : C^0(\mathfrak{g},\mathfrak{h};V) \to C^1(\mathfrak{g},\mathfrak{h};V)$ is nonzero: taking some $t \in T$, $d_1 t(w) = -i(w)(t) = -tw$ so $d_1 t \neq 0$. Thus $\dim \mathrm{Im}(d_1) = 1$.

For $n \geq 4$, $\dim C^1(\mathfrak{g},\mathfrak{h};V) = 1$ and it follows that $H^1(\mathfrak{g},\mathfrak{h};V) = 0$.

When $n = 3$, $\dim C^1(\mathfrak{g},\mathfrak{h};V) = 2$ while $d_1(C^0(\mathfrak{g},\mathfrak{h};V)) \subset \mathrm{Ker}(d_2) \subset C^1(\mathfrak{g},\mathfrak{h};V)$. We should check whether $d_2 = 0$. It is enough to check the value of d_2 on some non-scalar operator, say $J \in \mathrm{Hom}(W, W)^{\mathfrak{h}}$, which corresponds to $\frac{\pi}{2}$-rotation. Let w_1, w_2 be the standard basis of W. Then

$$d_2 J(g_1, g_2) = J([g_1, g_2]) - g_1 J(g_2) + g_2 J(g_1).$$

Since $\mathfrak{h} \subset \mathrm{Hom}(d_2 J)$ and $\mathfrak{g} = \mathfrak{h} \oplus W$, $d_2 J \neq 0 \iff d_2 J(w_1, w_2) \neq 0$. Now

$$[i(w_1), i(w_2)] = J \in \mathfrak{h}$$

so $J([i(w_1), i(w_2)]) = 0$. And

$$-i(w_1) J(i(w_2)) + i(w_2) J(i(w_1)) = i(w_1) w_1 + i(w_2) w_2 = (0, 0, 2)^T$$

so $d_2 J \neq 0$.

Thus $\dim \mathrm{Ker} d_2 = 1$ also for $n = 3$, and $H^1(\mathfrak{so}(n-1,1), \mathfrak{so}(n-1); V) = 0$ for all n. Q.E.D.

Now consider the exact sequence $0 \to L(V) \to \Gamma_{odd}(\widetilde{K}^{n,n-1}(V)) \to F_V \to 0$ where $L(V)$ is the space of linear sections on V (an n-dimensional space), and it is G-isomorphic to V. We have the long exact sequence of cohomology

$$0 \to L(V)^G \to \Gamma_{odd}(\widetilde{K}^{n,n-1}(V))^G \to F_V^G \to H^1(G; L(V)) = 0.$$

It follows that every G-invariant section of F_V lifts to a G-invariant section in $\Gamma_{odd}(\widetilde{K}^{n,n-1}(V))$.

3. Computing valuations on $SO(n-1)$-invariant unconditional bodies

Definition 3.1. The k-support function of a body $K \subset \mathbb{R}^n$, denoted $h_k(\Lambda; K) \in C(Gr(n, n-k))$, is the k-volume of the projection of K to $\Lambda^\perp$.

Recall that $K \in \mathcal{K}(\mathbb{R}^n)$ is an unconditional convex body if it is invariant to reflections w.r.t. any of the coordinate hyperplanes.

Let $L \subset \mathbb{R}^2$ be an unconditional convex body. Denote by $L^n \subset \mathbb{R}^n$ its rotation body around the vertical axis, namely

$$L^n = \{(x\omega, y) | \omega \in S^{n-2}, (x, y) \in L\}.$$

Since L^n is $SO(n-1)$-invariant, it follows that $h_k(\Lambda; L^n)$ only depends on the angle α between Λ and the axis $\widehat{x_n}$. Thus whenever this cannot lead to confusion, we may write $h_k(\Lambda; L^n) = h_k(\alpha; L^n)$ for $0 \leq \alpha \leq \frac{\pi}{2}$. By abuse of notation, we will consider $h_k(\bullet; L^n)$ to be a function both on the unit circle S^1 and on the sphere $S^{n-1} \subset \mathbb{R}^n$; we will write $h_k(\alpha; L^n)$ or $h_k(\omega; L^n)$ when we need to emphasize that the domain is S^1, resp. S^{n-1}. When considered as a function on S^1, or equivalently as a 2π-periodic function on the real line, the following identities hold: $h_k(\alpha, L^n) = h_k(\alpha + \pi, L^n) = h_k(-\alpha, L^n)$.

Denoting $R_{k+1} \in O(k+1)$ the reversal of time direction and $G_{k+1} = \langle SO(k), R_{k+1}\rangle \subset O(k+1)$, it is obvious that L^n is G_n-invariant.

Proposition 3.2. *Given an unconditional convex body $L \subset \mathbb{R}^2$, L^n is also an unconditional convex body, and $h_k(\alpha; L^n) = h_k(\alpha, L)$ for all $n > k$. Moreover, any G_n-invariant convex body equals L^n for some unconditional convex body $L \subset \mathbb{R}^2$.*

Proof. The Minkowski functional of L^n is $p_n(\omega x, y) = \|(x, y)\|_L$ for $x, y \in \mathbb{R}$, $\omega \in S^{n-1}$. Let us verify it is convex:

$$\begin{aligned}&p_n(\omega_1 x_1, y_1) + p_n(\omega_2 x_2, y_2)\\ &\qquad= \|(x_1, y_1)\|_L + \|(x_2, y_2)\|_L \geq \|(|x_1| + |x_2|, |y_1| + |y_2|)\|_L\end{aligned}$$

while

$$\begin{aligned}&p_n((\omega_1 x_1, y_1) + (\omega_2 x_2, y_2)) = p_n(\omega_1 x_1 + \omega_2 x_2, y_1 + y_2)\\ &\qquad= \|(|\omega_1 x_1 + \omega_2 x_2|, y_1 + y_2)\|_L \leq \|(|x_1| + |x_2|, |y_1| + |y_2|)\|_L\end{aligned}$$

by unconditionality of L. The unconditionality of L^n is obvious. Now $h_k(\alpha; L^n)$ can be computed as follows. Let $e_1, \ldots, e_n$ be the standard basis, and define $\Omega = \mathrm{Span}\{e_1, \ldots, e_k, e_n\}$. Let $\Lambda_\alpha \subset \Omega$ be a k-dimensional subspace forming angle α with the space-like coordinate hyperplane. Then $h_k(\alpha; L^{k+1}) = h_k(\alpha; \Omega \cap L^n) = \mathrm{vol}_k(\mathrm{Pr}_{\Lambda_\alpha}(\Omega \cap L^n))$ and by unconditionality of L, $\mathrm{Pr}_\Omega(L^n) = L^n \cap \Omega$, so

$$h_k(\alpha, L^n) = \mathrm{vol}_k(\mathrm{Pr}_{\Lambda_\alpha}(L^n)) = \mathrm{vol}_k(\mathrm{Pr}_{\Lambda_\alpha}\mathrm{Pr}_\Omega(L^n)) = h_k(\alpha; L^{k+1}).$$

Finally, given a G_n-invariant convex body K, it is immediate that its 2-dimensional x_1-x_n section L will be an unconditional convex body, and $K = L^n$, concluding the proof. Q.E.D.

REMARK 3.3. It follows that $L \mapsto L^n$ is a Hausdorff homeomorphism between the spaces of 2-dimensional unconditional convex bodies and $SO(n-1)$-invariant, unconditional convex bodies. In light of the proposition, the notation $h_k(\alpha, L) = h_k(\alpha, L^{k+1})$ for $0 \leq \alpha \leq \frac{\pi}{2}$, as well as $h_k(\omega, L) = h_k(\omega, L^{k+1})$ for $\omega \in S^k$, is well-defined.

Recall the cosine transform $T_k : C^\infty(S^k) \to C^\infty(S^k)$ given by

$$T_k(f)(y) = \int_{S^k} f(x)|\langle x, y\rangle| dx$$

is a self-adjoint isomorphism when restricted to even functions, and extends to an isomorphism of generalized even functions. It is well-known that $T_k(\sigma_k(\omega; L)) = h_k(\omega; L)$ where $\sigma_k \in C(S^k)^*$ is the surface-area measure of L^{k+1}.

Lemma 3.4. *If $f \in C^\infty(\mathbb{R})$ is even, then $f(|x|) \in C^\infty(\mathbb{R}^n)$.*

Proof. This is because $f(x) = g(x^2)$ for $g \in C^\infty[0, \infty)$. Q.E.D.

For the following, we recall the definition of Sobolev spaces. On the linear space $\mathbb{R}^k$, denote $f \mapsto \hat{f}$ the Fourier transform, and the p-Sobolev space is the completion of $C_c^\infty(\mathbb{R}^k)$ w.r.t. the norm $\|f\|_{L_p^2} = \|\hat{f}(\omega)(1 + |\omega|^p)\|_{L^2}$. For a compact smooth manifold X, $L_p^2(X) \subset C^{-\infty}(X)$ is defined by some choice of a finite atlas $\{U_\alpha\}$ for X and an attached partition of unity $\{\rho_\alpha\}$:

$$L_p^2(X) = \{\sum_\alpha \rho_\alpha f_\alpha : f_\alpha \in L_p^2(U_\alpha)\}$$

The resulting space $L_p^2(X)$ is independent of the choices made.

Proposition 3.5. *For all $k \geq 1$ and $\epsilon > 0$, $h_k(\omega; L) \in L^2_{\frac{3}{2}-\epsilon}(S^k)$. If $h_1(\alpha; L)$ is smooth in a neighborhood of the poles and the equator in S^1, then $h_k(\omega; L) \in L^2_{\frac{k}{2}+1-\epsilon}(S^k)$ is smooth near the poles, and $h_k(\alpha; L) \in L^2_{\frac{k}{2}+1-\epsilon}(S^1)$.*

Proof. Denote

$$\Box = \frac{1}{2\omega_{k-1}}(\Delta + k) : C^\infty_{even}(S^k) \to C^\infty_{even}(S^k)$$

where ω_{k-1} is the surface area of S^{k-1}. It is an invertible differential operator of order 2. Let $\mathcal{R}_k : C^\infty_{even}(S^k) \to C^\infty_{even}(S^k)$ denote the spherical Radon transform, which is an invertible Fourier integral operator of order $-\frac{k-1}{2}$ (see [**13**]). Then (see [**12**])

$$\Box T_k = \mathcal{R}_k \iff T_k = \Box^{-1}\mathcal{R}_k. \tag{3}$$

Therefore, the cosine transform T_k is an invertible (on even functions) Fourier integral operator of order $-\frac{k+3}{2}$, and it respects Sobolev spaces, i.e. for all $s \in \mathbb{R}$

$$T_k : L_s^2(S^k) \to L^2_{s+\frac{k+3}{2}}(S^k)$$

is an isomorphism. In particular, T_1 is invertible by a differential operator followed by a $\frac{\pi}{2}$-rotation.

For the first part, note that the surface area measure $\sigma_k \in C(S^k)^* \subset L^2_{-\frac{k}{2}-\epsilon}(S^k)$, so $h_k(\omega; L) = T_k(\sigma_k) \in L^2_{\frac{3}{2}-\epsilon}(S^k)$.

For the second part, note that $\sigma_1 = T_1^{-1}(h_1) \in C(S^1)^* \subset L^2_{-\frac{1}{2}-\epsilon}(S^1)$ is smooth in a neighborhood of the equator and of the poles of S^1, since h_1 is smooth there, and by equation (3). Let $\sigma_k = \pi^*\sigma_1$ be the surface area measure of L^{k+1}, where $\pi : S^k \to S^k/SO(k-1)$. Then σ_k is smooth near the poles from unconditionality of L and Lemma 3.4, so $\sigma_k \in L^2_{-\frac{1}{2}-\epsilon}(S^k)$; also σ_k is smooth near the equator $S^{k-1} \subset S^k$. Therefore, $h_k(\omega; L) = T_k(\sigma_k) \in L_d^2(S^k)$ where $d = -\frac{1}{2} - \epsilon + \frac{k+3}{2} = \frac{k}{2} + 1 - \epsilon$, and also h_k is smooth near the poles. Then $h_k(\alpha; L)$, which can be obtained by taking a vertical 2-dimensional restriction of $h_k(\omega; L)$, lies in $L^2_d(S^1)$ and is smooth near the poles, as required. Q.E.D.

Remark 3.6. It follows that under the assumptions of Proposition 3.5, $h_k \in C^{\lfloor \frac{k}{2} \rfloor}(S^1)$, and if $K_n \to K$ in the Hausdorff topology s.t. $h_1(\bullet; K_n)$ and $h_1(\bullet; K)$ are as above, then also $h_k(\alpha; K_n) \to h_k(\alpha; K)$ in the $C^{\lfloor \frac{k}{2} \rfloor}(S^1)$ topology.

In the following, the singular support of a generalized function $f \in C^{-\infty}(M)$ over a manifold M will be denoted by sing-supp (f).

Proposition 3.7. *Let $\phi \in Val_k^{ev}(\mathbb{R}^n)^{SO(n-1)}$ satisfy*

$$\phi(K^n) = \int_{S^1} f h_k(\alpha; K) d\alpha$$

for $SO(n-1)$-invariant convex bodies K^n with smooth $h_k(\bullet; K)$, where $f \in C^{-\infty}_{even}(S^1)$. Then $\phi(K^n) = \int_{S^1} f h_k(\alpha; K)$ for all $SO(n-1)$-invariant symmetric convex bodies K^n such that sing-supp$(h_k(\alpha; K))$ and sing-supp(f) are disjoint, and sing-supp$(h_k(\alpha; K))$ is disjoint from the poles.

Proof. Denote $G = SO(k+1)$, $H = SO(k)$. Write $S^k = H \backslash G$ for the space of orbits under left action. Let $d\mu$ be the Haar probability measure on G, $d\sigma$ the pushforward to S^k. Fix a positive approximate identity $F_N \in C^\infty(S^k)^H$ supported near the north pole (identified with its H-bi-invariant pullback to G). It can be obtained by fixing an approximate identity $\tilde{F}_N$ on G, and then taking

$$F_N(g) = \int_{H \times H} \tilde{F}_N(h_1 g h_2) dh_1 dh_2.$$

Note that $F_N(g) = F_N(g^{-1})$ by bi-invariance of F_N, and since $\langle gH, H \rangle = \langle H, g^{-1}H \rangle$ (considered as points on the sphere).

Convolution of functions is defined by

$$u * v(x) = \int_G u(g) v(g^{-1}x) d\mu(g) = \int_G v(g) u(xg^{-1}) d\mu(g)$$

so that $(L_h u) * v = L_h(u * v)$ and $R_h(u * v) = u * R_h v$ (here L_h and R_h denote the left and right actions respectively). In particular, for $u \in C^\infty(S^k)$, $v \in C^\infty(G)$, $u * v \in C^\infty(S^k)$, and if v is right H-invariant, so is $u * v$. The following properties hold:

1) Convolution with F_N on either side is self-adjoint: for $u, v \in C^\infty(S^k)$, $\langle F_N * u, v \rangle = \langle u, F_N * v \rangle$ and $\langle u * F_N, v \rangle = \langle u, v * F_N \rangle$. For

instance,

$$\langle F_N * u, v\rangle = \int_G d\mu(x)v(Hx)\int_G d\mu(g)u(Hg)F_N(xg^{-1})$$
$$= \int_{G\times G} d\mu(x)d\mu(g)v(Hx)u(Hg)F_N(xg^{-1})$$

and we can exchange x and g since $F_N(xg^{-1}) = F_N(gx^{-1})$. Similarly,

$$\langle u * F_N, v\rangle = \int_G d\mu(x)v(Hx)\int_G d\mu(g)u(Hg)F_N(g^{-1}x)$$
$$= \int_{G\times G} d\mu(x)d\mu(g)v(Hx)u(Hg)F_N(g^{-1}x).$$

2) For $u \in C^\infty(S^k)$, one has $F_N * u \to u$ in $C^\infty(S^k)$. For $u \in C^\infty(G/H)$, $u * F_N \to u$.

$$F_N * u(x) = \int_{H\times H} dh_1dh_2\int_G \tilde{F}_N(h_1gh_2)u(g^{-1}x)dg$$
$$= \int_{H\times H} dh_1dh_2\int_G \tilde{F}_N(h_1g)u(h_2^{-1}g^{-1}x)dg$$

by left H-invariance of u; this equals

$$\int_H dh\int_G \tilde{F}_N(hg)u(g^{-1}x)dg = \int_H dh\tilde{F}_N*u(hx)dh = \int_H L_h(\tilde{F}_N*u)(x)dh.$$

Since $L_h(\tilde{F}_N*u)(x) \to L_hu(x) = u(x)$ in $C^\infty(G)$, we conclude that $\int_H L_h(\tilde{F}_N*u)(x)dh \to u$ in $C^\infty(S^k)$. Similarly, for $u \in C^\infty(G/H)$,

$$u * F_N(x) = \int_{H\times H} dh_1dh_2\int_G \tilde{F}_N(h_1gh_2)u(xg^{-1})dg$$
$$= \int_H dh_2\int_G \tilde{F}_N(gh_2)u(xg^{-1})dg = \int_H dh\int_G \tilde{F}_N(g)u(xh^{-1}g^{-1})dg$$
$$= \int_H R_h(u*\tilde{F}_N)(x)dh$$

and again

$$R_h(u*\tilde{F}_N) \to R_hu = u.$$

implying the statement.

3) For $u \in C^{-\infty}(S^k)$, $F_N * u \to u$ for $u \in C^{-\infty}(S^k)$ and $u * F_N \to u$ for $u \in C^{-\infty}(G/H)$. This is a direct consequence of properties 1 and 2.

4) For $u \in C^{-\infty}(S^k)$, $T_k(u * F_N) = T_k(u) * F_N$. It is enough by self-adjointness of T_k and the convolution operator to verify this for

$u \in C^\infty(S^k)$:

$$T_k(u * F_N)(x) = \int_{S^k} dy|\langle x, y\rangle| \int_G dg F_N(g) u(yg^{-1})$$
$$= \int_G dg F_N(g) \int_{S^k} u(yg^{-1})|\langle x, y\rangle| dy = \int_G dg F_N(g) \int_{S^k} u(y)|\langle xg^{-1}, y\rangle| dy$$
$$= \int_G dg F_N(g) T_k u(xg^{-1}) = T_k u * F_N(x).$$

Note that $F_N * u \in C^{-\infty}(S^k)^H$ whenever $u \in C^{-\infty}(S^k)^H$.

Let $\sigma_k \in C^{-\infty}(S^k)^H$ be the surface area measure of K^{k+1}. Then by Minkowski's theorem, $\sigma_k * F_N$ is the surface area measure of a sequence of H-invariant bodies denoted K_N^{k+1} s.t. $K_N \to K$; therefore also $K_N^n \to K^n$ and $\phi(K_N^n) \to \phi(K^n)$. On the other hand,

$$T(\sigma_k * F_N) = h_k(\bullet; K) * F_N$$

so

$$\phi(K_N^n) = \int_{S^1} f \cdot T(\sigma_k * F_N) d\alpha = \int_{S^1} f(\alpha) \cdot (h_k(\bullet; K) * F_N)(\alpha) d\alpha.$$

Choose a cut-off function $\chi \in C^\infty(S^1)^{\mathbb{Z}_2}$ (the action is reflection w.r.t. the vertical axis; note that χ induces a smooth H-invariant function on S^k, also denoted χ) such that $\chi(\alpha) h_k(\bullet; K) \in C^\infty(S^k)$ and $(1 - \chi(\alpha)) f(\alpha) \in C^\infty(S^1)$, and $\chi = 1$ in a neighborhood of the poles. Now we can restrict $\chi(\alpha) h_k(\bullet; K)$ to a smooth function on S^1, and

$$\chi(\alpha)(F_N * h_k(\bullet; K))(\alpha) \to \chi(\alpha) h_k(\alpha; K)$$

in $C^\infty(S^k)$ and also in $C^\infty(S^1)$ (by restriction). Then

$$\int_{S^1} f(\alpha)(h_k(\bullet; K) * F_N)(\alpha) d\alpha = \int_{S^1} f(\alpha) \Big(\chi(\alpha)(h_k(\bullet; K) * F_N)(\alpha)\Big) d\alpha$$
$$+ \int_{S^1} \Big((1 - \chi(\alpha)) f(\alpha)\Big)(h_k(\bullet; K) * F_N)(\alpha) d\alpha.$$

The first summand converges to

$$\int_{S^1} f(\alpha) \chi(\alpha) h_k(\alpha; K) d\alpha.$$

Also, $(1 - \chi(\alpha)) f(\alpha)$ can be pulled back to a smooth function on S^k since $1 - \chi = 0$ near the poles. In particular, we will have

$$\int_{S^1} \Big((1 - \chi(\alpha)) f(\alpha)\Big)(h_k(\bullet; K) * F_N)(\alpha) d\alpha$$
$$\to \int_{S^1} \Big((1 - \chi(\alpha)) f(\alpha)\Big) h_k(\alpha; K)(\alpha) d\alpha.$$

And so the sum converges to $\int_{S^1} f(\alpha)h_k(\alpha;K)d\alpha$, as required. Q.E.D.

4. Finding the generalized invariant valuations

From now on, $n \geq 3$, and $G = SO^+(n-1,1)$. Let us recall some definitions and facts and introduce notation.

The space of smooth valuations, denoted $Val^\infty(V)$, consists of the smooth vectors of the representation of $GL(V)$ in $Val(V)$.

Define the bundle $E^{n,k}$ over $Gr(V,n-k)$ with fiber over $\Lambda \in Gr(V,n-k)$ equal to $E^{n,k}|_\Lambda = D(V/\Lambda) \otimes D(T_\Lambda Gr(V,n-k))$. We will sometimes refer to it as the Crofton bundle, and we call its (generalized) sections (generalized) Crofton measures. It will be used later to give explicit formulas describing certain valuations. Also, recall Klain's bundle $K^{n,k}$ over $Gr(V,k)$ that has fiber $D(\Lambda)$ over $\Lambda \in Gr(V,k)$. Klain's imbedding $Kl : Val_k^{ev}(V) \to \Gamma(K^{n,k})$ is $GL(V)$-equivariant, and maps smooth valuations to smooth sections; see [**1**].

Observe that given $\Lambda \in Gr(V,n-k)$, the fibers of the two bundles over it satisfy

$$\begin{aligned} E^{n,k}|_\Lambda \otimes K^{n,n-k}|_\Lambda &= D(V/\Lambda) \otimes D(T_\Lambda Gr(V,n-k)) \otimes D(\Lambda) \\ &= D(V) \otimes D(T_\Lambda Gr(V,n-k)) \end{aligned}$$

so integration over $Gr(V,n-k)$ gives a natural bilinear non-degenerate pairing

$$\Gamma^{\pm\infty}(E^{n,k}) \times \Gamma^{\mp\infty}(K^{n,n-k}) \to D(V).$$

The $GL(V)$-equivariant cosine transform

$$T_{n-k,k} : \Gamma^\infty(E^{n,k}) \to \Gamma^\infty(K^{n,k})$$

is given as follows. For $\gamma \in \Gamma^\infty(E^{n,k})$, the value of $T_{n-k,k}(\gamma) \in D(\Lambda)$ is a Lebesgue measure on Λ, normalized by fixing some ellipsoid $D_\Lambda \subset \Lambda$, and setting

$$T_{n-k,k}(\gamma)(D_\Lambda) = \int_{\Omega \in Gr(n,n-k)} \gamma \otimes Pr_{V/\Omega}(D_\Lambda).$$

We will write $T_{n-k,k} : C^\infty(Gr(n,n-k)) \to C^\infty(Gr(n,k))$ also for the cosine transform after a Euclidean trivialization, and also $T_{n-k,k} : \Gamma^{-\infty}(E^{n,k}) \to \Gamma^{-\infty}(K^{n,k})$ for the adjoint operator to

$$T_{k,n-k} : \Gamma^\infty(E^{n,n-k}) \to \Gamma^\infty(K^{n,n-k}).$$

It extends the cosine transform on smooth sections.

4.1. Some representation theory. We make use of the following facts (see [**4**]):

1) The highest weights of $SO(n)$ are parametrized by sequences of integers $\lambda = (\lambda_1, \dots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ with $\lambda_1 \geq \dots \geq \lambda_{\lfloor \frac{n}{2} \rfloor} \geq 0$ for odd n, and $\lambda_1 \geq \dots \geq \lambda_{\lfloor \frac{n}{2} \rfloor - 1} \geq |\lambda_{\lfloor \frac{n}{2} \rfloor}|$ for even $n > 2$.

2) The irreducible components of $C^\infty(Gr(n,k))$ (considered as a representation of $SO(n)$) are of multiplicity one, with highest weights $\lambda \in \Lambda_k^+ \cap \Lambda_{n-k}^+$. Here $\Lambda_j = \{\lambda : \lambda_i = 0\,\forall i > j,\ \lambda_i \equiv 0 \mod 2\,\forall i\}$.
3) The image of $T_k : C^\infty(Gr(n,n-k)) \to C^\infty(Gr(n,k))$ consists of representations with highest weights $\lambda \in \Lambda_k^+ \cap \Lambda_{n-k}^+$, $|\lambda_2| \le 2$. The kernel is thus $\mathrm{Ker}T_k = \oplus\rho_\lambda$ with $\lambda \in \Lambda_k^+ \cap \Lambda_{n-k}^+$, $|\lambda_2| \ge 4$. The image of T_k is closed.
4) The irreducible representations of $SO(n)$ which contain an $SO(n-1)$-invariant element are precisely those corresponding to spherical harmonics. Their highest weight is $(d,0,\dots,0)$ (for degree d spherical harmonics). The spherical harmonics appearing in $C^\infty(G(n,n-k))$ are precisely those of even degree d.
5) In particular, $C^\infty(Gr(n,n-k))^{SO(n-1)} \cap \mathrm{Ker}T_{n-k,k} = 0$. Thus

$$T_{n-k,k} : C^\infty(Gr(V,n-k))^{SO(n-1)} \to C^\infty(Gr(V,k))^{SO(n-1)} \tag{4}$$

is an isomorphism: It is injective and has dense image (by Schur's Lemma), and also

$$T_{n-k,k}\Big((C^\infty(Gr(V,n-k)))^{SO(n-1)}\Big) = \Big(T_{n-k,k}(C^\infty(Gr(V,n-k)))\Big)^{SO(n-1)} \tag{5}$$

implying the image is closed. Equation (5) holds because $T_{n-k,k}$ obviously maps $SO(n-1)$-invariant vectors to $SO(n-1)$-invariant vectors, and if $v \in T_{n-k,k}(C^\infty(Gr(V,n-k)))$ is $SO(n-1)$-invariant, then $v = T_{n-k,k}u$ for some $u \in C^\infty(Gr(V,n-k))$ such that $v = T_{n-k,k}(gu)$ for all $g \in SO(n-1)$, implying $v = T_{n-k,k}(\int_{SO(n-1)} gu \cdot dg)$.
6) In particular,

$$T_{n-k,k} : C^{-\infty}(Gr(V,n-k))^{SO(n-1)} \to C^{-\infty}(Gr(V,k))^{SO(n-1)}$$

is also an isomorphism, since $T_{n-k,k}$ is a symmetric operator (after the obvious identification $Gr(V,k) = Gr(V,n-k)$).

Note that the action of $SO(n-1)$ on $\Gamma^{-\infty}(E^{n,k})$ and $\Gamma^{-\infty}(K^{n,k})$ (after a Euclidean trivialization) and on $C^{-\infty}(Gr(V,n-k))$ resp. $C^{-\infty}(Gr(V,k))$ coincides. We deduce the following

Corollary 4.1. *The map*

$$T_{n-k,k} : \Gamma^{-\infty}(E^{n,k})^{SO^+(n-1,1)} \to \Gamma^{-\infty}(K^{n,k})^{SO^+(n-1,1)}$$

is injective.

Let us prove the following

Proposition 4.2. *It holds that*

$$C^\infty(Gr(n,k)) \cap T_{n-k,k}(C^{-\infty}(Gr(n,n-k))) = C^\infty(Gr(n,k)).$$

Proof. Assume $h = T_{k,n-k}(\sigma)$ for some $\sigma \in C^{-\infty}(Gr(n,k))$ and $h \in C^\infty(Gr(n,n-k))$. Choose an approximate identity $\mu_N \in \mathcal{M}^\infty(SO(n))$. Then $T_{k,n-k}(\sigma * \mu_N) = T_{k,n-k}(\sigma) * \mu_N = h * \mu_N \to h$ in the C^∞-topology. Since $\sigma * \mu_N \in C^\infty(Gr(n,k))$, and the image of $T_{k,n-k}$ is closed in the C^∞ topology, it follows that $h \in T_{n-k,k}(C^\infty(Gr(n,k)))$, as claimed. Q.E.D.

4.2. Translation-invariant generalized valuations. The space $Val_k^{ev,-\infty}(V)$ of generalized k-homogeneous even valuations is defined by

$$Val_k^{ev,-\infty}(V) = \left(Val_{n-k}^{ev,\infty}(V)\right)^* \otimes D(V) = \left(Val_{n-k}^{ev,\infty}(V) \otimes D(V)^*\right)^*$$

By the Alesker-Poincare duality, there is a natural inclusion $Val_k^{ev,\infty}(V) \subset Val_k^{ev,-\infty}(V)$.

Let us write this inclusion explicitly. Recall that a Crofton measure $\mu_\phi \in \Gamma^\infty(Gr(V,n-k), E^{n,k})$ for $\phi \in Val_k^{ev,\infty}(V)$ is any section such that $T_{n-k,k}(\mu_\phi) = Kl(\phi)$, which always exists by [**4**]. It is equivalent to a smooth, translation-invariant measure on the affine Grassmannian $\overline{Gr}(V,n-k)$.

For $\phi \in Val_k^{ev,\infty}(\mathbb{R}^n)$ and $\psi \in Val_{n-k}^{ev,\infty}(\mathbb{R}^n)$, the duality map is given by

$$\langle \phi, \psi \rangle = \langle Kl(\phi), \mu_\psi \rangle.$$

Equivalently,

$$\langle \phi, \psi \rangle(\bullet) = \int_{\overline{Gr}(V,k)} \phi(\bullet \cap E) d\mu_\psi(E) \in D(V).$$

We have the surjective map

$$Cr_k : \Gamma^\infty(E^{n,n-k}) \to Val_{n-k}^{ev,\infty}(V)$$

given by

$$Cr_k(s)(K) = \int_{\Lambda \in Gr(V,k)} s(Pr_{V/\Lambda}(K)).$$

We will need the following

Claim 4.3. Let $T : X \to Y$ be a bounded linear map between Frechet spaces X, Y such that $Im(T) \subset Y$ is closed. Then $Im(T^*) \subset X^*$ is also closed.

Proof. By Banach's open mapping theorem, $T : X/\mathrm{Ker}(T) \to \mathrm{Im}(T)$ is an isomorphism of Frechet spaces. Therefore,

$$T^* : \mathrm{Im}(T)^* \to (X/\mathrm{Ker}(T))^* = \mathrm{Ker}(T)^\perp$$

is also an isomorphism. It remains to observe that $T^* : Y^* \to X^*$ factorizes as $Y^* \twoheadrightarrow \mathrm{Im}(T)^* \simeq \mathrm{Ker}(T)^\perp \hookrightarrow X^*$ and the last inclusion is closed. Q.E.D.

Proposition 4.4. *There is a unique extension by continuity of Klain's imbedding, $Kl_k : Val_k^{ev,-\infty}(V) \to \Gamma^{-\infty}(K^{n,k})$, which is an imbedding with closed image.*

Consider the adjoint map of Cr_k:

$$Cr_k^* : Val_k^{ev,-\infty}(V) \otimes D(V)^* \to \Gamma^{-\infty}(K^{n,k}) \otimes D(V)^*$$

which gives a map

$$A : Val_k^{ev,-\infty}(V) \to \Gamma^{-\infty}(K^{n,k})$$

s.t. $Cr_k^* = A \otimes Id$. Let us verify that A extends Klain's imbedding $Kl_k : Val_k^{ev,\infty}(V) \to \Gamma^{\infty}(K^{n,k})$. For $\gamma \in \Gamma^{\infty}(E^{n,n-k})$, one has the obvious Crofton measure $\mu_{Cr_k(\gamma)} = \gamma$, so for all $\psi \in Val_k^{ev,\infty}(V)$

$$\begin{aligned} A(\psi)(\gamma) = \langle Cr_k(\gamma), \psi \rangle &= \int_{Gr(V,k)} \mu_{Cr_k(\gamma)} Kl_k(\psi) \\ &= \int_{Gr(V,k)} \gamma Kl_k(\psi) = \langle \gamma, Kl_k(\psi) \rangle \end{aligned}$$

as required. Moreover, $\mathrm{Ker} A = 0$, since Cr_k is surjective, and by Claim 4.3 the image of A is closed.

Proposition 4.5. *The map Cr_k admits a unique extension by continuity $Cr_k : \Gamma^{-\infty}(E^{n,n-k}) \to Val_{n-k}^{ev,-\infty}(V)$, which is surjective. It holds that $Kl_{n-k} \circ Cr_k = T_{k,n-k}$.*

Consider the dual to Klain's imbedding $Kl_k : Val_k^{ev,\infty}(V) \to \Gamma^{\infty}(K^{n,k})$, tensored with the identity on $D(V)$: It is given by

$$B : \Gamma^{-\infty}(E^{n,n-k}) \to Val_{n-k}^{ev,-\infty}(V)$$

where

$$B(s)(\psi) = \langle s, Kl_k(\psi) \rangle$$

for all $\psi \in Val_k^{ev,\infty}(V)$. Then B extends the Crofton surjection: for $\gamma \in \Gamma^{\infty}(E^{n,n-k})$ and $\psi \in Val_k^{ev,\infty}(V)$,

$$B(\gamma)(\psi) = \langle \gamma, Kl_k(\psi) \rangle = \langle Cr_k(\gamma), \psi \rangle.$$

Let us verify it is surjective: the image of B is dense since Kl_k is injective. The image of B is closed by Claim 4.3 since $\mathrm{Im}(Kl_k)$ is closed. Note that

$$Cr_{n-k}^* \circ Kl_k^* = (Kl_k \circ Cr_{n-k})^* = T_{n-k,k}^* = T_{k,n-k}$$

implying $Kl_{n-k} \circ B = T_{k,n-k}$.

Definition 4.6. A generalized Crofton measure for $\phi \in Val_k^{ev,-\infty}(V)$ is any $\mu \in \Gamma^{-\infty}(E^{n,n-k})$ s.t. $Cr_k(\mu) = \phi$. We proved that such μ necessarily exists.

4.3. Reconstructing a continuous valuation from its generalized Crofton measure.

Lemma 4.7. *Let W be a linear space, $\phi \in Val_k^{ev}(W)$ a continuous valuation, and $\mu_\phi \in \Gamma^{-\infty}(E^{n,k})$ a generalized Crofton measure for ϕ. Let K be a convex body such that $|Pr_{W/\Lambda}(K)| \in \Gamma^\infty(K^{n,n-k}) \otimes D(W)^*$. Then*

$$\phi(K) = \int_{Gr(n,n-k)} |Pr_{W/\Lambda}(K)| \mu_\phi(\Lambda).$$

Proof. A convex body $K \subset W$ is naturally an element of $Val_k^{ev,\infty}(W)^* = Val_{n-k}^{ev,-\infty}(W) \otimes D(W)^*$; denote the corresponding element by $\psi_{K,n-k}$. Then $\psi_{K,n-k} = Kl_k^*(\gamma_{K,n-k}) = (Cr_k \otimes Id)(\gamma_{K,n-k})$ for some $\gamma_{K,n-k} \in \Gamma^{-\infty}(E^{n,n-k}) \otimes D(W)^*$, and so

$$\begin{aligned} Cr_{n-k}^*(\psi_{K,n-k}) &= (Kl_{n-k} \otimes Id)(\psi_{K,n-k}) \\ &= (T_{k,n-k} \otimes Id)(\gamma_{K,n-k}) \in \Gamma^{-\infty}(K^{n,n-k}) \otimes D(W)^* \end{aligned}$$

In particular, $Cr_{n-k}^*(\psi_{K,n-k})$ lies in the image of the cosine transform. Let us verify that $Cr_{n-k}^*(\psi_{K,n-k})$ is continuous and $Cr_{n-k}^*(\psi_{K,n-k})(\Lambda) = |Pr_{W/\Lambda}(K)| \in \Gamma(K^{n,n-k}) \otimes D(W)^*$, where $\Lambda \in Gr(V, n-k)$.

Take any smooth Crofton measure $\gamma \in \Gamma^\infty(E^{n,k})$. Then

$$\begin{aligned} \langle Cr_{n-k}^*(\psi_{K,n-k}), \gamma\rangle &= \langle \psi_{K,n-k}, Cr_{n-k}(\gamma)\rangle = Cr_{n-k}(\gamma)(K) \\ &= \int_{Gr(n,n-k)} |Pr_{W/\Lambda}(K)|\gamma, \end{aligned}$$

that is, $Cr_{n-k}^*(\psi_{K,n-k}) = |Pr_{W/\Lambda}(K)|$, so

$$|Pr_{W/\Lambda}(K)| \in T_{k,n-k}(\Gamma^{-\infty}(E^{n,n-k})) \otimes D(W)^*.$$

By Proposition 4.2, it follows that $|Pr_{W/\Lambda}(K)| = T_{k,n-k}(\sigma)$ for some $\sigma \in \Gamma^\infty(E^{n,n-k}) \otimes D(W)^*$.

Now fix some Euclidean structure on W. We know that $T_{n-k,k}(\mu_\phi) = Kl_k(\phi)$. Choose a sequence $\phi_j \in Val_k^{ev,\infty}(W)$ s.t. $\phi_j \to \phi$ in $Val_k^{ev}(W)$, so $\phi_j(K) \to \phi(K)$. Choose Crofton measures $\mu_j \in \Gamma^\infty(E^{n,k})$ s.t. $T_{n-k,k}(\mu_j) = Kl_k(\phi_j)$. Then since $T_{k,n-k}^* = T_{n-k,k}$,

$$\phi_j(K) = \int_{Gr(n,n-k)} |Pr_{\Lambda^\perp}(K)|\mu_j(\Lambda) = \int_{Gr(n,k)} \sigma T_{n-k,k}(\mu_j)$$

$$= \int_{Gr(n,k)} \sigma Kl_k(\phi_j) \to \int_{Gr(n,k)} \sigma Kl_k(\phi) = \int_{Gr(n,k)} \sigma T_{n-k,k}(\mu_\phi)$$

and since σ is smooth and $T_{k,n-k}^* = T_{n-k,k}$, this equals

$$\int_{Gr(n,n-k)} |Pr_{\Lambda^\perp}(K)|\mu_\phi(\Lambda)$$

as claimed. Q.E.D.

Thus, given a generalized section $s \in \Gamma^{-\infty}(E^{n,k})^{SO^+(n-1,1)}$, we may consider $\phi = Cr_{n-k}(s)$, which is an even, k-homogeneous, Lorentz-invariant generalized valuation. Then one may ask whether a continuous extension to all convex bodies of ϕ exists. According to the lemma, its value (as a continuous valuation) on all convex bodies with smooth k-support function should be given by the formula

$$\phi(K) = \int_{\Lambda \in Gr(V,n-k)} s(Pr_{V/\Lambda}(K)).$$

4.4. Finding the invariant generalized sections. This subsection provides a technical tool for classifying generalized sections supported on a closed submanifold. These results are well-known, and appear here with proofs for the convenience of the reader.

Let X be a smooth manifold, and $Y \subset X$ a closed submanifold. Let E be a smooth vector bundle over X. Let $|\omega_X|$ denote the line bundle of densities over X. Define the space

$$J_Y^q(X) = \Big\{ f \in C^\infty(X) : \forall p \leq q-1,\, X_1, \ldots, X_p \in \Gamma^\infty(TX), \\ \Big(L_{X_1} \ldots L_{X_p} f \Big)\Big|_Y = 0 \Big\}$$

where L_{X_j} is the Lie derivative, so that $J_Y^0(X) = C^\infty(X)$ and $J_Y^1(X) = \{f \in C^\infty(X) : f|_Y = 0\}$. Then define $\mathcal{M}_Y^q(E^*) = J_Y^q(X) \cdot \Gamma^\infty(X, E^* \otimes |\omega_X|)$, and

$$\Gamma_Y^{-\infty,q}(E) = (\mathcal{M}_Y^{q+1}(E^*))^o \subset \Big(\Gamma^\infty(X, E^* \otimes |\omega_X|) \Big)^* = \Gamma^{-\infty}(X, E)$$

is the annihilator of $\mathcal{M}_Y^{q+1}(E^*)$. Note that $J_Y^{q+1}(X) \subset C^\infty(X)$ is a closed ideal, and therefore $\mathcal{M}_Y^{q+1}(E^*) \subset \Gamma^\infty(X, E^* \otimes |\omega_X|)$ is a closed subspace. It is also easy to see that in fact $\Gamma_Y^{-\infty,q}(E) \subset \Gamma_Y^{-\infty}(E)$, the space of generalized sections of E supported on Y.

The following fact is well known.

Fact. *Given a generalized distribution $\mu \in \mathcal{M}^{-\infty}(\mathbb{R}^n)$ supported on $\mathbb{R}^k \subset \mathbb{R}^n$, and $K \subset \mathbb{R}^n$ compact, one can choose some integer $q \geq 0$ s.t.*

$$\langle \mu, f \rangle = \sum_{p=0}^{q} \sum_{|I|=p} \left\langle \mu_I, \frac{\partial f}{\partial x_I} \right\rangle \tag{6}$$

for all $f \in C_c^\infty(K)$, where $I = (i_1, \ldots, i_p)$ ranges over multi-indices of size p with $|i_j| \geq k+1$, and $\mu_I \in C^{-\infty}(\mathbb{R}^k)$. The representation is unique.

It follows from this fact that for compact Y,

$$\Gamma_Y^{-\infty,0}(E) \subset \cdots \subset \Gamma_Y^{-\infty,q-1}(E) \subset \Gamma_Y^{-\infty,q}(E) \subset \cdots$$

is a filtration of $\Gamma_Y^{-\infty}(E)$.

Let F^q denote the vector bundle over Y with fiber

$$F^q|_x = Sym^q(N_xY) \otimes D^*(N_xY) \otimes E|_x$$

where $N_xY = T_xX/T_xY$ is the normal space to Y at x, and $D^*(N_xY)$ the dual to the space of densities on N_xY.

Proposition. *There is a natural isomorphism*

$$\Gamma_Y^{-\infty,q}(E)/\Gamma_Y^{-\infty,q-1}(E) \simeq \Gamma^{-\infty}(Y, F^q).$$

Proof. Write

$$\Gamma_Y^{-\infty,q}(E)/\Gamma_Y^{-\infty,q-1}(E) \simeq \Big(\mathcal{M}_Y^q(E^*)/\mathcal{M}_Y^{q+1}(E^*)\Big)^*.$$

We will construct a natural isomorphism

$$\mathcal{M}_Y^q(E^*)/\mathcal{M}_Y^{q+1}(E^*) \simeq \Gamma^\infty(Y, Q^q)$$

where Q^q is the vector bundle over Y with fiber over $x \in Y$ equal to $Q^q|_x = F^*|_x \otimes D(T_xY) = Sym^q(N_x^*Y) \otimes D(T_xX) \otimes E^*|_x$, so that $\Gamma^\infty(Y, Q^q)^* \simeq \Gamma^{-\infty}(Y, F^q)$.

This reduces to finding a natural isomorphism

$$J_Y^q(X)/J_Y^{q+1}(X) \simeq \Gamma^\infty(Y, Sym^q(N^*Y)).$$

Indeed, there is a natural map $A : J_Y^q(X) \to \Gamma^\infty(Y, Sym^q(N^*Y))$. To see this, given a function $f \in J_Y^q(X)$, and vectors $v_1, \ldots, v_q \in N_xY$, choose arbitrary lifts $X_1, \ldots, X_q \in T_xY$, and define $Af(x) \in Sym^q(N_xY)^*$ by $Af(x)(v_1, \ldots, v_q) = L_{X_1} \ldots L_{X_q} f(x)$. First note that $L_{X_1} \ldots L_{X_q} f(x) \in \mathbb{C}$ is well-defined, since derivatives of order $\leq q-1$ of f at x vanish. Moreover, given a vector field $V \in \Gamma^\infty(X, TX)$ s.t. $V|_Y \in \Gamma^\infty(Y, TY)$, it holds that $L_V f \in J_Y^q(X)$. It follows that $L_{X_1} \ldots L_{X_q} f(x)$ does not depend on the choice of lifts X_j, so that A is well-defined. It is immediate that $\mathrm{Ker} A = J_Y^{q+1}(X)$.

It remains to show that A is onto. First let us prove it locally, namely, assume $Y = \mathbb{R}^k$ and $X = \mathbb{R}^n$. Let $\{e_i\}$ be the standard basis. The standard scalar product in $\mathbb{R}^n$ gives identifications $N^*\mathbb{R}^k = N\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then, given

$$s(x_1, \ldots, x_k) = \sum_{I=(i_1,\ldots,i_q)} f_I(x_1, \ldots, x_k) e_I \in \Gamma^\infty(\mathbb{R}^k, Sym^q(\mathbb{R}^{n-k}))$$

where the sum ranges over multi-indices $I = (i_1, \ldots, i_q)$ with $k+1 \leq i_1 \leq \cdots \leq i_q \leq n$ and $e_I = e_{i_1} \otimes \cdots \otimes e_{i_q}$, take

$$f(x_1, \ldots, x_n) = \rho(x_1, \ldots, x_n) \sum_{I=(i_1,\ldots,i_q)} c_I f_I(x_1, \ldots, x_k) x_{i_1} \cdot \cdots \cdot x_{i_q}$$

where c_I is defined by the equality $c_I^{-1} = \frac{\partial^q}{\partial x_I}(x_{i_1} \cdot \ldots \cdot x_{i_q})$, and $\rho \in C^\infty(\mathbb{R}^n)$ is any function that is identically 1 near $\mathbb{R}^k$. Then evidently $f \in J^q_{\mathbb{R}^k}(\mathbb{R}^n)$, and $A(f) = s$. Note that if s is compactly supported, we may choose ρ so that f is also compactly supported, and $\mathrm{Supp}(f) \cap \mathbb{R}^k = \mathrm{Supp}(s)$.

For the global surjectivity of A, choose a partition of unity $\{\rho_j\} \subset C^\infty(X)$ attached to a locally finite open cover of Y by sufficiently small open sets $\{U_j\}$, such that $\mathrm{Supp}\rho_j \subset U_j$. Given $s \in \Gamma^\infty(Y, Sym^q(N^*Y))$, by local surjectivity of A we can find compactly supported $f_j \in J^q_{Y,c}(U_j) \subset J^q_Y(X)$ for all j (here $J^q_{Y,c}(U_j)$ stands for the compactly supported elements of $J^q_Y(U_j)$) such that $A(f_j) = \rho_j s$. Then $A(\sum f_j) = s$, concluding the proof. Q.E.D.

Remark 4.8. We thus obtain a useful tool for finding the G-invariant generalized sections of a vector bundle:

Proposition 4.9. *Let G be a group, X a manifold equipped with G-action, E over X a G-equivariant vector bundle, and $Y \subset X$ a closed orbit of G. Then there is a canonical injective map*

$$\Gamma_Y^{-\infty,q}(E)^G / \Gamma_Y^{-\infty,q-1}(E)^G \to \Gamma^\infty(Y, F^q)^G.$$

In particular, if $\dim \Gamma_Y^{-\infty}(E)^G \neq 0$, *then* $\dim \Gamma^\infty(Y, F^q)^G \neq 0$ *for some* $q \geq 0$.

Proof. Taking the G-invariant elements of a G-module is a left exact functor. Therefore, the exact sequence

$$0 \to \Gamma_Y^{-\infty,q-1}(E) \to \Gamma_Y^{-\infty,q}(E) \to \Gamma^{-\infty}(Y, F^q)$$

gives an injection

$$\Gamma_Y^{-\infty,q}(E)^G / \Gamma_Y^{-\infty,q-1}(E)^G \to \Gamma^{-\infty}(Y, F^q)^G.$$

It remains to verify that in fact $\Gamma^{-\infty}(Y, F^q)^G \subset \Gamma^\infty(Y, F^q)$. This holds because G acts transitively on Y: choose any smooth, compactly supported probability measure $\mu \in \mathcal{M}_c^\infty(G)$. Then $\forall f \in \Gamma^{-\infty}(Y, F^q)^G$, $f = f * \mu = \int_G g^* f d\mu(g) \in \Gamma^\infty(Y, F^q)^G$.

Finally, since any element of $\Gamma_Y^{-\infty}(E)$, restricted to a sufficiently small open set, lies in $\Gamma_Y^{-\infty,q}(E)$ for some $q \geq 0$, it follows by G-invariance that $\Gamma_Y^{-\infty}(E)^G = \cup_{q=0}^\infty \Gamma_Y^{-\infty,q}(E)^G$. Q.E.D.

4.4.1. Construction of some generalized functions on the unit circle. For the following, define $c_j(\lambda)$ by

$$\left(\frac{\sin x}{x}\right)^\lambda = \sum_{j=0}^\infty c_j(\lambda) x^{2j}$$

The series converge locally uniformly in $x \in (-\pi, \pi)$ for every $\lambda \in \mathbb{C}$; in particular $\sum_{j=0}^\infty |c_j(\lambda)|$ converges. The coefficients $c_j(\lambda)$ are polynomial functions of $\lambda \in \mathbb{C}$: $c_0(\lambda) = 1$, $c_1(\lambda) = -\frac{\lambda}{3!}$, $c_2(\lambda) = \frac{\lambda}{5!} + \frac{\lambda(\lambda-1)}{2\cdot 3!^2}$, $c_3(\lambda) = -\frac{\lambda}{7!} - \frac{\lambda(\lambda-1)}{3!5!} - \frac{\lambda(\lambda-1)(\lambda-2)}{6\cdot 3!^3}$, and so on.

Lemma 4.10. *For every $k \in \mathbb{Z}$, the function $I_k : \{Re\lambda > 0\} \to \mathbb{C}$ given by*

$$I_k(\lambda) = \int_0^1 x^k |\sin x|^\lambda dx$$

admits a meromorphic extension to the complex plane, with simple poles at $\lambda = -(k+2j+1)$, $j = 0, 1, 2, \dots$ and residues $Res(I_k, -k-2j-1) = c_j(-k-2j-1)$.

Proof. Write

$$\begin{aligned} I_k(\lambda) &= \int_0^1 x^{k+\lambda}\Big(\frac{\sin x}{x}\Big)^\lambda dx \\ &= \sum_{j=0}^\infty c_j(\lambda)\frac{1}{\lambda+k+2j+1} \end{aligned}$$

which is meromorphic, with simple poles at $\lambda = -k-2j-1$, $j \geq 0$, and residues as claimed. Q.E.D.

Lemma 4.11. *There exists a meromorphic map $\sin_+^\lambda x : \mathbb{C} \to C^{-\infty}(-\pi, \pi)$ with simple poles at $\lambda = -1, -2, \dots$ and residues*

$$Res(\sin_+^\lambda, -k) = \begin{cases} \sum_{j=0}^m \frac{1}{(2j)!}c_{m-j}(-k)\delta_0^{(2j)}, k = 2m+1 \\ -\sum_{j=0}^{m-1} \frac{1}{(2j+1)!}c_{m-1-j}(-k)\delta_0^{(2j+1)}, k = 2m \end{cases}$$

s.t. for all $\lambda \notin \mathbb{Z}_{<0}$, $\sin_+^\lambda x(\phi dx) = \int_0^\pi \phi(x)\sin^\lambda x dx$ for $\phi \in C_c^\infty(-\pi, \pi)$ that vanishes in a neighborhood of 0.

Proof. For $Re(\lambda) > -1$, $\sin_+^\lambda x$ is locally integrable near 0 and so $\sin_+^\lambda x \in C^{-\infty}(-1, 1)$ is well-defined and analytic in λ. A meromorphic continuation with the desired properties in the region $Re(\lambda) > -(k+1)$ is given for $\phi \in C_c(-\pi, \pi)$ by

$$\begin{aligned} &\sin_+^\lambda x(\phi dx) = \int_1^\pi \phi(x)\sin^\lambda x dx \\ &\quad + \int_0^1 \sin^\lambda x(\phi(x) - \phi(0) - x\phi'(0) - \dots - \frac{1}{(k-1)!}x^{k-1}\phi^{(k-1)}(0))dx \\ &\quad + \phi(0)I_0(\lambda) + \phi'(0)I_1(\lambda) + \dots + \frac{1}{(k-1)!}\phi^{(k-1)}(0)I_{k-1}(\lambda) \end{aligned}$$

By the Lemma above, this is a well-defined generalized function, meromorphic in λ, with simple poles at $\lambda = -1, -2, \dots$ and residues as claimed. Q.E.D.

We define also $\sin_-^\lambda x \in C^{-\infty}(-\pi,\pi)$ by

$$\langle \sin_-^\lambda x, \phi(x)dx\rangle = \langle \sin_+^\lambda x, \phi(-x)dx\rangle.$$

Then

$$\mathrm{Res}(\sin_-^\lambda x, -k) = \begin{cases} \sum_{j=0}^{m} \frac{1}{(2j)!} c_{m-j}(-k)\delta^{(2j)}, k = 2m+1 \\ \sum_{j=0}^{m-1} \frac{1}{(2j+1)!} c_{m-1-j}(-k)\delta^{(2j+1)}, k = 2m \end{cases}$$

Before formulating the main result of this subsection, recall the following

Claim. Let $f : \mathbb{C} \to C^{-\infty}(X)$, $\lambda \mapsto f_\lambda(x)$ be meromorphic, where X is a smooth manifold. Assume that λ_0 is a simple pole, and $h(x) \in C(X)$ positive s.t. $f_\lambda(gx) = h(x)^\lambda f_\lambda(x)$ in the holomorphic domain of f_λ, for some $g \in \mathrm{Diff}(X)$. Then $r(x) = \mathrm{Res}(f_\lambda; \lambda_0)$ satisfies the same equation.

Proof. Indeed, write $f_\lambda(x) = \frac{a_{-1}(x)}{\lambda-\lambda_0} + a_0(x) + \cdots$, so that $r(x) = a_{-1}(x)$. Then

$$\begin{aligned} &f_\lambda(gx) = h(x)^\lambda f_\lambda(x) \\ &\qquad \Rightarrow \frac{a_{-1}(gx)}{\lambda-\lambda_0} + a_0(gx) + \cdots = \frac{a_{-1}(x)h(x)^\lambda}{\lambda-\lambda_0} + a_0(x)h(x)^\lambda + \cdots \end{aligned}$$

Developing $h(x)^\lambda$ into power series near $\lambda = \lambda_0$, we see that

$$a_{-1}(gx) = a_{-1}(x)h(x)^{\lambda_0}$$

as claimed. Q.E.D.

Recall the Lorentz form Q on $\mathbb{R}^2$, which we now restrict to the unit circle S^1. Then $\{Q \geq 0\} = \{-\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4}\} \cup \{\frac{3\pi}{4} \leq \alpha \leq \frac{5\pi}{4}\}$ and $\{Q \leq 0\} = \{\frac{\pi}{4} \leq \alpha \leq \frac{3\pi}{4}\} \cup \{\frac{5\pi}{4} \leq \alpha \leq \frac{7\pi}{4}\}$.

Corollary 4.12. *(a) For any sign $\epsilon \in \{+,-\}$ there is a generalized function f_λ^ϵ on S^1, namely $\cos_\epsilon^\lambda(2\alpha)$, which is meromorphic in λ, with simple poles at $\lambda = -1,-2,\ldots$. It is supported on $\{\alpha : signQ(\alpha) \in \{0,\epsilon\}\}$. It satisfies the equation*

$$\langle f_\lambda^\epsilon, \phi(\alpha)d\alpha\rangle = \int_{signQ(\alpha)=\epsilon} |\cos 2\alpha|^\lambda \phi(\alpha)d\alpha$$

for every $\phi \in C^\infty(S^1)$ vanishing in a neighborhood of the light cone. It also satisfies the equation

$$(g^{-1})^*(f_\lambda)(t) = \kappa^\lambda \Big(\frac{1+\kappa^2 t^2}{1+t^2}\Big)^{-\lambda} f_\lambda(t) \tag{7}$$

for $g = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$, where $(g^{-1})^*(f_\lambda) = f_\lambda \circ g$, $\kappa = e^{-2\theta}$, $t = \tan(\frac{\pi}{4} - \alpha)$.

(b) For $\lambda = -k$, $k = 1,2,\ldots$, the linear combination

$$f_\lambda^+ + (-1)^k f_\lambda^-$$

is holomorphic at $\lambda = -k$ and satisfies equation (7). Also, the residue $Res(f^\epsilon_\lambda; -k)$ satisfies equation (7). $Res(f^\epsilon_\lambda; -k)$ is given explicitly for odd $k = 2m+1$ by

$$\sum_{j=0}^{m} \frac{1}{(2j)!2^{2j}} c_{m-j}(-k)(\delta^{(2j)}_{\alpha=\pi/4} + \delta^{(2j)}_{\alpha=5\pi/4} - \delta^{(2j)}_{\alpha=3\pi/4} - \delta^{(2j)}_{\alpha=7\pi/4})$$

and for even $k = 2m$ by

$$-\epsilon \sum_{j=0}^{m-1} \frac{1}{(2j+1)!2^{2j+1}} c_{m-1-j}(-k)(\delta^{(2j+1)}_{\alpha=\pi/4} + \delta^{(2j+1)}_{\alpha=5\pi/4} - \delta^{(2j+1)}_{\alpha=3\pi/4} - \delta^{(2j+1)}_{\alpha=7\pi/4})$$

Proof. (a) This can be verified directly for $\mathrm{Re}\lambda > 0$, similarly to equation (9). Then, both sides of the equation are meromorphic maps $\mathbb{C} \to C^{-\infty}(S^1)$ so uniqueness of meromorphic extension applies. For statement (b) concerning residues (the second half is immediate from (a)), we use the claim above. Q.E.D.

Remark 4.13. All the generalized functions on S^1 that we defined are even, and so define generalized functions on $\mathbb{RP}^1$. Let Q denote the Lorentz quadratic form on $\mathbb{R}^2$. The Q-orthogonal complement of a line in $\mathbb{R}^2$ (which is the same as reflection w.r.t. to the light cone) induces a $\mathbb{Z}_2$-action on $\mathbb{RP}^1$ and so also on $C^{-\infty}(\mathbb{RP}^1)$. We call $f \in C^{-\infty}(\mathbb{RP}^1)$ cone-symmetric or cone-antisymmetric according to the action of $\mathbb{Z}_2$ on it. Then for $\lambda \neq -k$, $\cos^\lambda_+(2\alpha) + \cos^\lambda_-(2\alpha)$ is cone-symmetric and $\cos^\lambda_+(2\alpha) - \cos^\lambda_-(2\alpha)$ is cone-antisymmetric; for $\lambda = -k$, there are two cases:

- k is odd, then $\mathrm{Res}(\cos^\lambda_\pm(2\alpha), -k)$ is cone-symmetric and $\cos^\lambda_+(2\alpha) - \cos^\lambda_-(2\alpha)$ is cone-antisymmetric.
- k is even, then $\mathrm{Res}(\cos^\lambda_\pm(2\alpha), -k)$ is cone-antisymmetric and $\cos^\lambda_+(2\alpha) + \cos^\lambda_-(2\alpha)$ is cone-symmetric.

We will denote the cone-symmetric and cone-antisymmetric functions corresponding to λ by $f^+_\lambda(\alpha)$ and $f^-_\lambda(\alpha)$, respectively, normalized so that $f^+_{-(2j+1)} = \mathrm{Res}(f^+_\lambda, -(2j+1))$ and $f^-_{-2j} = \mathrm{Res}(f^-_\lambda, -2j)$. Note that $f^\pm_\lambda$ is invariant to reflection w.r.t. the origin and to both coordinate axes.

For non-integer λ, we write f^T_λ and f^S_λ for the functions corresponding to $\cos^\lambda_-(2\alpha)$ and $\cos^\lambda_+(2\alpha)$, resp. (standing for the time-like and space-like support of the function).

Remark 4.14. Note that the generalized functions supported on the light cone correspond to the residues, and they are given by derivatives of order $k-1$ for $\lambda = -k$ since $c_0(\lambda) \equiv 1$.

We will now construct generalized functions $f^\pm_{n,k,\lambda} \in C^{-\infty}(Gr(n,k))$ that are $SO(n-1)$-invariant, have singular support on the light cone, and satisfy the following transformation law under the Lorentz group: Fix any $(k-1)$-dimensional $\tilde{\Lambda} \subset \mathbb{R}^{n-1}$ (the space coordinate plane), and $v \in \mathbb{R}^{n-1}$ orthogonal to $\tilde{\Lambda}$. Denote $\Pi = \mathrm{Span}\{v, e_n\}$. Let $g \in G$ be a θ-boost in Π, namely

$$g|_{\Pi} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

while $g|_{\Pi^\perp} = Id$. Denote

$$\Lambda_\alpha = \tilde{\Lambda} + R_\alpha v$$

where R_α denotes rotation by α in Π, extended by the identity in the orthogonal directions. Then

$$(g^{-1})^*(f^{\pm}_{n,k,\lambda})(\Lambda_\alpha) = \kappa^\lambda \Big(\frac{1+\kappa^2 t^2}{1+t^2}\Big)^{-\lambda} f^{\pm}_{n,k,\lambda}(\Lambda_\alpha) \tag{8}$$

where $\langle (g^{-1})^*(f_\lambda), \mu\rangle = \langle f_\lambda, (g^{-1})_*\mu\rangle$, $\kappa = e^{-2\theta}$, $t = \tan(\frac{\pi}{4} - \alpha)$.

Here and in the following, $\alpha : Gr(n,k) \to [0, \frac{\pi}{2}]$ is the elevation angle of $\Lambda \in Gr(n,k)$ above the space coordinate hyperplane.

This is achieved as follows: choose a smooth function $\chi \in C^\infty(S^1)$ invariant to reflection w.r.t both coordinate axes, s.t. χ vanishes in a 2ϵ-neighborhood of the poles and of the equator, and equals 1 outside a 3ϵ-neighborhood of the poles and equator. Let $f \in C^{-\infty}(S^1)$ be any generalized function smooth near the poles and the equator, and invariant to reflections w.r.t. both axes.

Define $C_\epsilon = \{\Lambda \in Gr(n,k) : \alpha(\Lambda) \geq \frac{\pi}{2} - \epsilon\}$ and $E_\epsilon = \{\Lambda \in Gr(n,k) : \alpha(\Lambda) \leq \epsilon\}$. Outside $C_\epsilon \cup E_\epsilon$, one has the well-defined smooth submersion $\alpha : Gr(n,k) \setminus (C_\epsilon \cup E_\epsilon) \to (\epsilon, \frac{\pi}{2} - \epsilon)$. So we may pull-back χf as follows: define $u = \alpha^*(\chi f) \in C^{-\infty}(Gr(n,k))^{SO(n-1)}$ (which we extend to $C_\epsilon \cup E_\epsilon$ by zero).

Now observe that α^2 is a smooth function on $E_{3\epsilon}$: this can be seen by writing

$$\sin^2\alpha = \sum_{j=1}^{k} \langle v_j, e_n\rangle^2$$

where $\{v_j\}$ is any orthonormal basis of Λ, and e_n the unit vector in the time direction. Also, $(\frac{\pi}{2} - \alpha)^2$ is smooth in $C_{3\epsilon}$. Since the function $(1-\chi)f \in C^\infty(S^1)$ is smooth and invariant to reflections w.r.t. both coordinate axes, by Lemma 3.4 (applied separately near $\alpha = 0$ and $\alpha = \frac{\pi}{2}$) one may define a smooth $SO(n-1)$-invariant function $v(\Lambda) = ((1-\chi)f)(\alpha(\Lambda)) \in C^\infty(Gr(n,k))^{SO(n-1)}$ supported in $C_{3\epsilon} \cup E_{3\epsilon}$. Now define $Gr_{n,k}(f) = u + v \in C^{-\infty}(Gr(n,k))^{SO(n-1)}$.

We now define $f^{\pm}_{n,k,\lambda} = Gr_{n,k}(f^{\pm}_\lambda)$ for non-integer λ. Then for values of λ satisfying $\mathrm{Re}\lambda > 0$, verifying that $f^{\pm}_{n,k,\lambda}$ satisfies equation (8) amounts to testing the numerical equation given by (7). As before for S^1, we conclude by meromorphic extension that the equation is satisfied for all values of λ that are not odd resp. even negative integers for $f^+_{n,k,\lambda}$ resp. $f^-_{n,k,\lambda}$. Finally we define $f^-_{n,k,-2j}$ and $f^+_{n,k,-(2j+1)}$ by taking the respective residues.

Let us write an explicit formula for $f^\pm_{n,k,\lambda}(\mu)$ for $\mu \in \mathcal{M}^\infty(Gr(n,k))^{SO(n-1)}$. Writing $\mu = \phi(\alpha)d\Lambda$ where $d\Lambda$ is the unique $SO(n)$-invariant probability measure on $Gr(n,k)$, we claim that

$$f^\pm_{n,k,\lambda}(\mu) = f^\pm_\lambda(\phi(\alpha)g_{n,k}(\alpha)d\alpha)$$

with $g_{n,k}(\alpha) = C_{n,k}\cos^{n-k-1}\alpha\sin^{k-1}\alpha$.

Indeed, by uniqueness of meromorphic continuation it is enough to verify the formula for $\mathrm{Re}\lambda > 0$. Then $f^\pm_\lambda$ is continuous and $f^\pm_{n,k,\lambda}(\Lambda) = f^\pm_\lambda(\alpha(\Lambda))$. So we may write

$$f^\pm_{n,k,\lambda}(\mu) = \int_{Gr(n,k)} f^\pm_\lambda(\alpha(\Lambda))\phi(\alpha(\Lambda))d\Lambda$$

and integrate along submanifolds of constant elevation. It remains to see that $\alpha_*(d\Lambda) = g_{n,k}(\alpha)d\alpha$. The angle $\beta = \frac{\pi}{2} - \alpha$ between a random (w.r.t. the Haar measure on $Gr(n,k)$) k-dimensional subspace and a fixed direction is distributed as the angle between a random vector $x \in S^{n-1}$ (w.r.t. the Haar measure) and a fixed k-subspace. Since

$$\begin{aligned}\{x \in S^{n-1} : \angle(v, \mathbb{R}^k) = \beta\} &= \{x \in S^{n-1} : x_1^2 + \cdots + x_k^2 = \cos^2\beta\} \\ &= \Big(\cos\beta S^{k-1}\Big) \times \Big(\sin\beta S^{n-k-1}\Big),\end{aligned}$$

we get

$$g_{n,k}(\alpha) = C_{n,k}\cos^{k-1}\beta\sin^{n-k-1}\beta = C_{n,k}\cos^{n-k-1}\alpha\sin^{k-1}\alpha.$$

4.4.2. The case $k = 1$. We will denote $X = Gr(V, 1)$; $M \subset X$ will be the set of Q-degenerate subspaces, referred to as the light cone in X. We denote by α the angle between a line $\Lambda \in X$ and the space coordinate hyperplane. To describe the action of G on the various bundles, we will fix a set of generators of G. For this, fix a plane $\Pi = Sp\{e_1, e_n\}$. As a group, G is generated by $SO(n-1) \subset G$, which fixes e_n, and θ-boosts in Π, namely, transformations of the form

$$g_\theta|_\Pi = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

while $g_\theta|_{\Pi^\perp} = Id$. We start by proving

Proposition 4.15. *The G-invariant generalized sections of $K^{n,1}$ are spanned by $|\cos 2\alpha|^{\frac{1}{2}} s_0$ and $sign(\cos 2\alpha)|\cos 2\alpha|^{\frac{1}{2}} s_0$, where s_0 is the Euclidean section.*

Proof. We should only prove that there are no G-invariant sections supported on the light cone, which is denoted M.

The action of $SO(n-1) \subset G$ on $X = Gr(n,1)$ keeps α invariant. For g_θ a θ-boost as above, the action is given by

$$\tan\beta = \frac{\tan\alpha + \tanh\theta}{1 + \tan\alpha\tanh\theta}$$

where $\beta = g_\theta\alpha$ is the angle between $g_\theta\Lambda$ and the space coordinate hyperplane. In particular,

$$d\beta = \frac{d\alpha}{\cosh 2\theta + \sin 2\alpha \sinh 2\theta}$$

The action of G on the fibers is given by

$$(g_\theta)_*(\phi s_0)(\beta) = \phi(\alpha)\frac{|\cos 2\beta|^{\frac{1}{2}}}{|\cos 2\alpha|^{\frac{1}{2}}} s_0(\beta)$$

(with the value at $\alpha = \beta = \frac{\pi}{4}$ understood in the limit sense). We change the coordinates as follows: $\epsilon = \frac{\pi}{4} - \alpha$, $\eta = \frac{\pi}{4} - \beta$ and $t = \tan\epsilon$, $s = \tan\eta$. Also, denote

$$\kappa = \frac{1-\tanh\theta}{1+\tanh\theta} = \frac{1}{(\cosh\theta + \sinh\theta)^2} = e^{-2\theta}.$$

This corresponds to

$$s = \kappa t$$

and

$$g_\theta(\phi s_0)(s) = \phi(t)\kappa^{\frac{1}{2}}\Big(\frac{1+\kappa^2 t^2}{1+t^2}\Big)^{-\frac{1}{2}} s_0(s). \tag{9}$$

Now the existence of an invariant generalized section supported on M (corresponding to $t_0 = 0$) would imply according to Proposition 4.9 the existence for some $q \geq 0$ of a non-zero invariant section over M of

$$F = D^*(NM) \otimes Sym^q(NM) \otimes K^{n,1}|_M = D^*(NM) \otimes (NM)^{\otimes q} \otimes K^{n,1}|_M$$

(for the last equality note that NM is a line bundle).

Note that for $l \in M$,

$$N_l M = T_l X / T_l M = (l^* \otimes (V/l))/(l^* \otimes (l^Q/l)) \backsimeq l^* \otimes (V/l^Q),$$

where l^Q is the Q-orthogonal complement of l, and $l \in M \iff l \subset l^Q$. Applying a θ-boost fixing l, the resulting transformation of the fiber of $F|_l$ is multiplication by

$$\kappa \cdot \kappa^q \cdot \kappa^{1/2}$$

for $\kappa = e^{-2\theta}$, which cannot equal 1 for any q. We conclude there are no invariant sections supported on the light cone. Q.E.D.

When $k=1$, the Crofton fiber $E^{n,1}|_\Lambda$ is canonically isomorphic (in particular, as G-equivariant bundles) to $D(V)^n \otimes D(\Lambda)^{*(n+1)}$:

$$\begin{aligned} D(V/\Lambda)\otimes D(T_\Lambda Gr(n,n-1)) &= D(V/\Lambda)\otimes |\wedge^{top}((V/\Lambda)^*\otimes\Lambda)| \\ &= D(V/\Lambda)\otimes D((V/\Lambda)^{\otimes(n-1)})\otimes|\Lambda^{\wedge top}| \\ &= D(V/\Lambda)^n\otimes D(\Lambda)^* \\ &= D(V)^n\otimes D(\Lambda)^{*(n+1)} \end{aligned}$$

Let α be the angular altitude on the sphere, and z_0 be the Euclidean section of the bundle $E^{n,1}$. The transformation rule under the G-action for a θ-boost g_θ is therefore

$$(g_\theta)_*(\phi z_0)(\beta)=\phi(\alpha)\frac{|\cos 2\beta|^{-\frac{n+1}{2}}}{|\cos 2\alpha|^{-\frac{n+1}{2}}}s_0(\beta)$$

or equivalently

$$g_\theta(\phi z_0)(s)=\phi(t)\kappa^{-\frac{n+1}{2}}\left(\frac{1+\kappa^2t^2}{1+t^2}\right)^{\frac{n+1}{2}}z_0(s) \tag{10}$$

where $t=\tan(\frac{\pi}{4}-\alpha)$, $s=\tan(\frac{\pi}{4}-\beta)$, $\beta=g_\theta\alpha$, $s=\kappa t$, and $\kappa=e^{-2\theta}$.

Let f be a G-invariant generalized section of $E^{n,1}$. When restricted to an open orbit, such a section must be smooth (since an open orbit is a homogeneous manifold for G). Therefore, on the open orbits $f=C|\cos 2\alpha|^{-\frac{n+1}{2}}z_0$, where C is a locally constant function on $Gr(V,n-1)$.

In light of Corollary 4.1, we get

Corollary 4.16. *The space $\Gamma^{-\infty}(Gr(n,n-1),E^{n,1})^G$ is at most 2-dimensional.*

We will now turn to constructing two independent sections of this space, proving it is in fact 2-dimensional. Let us first remark that applying Proposition 4.9 for this manifold (this time $T_\Lambda M=(\Lambda/\Lambda^Q)^*\otimes(V/\Lambda)$), one can see that an invariant generalized section supported on the light cone can exist only if

$$q+1-\frac{n+1}{2}=0 \iff n=2q+1$$

where q is the order of the section (as a differential operator). We will show that such sections do indeed exist.

Proposition 4.17. $\dim\Gamma^{-\infty}(E^{n,1})^G=2$. *For odd n, there is a one-dimensional subspace of generalized sections supported on the light cone. For even n, none are supported on the light cone.*

Proof. The sections are associated with the generalized functions on $Gr(n,n-1)$ constructed in 4.4.1.

According to equations (8) and (10), they are given (after a Euclidean trivialization) by $f^{\pm}_{n,n-1,\lambda}$ with $\lambda = -\frac{n+1}{2}$. The support properties follow immediately from the corresponding properties for $f^{\pm}_{\lambda}$. Q.E.D.

Those sections will be denoted $f^{\pm}_{n,1}$.

Let us write explicit formulas for those sections in some dimensions: For $n = 3$, the cone-symmetric section $f^{+}_{3,1}$ (after rescaling) is given by

$$\phi(\alpha,\psi)d\sigma \mapsto \int_{\epsilon=0}^{\frac{\pi}{4}}\int_{\psi=0}^{2\pi}\frac{\phi(\frac{\pi}{4}+\epsilon,\psi)+\phi(\frac{\pi}{4}-\epsilon,\psi)-2\phi(\frac{\pi}{4},\psi)}{|\sin 2\epsilon|^2}\sin(\epsilon+\frac{\pi}{4})d\epsilon d\psi + \sqrt{2}I_0(-2)\int_{\psi=0}^{2\pi}\phi(\frac{\pi}{4},\psi)d\psi$$

and the cone-antisymmetric section $f^{-}_{3,1}$ is given by

$$\phi(\alpha,\psi)d\sigma \mapsto \left.\frac{\partial}{\partial\alpha}\right|_{\alpha=\frac{\pi}{4}}(\sin\alpha\int_{S^1}d\psi\phi(\alpha,\psi)).$$

For higher odd values of n, the cone-antisymmetric section is given by

$$\phi(\alpha,\psi)d\sigma \mapsto \left.\frac{\partial^m}{\partial\alpha^m}\right|_{\alpha=\frac{\pi}{4}}(\sin^{n-2}\alpha\int_M\phi(\alpha,\psi)d\psi)+\text{lower order derivatives}$$

where $m = \frac{n-1}{2}$.

4.4.3. Case of general k. Denote $X = Gr(V,k)$, M the set of Q-degenerate subspaces.

Proposition 4.18. *There are no G-invariant sections over M of the bundle with fiber over Λ equal to $D^*(N_\Lambda M)\otimes Sym^q(N_\Lambda M)\otimes K^{n,k}|_\Lambda$.*

Proof. Fix $\Lambda \in M$ touching the light cone C along the line $l = \Lambda\cap\Lambda^Q$. Denote also $\Omega = \Lambda + \Lambda^Q = l^Q$. Write $N_\Lambda M = T_\Lambda X/T_\Lambda M$. Then

$$N_\Lambda M = l^*\otimes(V/\Omega).$$

Thus as in the case $k = 1$, for $g = g_\theta \in \mathrm{Stab}(\Lambda)$, the action on $D^*(N_\Lambda M)\otimes Sym^q(N_\Lambda M)\otimes K^{n,k}|_\Lambda$ is by multiplication by $\kappa^{q+1}\kappa^{1/2}$ where $\kappa = e^{-2\theta}$. So again by Proposition 4.9, there are no invariant generalized sections of $K^{n,k}$ supported on the light cone. Q.E.D.

Therefore by Proposition 2.3, $\dim\Gamma^{-\infty}(X,K^{n,k})^G = 2$.

Proposition 4.19. $\dim\Gamma^{-\infty}(E^{n,k})^G = 2$ *for all* $1 \le k \le n-1$. *For odd n, there is a one-dimensional subspace of generalized sections supported on the light cone. For even n, none are supported on the light cone.*

Proof. Again by Corollary 4.1, $\dim \Gamma^{-\infty}(E^{n,k})^G \leq 2$. Let us find two independent sections explicitly. This time $\Lambda \in Gr(V, n-k)$ and

$$E_\Lambda = D(V/\Lambda) \otimes D(T_\Lambda Gr(V, n-k)) = D(V) \otimes D(\Lambda)^* \otimes D(\Lambda^* \otimes V/\Lambda)$$
$$= D(V) \otimes D^*(\Lambda) \otimes D(V)^{n-k} \otimes D^*(\Lambda)^n = D(V)^{n-k+1} \otimes D(\Lambda)^{*(n+1)}$$

So, similarly to the case $k = 1$, the invariant sections $f^{\pm}_{n,k}$ of $E^{n,k}$ are given, after the Euclidean trivialization, by $f^{\pm}_{n,n-k,\lambda}$, with $\lambda = -\frac{n+1}{2}$. Q.E.D.

For even values of n, we will also use the basis $f^S_{n,k}, f^T_{n,k}$ corresponding to f^S_λ, f^T_λ.

Recall that for $\mu \in \mathcal{M}^\infty(Gr(n,k))^{SO(n-1)}$ such that $\mu = \phi(\alpha)d\Lambda$ where $d\Lambda$ is the unique $SO(n)$-invariant probability measure on $Gr(n, n-k)$, we have

$$f^{\pm}_{n,k}(\mu) = f^{\pm}_\lambda(\phi(\alpha)g_{n,n-k}(\alpha)d\alpha)$$

where $g_{n,n-k}(\alpha) = C_{n,k}\sin^{n-k-1}\alpha\cos^{k-1}\alpha$ and $\lambda = -\frac{n+1}{2}$. From now on, we renormalize $f^{\pm}_{n,k}$ so that $C_{n,k} = 1$.

Theorem 4.20. *For all $1 \leq k \leq n-1, \dim Val_k^{ev,-\infty}(\mathbb{R}^n)^{SO^+(n-1,1)} = 2$.*

Proof. According to Proposition 4.4, $\left(Val_k^{ev,-\infty}(V)\right)^G$ is naturally a subspace of $\left(\Gamma^{-\infty}(K^{n,k})\right)^G$. In particular, $\dim\left(Val_k^{ev,-\infty}(V)\right)^G \leq 2$. Then by Proposition 4.5,

$$\mathrm{Ker}\Big(Cr_{n-k} : \Gamma^{-\infty}(E^{n,k}) \to Val_k^{ev,-\infty}(V)\Big) \subset \mathrm{Ker}T_{n-k,k}$$

so by Corollary 4.1 one has $\dim Val_k^{ev,-\infty}(V)^G \geq 2$. Thus, we get equality. Q.E.D.

It follows that every $SO^+(n-1,1)$-invariant continuous valuation $\phi \in Val_k(V)$ is determined by its uniquely-defined $SO^+(n-1,1)$-invariant generalized Crofton measure.

5. The non-existence of even Lorentz-invariant valuations for $1 \leq k \leq n-2$

We now proceed to show that the generalized valuations $\phi = Cr(f^{\pm}_{n,k})$, corresponding to the sections $f^{\pm}_{n,k} \in \Gamma^{-\infty}(E^{n,k})^G$ that we found, are not continuous valuations. In fact, we will show those valuations cannot be extended by continuity to the double cone. By Lemma 4.7, it follows that for an $SO(n-1)$-invariant smooth convex unconditional body K^n with k-support function $h_k(\alpha; K)$, those valuations are given by

$$\phi(K^n) = f^{\pm}_\lambda(h_k(\alpha; K)g_{n,n-k}(\alpha)d\alpha)$$

with $\lambda = -\frac{n+1}{2}$. Then by 3.7, the same formula holds as long as $h_k(\alpha; K)$ is smooth near the light cone.

5.1. Computations related to the double cone. In the following, $C \subset \mathbb{R}^2$ is the unit ball of the l_1 norm. We will write $h_k(\alpha) = h_k(\alpha; C^{k+1})$ (where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ is the angle between the normal to the hyperplane to which C^{k+1} is projected, and the space-like coordinate hyperplane). It can be computed as follows: fix $u = (\cos\alpha, 0, \dots, 0, \sin\alpha)$ the normal to the hyperplane, and $v = (\cos\beta w, \sin\beta)$, $w \in S^{k-1}$. The surface area measure of C^{k+1} is $\sigma_{C^{k+1}}(v) = \delta_{\frac{\pi}{4}}(\beta) + \delta_{-\frac{\pi}{4}}(\beta)$ and

$$\begin{aligned} h_k(\alpha) &= T_k(\sigma_{C^{k+1}}(\beta))(\alpha) \\ &= \int_{S^k} (\delta_{\frac{\pi}{4}}(\beta) + \delta_{-\frac{\pi}{4}}(\beta)) |\langle u, v\rangle| \cos^{k-1}\beta d\beta d\sigma_{k-1}(w). \end{aligned}$$

If $k > 2$, we take $0 - \frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ to be the elevation angle of $w \in S^{k-1}$. If $k = 2$, $-\pi \le \phi \le \pi$. Let us write $a_k \le \phi \le b_k$ for both cases. Then

$$\begin{aligned} h_k(\alpha) = C_k &\int_{a_k}^{b_k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\delta_{\frac{\pi}{4}}(\beta) + \delta_{-\frac{\pi}{4}}(\beta)) |\sin\beta \sin\alpha \\ &\quad + \cos\beta\cos\alpha\sin\phi| \cos^{k-1}\beta \cos^{k-2}\phi d\beta d\phi \\ &= \frac{C_k}{2^{k/2}} \int_{a_k}^{b_k} (|\sin\alpha + \cos\alpha\sin\phi| + |\sin\alpha - \cos\alpha\sin\phi|) \cos^{k-2}\phi d\phi \\ &= \frac{2C_k}{2^{k/2}} \int_{a_k}^{b_k} |\sin\alpha - \cos\alpha\sin\phi| \cos^{k-2}\phi d\phi \\ &= \begin{cases} h_k^+(\alpha), & \frac{\pi}{4} \le \alpha \le \frac{\pi}{2} \\ h_k^-(\alpha), & 0 \le \alpha \le \frac{\pi}{4} \end{cases}. \end{aligned}$$

Denoting $A_k = \int_{-\pi/2}^{\pi/2} \cos^{k-2}\phi d\phi$, and replacing C_2 by $2C_2$ for $k = 2$, we get

$$h_k^+(\alpha) = \frac{2C_k}{2^{k/2}} A_k \sin\alpha$$

and

$$\begin{aligned} &h_k^-(\alpha) \\ &= \frac{2C_k}{2^{k/2}} \left(A_k \sin\alpha - 2\sin\alpha \int_{\arcsin\tan\alpha}^{\pi/2} \cos^{k-2}\phi d\phi + \frac{2}{k-1} \frac{(\cos 2\alpha)^{\frac{k-1}{2}}}{(\cos\alpha)^{k-2}} \right) \\ &= h_k^+(\alpha) + \frac{2C_k}{2^{k/2}} \left(\frac{2}{k-1} \frac{(\cos 2\alpha)^{\frac{k-1}{2}}}{(\cos\alpha)^{k-2}} - 2\sin\alpha \int_{\tan\alpha}^{1} (1-t^2)^{\frac{k-3}{2}} dt \right) \end{aligned}$$

with the exception

$$h_1(\alpha) = \frac{1}{\sqrt{2}} (|\sin(\alpha + \frac{\pi}{4})| + |\cos(\alpha + \frac{\pi}{4})|) = \max(|\sin\alpha|, |\cos\alpha|).$$

For $\epsilon > 0$ and every n, define the $\epsilon-$ stretching of $\mathbb{R}^n$, S_ϵ to be the diagonal $n\times n$ matrix $c_\epsilon \mathrm{diag}(1,\dots,1,\tan(\frac{\pi}{4}+\epsilon))$ where $c_\epsilon\to 1$ as $\epsilon\to 0$ will be specified shortly. In the following, we will denote $\eta=\tan(\frac{\pi}{4}+\epsilon)$. We replace the double cone with its $\epsilon-$ stretching $C_{n,\epsilon}=S_\epsilon C^n$, and take c_ϵ such that $h_k(\frac{\pi}{4};C_\epsilon)=\eta h_k(\frac{\pi}{4};C)$. We will write in the following $h_{k,\epsilon}(\alpha)=h_k(\alpha;C_{n,\epsilon})$, omitting ϵ when $\epsilon=0$. Again for all $k\leq n-1$

$$\begin{aligned}h_{k,\epsilon}(\alpha)=c_\epsilon C_k\int_{a_k}^{b_k}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}&(\delta_{\frac{\pi}{4}+\epsilon}(\beta)+\delta_{-\frac{\pi}{4}-\epsilon}(\beta))|\sin\beta\sin\alpha\\&+\cos\beta\cos\alpha\sin\phi|\cos^{k-1}\beta\cos^{k-2}\phi d\beta d\phi.\end{aligned}$$

Let us write

$$h_{k,\epsilon}(\alpha)=\begin{cases}h^+_{k,\epsilon}(\alpha),\ \alpha\geq\frac{\pi}{4}-\epsilon\\ h^-_{k,\epsilon}(\alpha),\ 0\leq\alpha\leq\frac{\pi}{4}-\epsilon\end{cases}$$

where for $k\geq 2$ (again the definition of C_k for $k=2$ is twice the definition of C_k for $k\geq 3$)

$$h^+_{k,\epsilon}(\alpha)=\frac{2C_k}{2^{k/2}}A_k\eta\cdot\sin\alpha$$

and

$$\begin{aligned}h^-_{k,\epsilon}(\alpha)&=\frac{2C_k}{2^{k/2}}\left(\eta\sin\alpha\Big(A_k-2\int_{\arcsin(\eta\tan\alpha)}^{\pi/2}\cos^{k-2}\phi d\phi\Big)\right.\\&\qquad\left.+\frac{2}{k-1}\cos\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}\right)\\&=h^+_{k,\epsilon}(\alpha)+\frac{2C_k}{2^{k/2}}\left(\frac{2}{k-1}\cos\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}\right.\\&\qquad\left.-2\eta\sin\alpha\int_{\eta\tan\alpha}^{1}(1-t^2)^{\frac{k-3}{2}}dt\right),\end{aligned}$$

while

$$h_1(\alpha;C_{n,\epsilon})=\max(\eta|\sin\alpha|,|\cos\alpha|).$$

By rescaling the bodies, and since we will be only considering a single value of k at a time, we may assume in the subsequent computations that $\frac{2C_k}{2^{k/2}}=1$ for all h_k.

REMARK 5.1. In this computation, α is the angle between the normal in $\mathbb{R}^{k+1}$ to the hyperplane to which we project, and the space coordinate hyperplane. The value of the even k-homogeneous cone-symmetric/antisymmetric valuation in $\mathbb{R}^n$ on $C_{n,\epsilon}$ when $\epsilon\neq 0$ is given by $f^{\pm}_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)d\alpha)$ by Proposition 3.7, since the singular support (in fact, the support) of the surface area measure of $C_{n,\epsilon}$ is disjoint from the light cone.

REMARK 5.2. We observe for the following that h_k^+ admits a real analytic extension to S^1, and if k is odd then also h_k^- admits a real analytic extension to $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The same holds for $h_{k,\epsilon}^{\pm}$, and in the corresponding cases it holds in the C^∞ topology that

$$\lim_{\epsilon \to 0^\pm} h_{k,\epsilon}^{\pm} = h_k^{\pm}.$$

It follows that for any continuous valuation ϕ with generalized Crofton measure $f_{n,k}^{\pm}$, one may write

$$\begin{aligned}\phi(C^n) &= \lim_{\varepsilon \to 0^+} \phi(C_{n,\varepsilon}) = \lim_{\varepsilon \to 0^+} f^{\pm}_{-\frac{n+1}{2}}(h_{k,\epsilon}^+(\alpha) g_{n,n-k}(\alpha)) \\ &= f^{\pm}_{-\frac{n+1}{2}}(h_k^+(\alpha) g_{n,n-k}(\alpha))\end{aligned}$$

and if n is odd then also

$$\phi(C^n) = f^{\pm}_{-\frac{n+1}{2}}(h_k^-(\alpha) g_{n,n-k}(\alpha)).$$

5.2. Applying the generalized valuations to the double cone.

Proposition 5.3. *(Reduction to $k = n-2$) If for every $n \geq 3$ there exists no continuous even G-invariant $(n-2)$-homogeneous valuation, then there exists no continuous even G-invariant j-homogeneous valuation for $j < n-2$.*

Proof. Let $\phi \in Val_j^+(\mathbb{R}^n)^{SO^+(n-1,1)}$ be such a valuation with $j < n-2$. By our assumption, if Λ is any $(n-2)$-subspace s.t. $Q|_\Lambda$ has mixed signature, then $\phi|_\Lambda = 0$. Since every j-dimensional subspace is contained in some Λ as above, we conclude that $Kl_j(\phi) = 0$, and therefore $\phi = 0$. Q.E.D.

Thus we may assume from now on that $k = n-2$, and prove non-extendibility of the corresponding valuations.

Proposition 5.4. *(Odd n, light cone support). For odd n, an $n-2$-homogeneous even valuation ϕ on $\mathbb{R}^n$ having generalized Crofton measure $f \in \Gamma^{-\infty}(E^{n,k})^G$ supported on the light cone, cannot be extended by continuity to all $SO(n-1)$-invariant compact convex bodies.*

Proof. Assume, on the contrary, that this can be accomplished. We will show that ϕ does not extend to the double cone by continuity. Recall from Proposition 4.19 that a valuation ϕ as above can occur only for odd n, and by Remark 4.13, it is cone-symmetric if $n \equiv 1 \mod 4$ and cone-antisymmetric otherwise. By Remark 5.1, we may evaluate the valuation on $C_{n,\epsilon}$ by $\phi(C_{n,\epsilon}) = f(h_k(\alpha; C_{n,\epsilon}) g_{n,n-k})$. Therefore,

$$\phi(C^n) = \lim_{\epsilon \to 0} f(h_k(\alpha; C_{n,\epsilon}) g_{n,n-k}).$$

Write

$$f(h g_{n,n-k}) = \sum_{j=0}^{m} c_j h^{(j)} \left(\frac{\pi}{4}\right)$$

with $m = \frac{n-1}{2}$ (note that the derivatives of $g_{n,n-k}$ are now incorporated into the coefficients c_j). Note that $c_m \neq 0$ since $g_{n,n-k}(\frac{\pi}{4}) \neq 0$. We will show that the two limits $\lim_{\epsilon\to 0^+} f(h_{k,\epsilon}(\alpha,C)g_{n,n-k})$ and $\lim_{\epsilon\to 0^-} f(h_{k,\epsilon}(\alpha,C)g_{n,n-k})$ are finite and different from one another, thus arriving at a contradiction. Equivalently, since $\lim_{\epsilon\to 0^+} h_{k,\epsilon} = h_k^+$ in the $C^\infty[-\frac{3\pi}{8},\frac{3\pi}{8}]$ topology, we will show that

$$\begin{aligned}\lim_{\epsilon\to 0^-}&\Big(f(h^+_{k,\epsilon}(\alpha,C)g_{n,n-k}) - f(h^-_{k,\epsilon}(\alpha,C)g_{n,n-k})\Big)\\ &= \lim_{\epsilon\to 0^-} f\Big((h^+_{k,\epsilon}(\alpha,C) - h^-_{k,\epsilon}(\alpha,C))g_{n,n-k})\Big)\end{aligned}$$

is non-zero. Define the functions $v_\epsilon(\alpha) = h^+_{k,\epsilon}(\alpha,C) - h^-_{k,\epsilon}(\alpha,C)$ and $u_\epsilon(\alpha) = (\sin\alpha)^{-1}v_\epsilon(\alpha)$.

Consider first the case $n > 3$. Then

$$u_\epsilon(\alpha) = 2\eta\int_{\eta\tan\alpha}^1 (1-t^2)^{\frac{k-3}{2}}dt - \frac{2}{k-1}\cot\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}$$

where as before $\eta = \tan(\frac{\pi}{4}+\epsilon)$. It suffices to prove that $\lim_{\epsilon\to 0^-} u_\epsilon^{(j)}(\frac{\pi}{4}) = 0$ for $j \le m-1$, and is non-zero for $j = m$. Indeed, $\lim_{\epsilon\to 0^-} u_\epsilon(\frac{\pi}{4}) = 0$, and

$$u'_\epsilon(\alpha) = \frac{2}{k-1}\frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}}{\sin^2\alpha}.$$

Since $k = n-2$, the numerator is a polynomial in $\tan^2\alpha$ with coefficients depending on ϵ, and we conclude that $u'_\epsilon \to \frac{2}{k-1}\frac{1}{\sin^2\alpha}(1-\tan^2\alpha)^{\frac{k-1}{2}}$ in $C^\infty[-\frac{3\pi}{8},\frac{3\pi}{8}]$. Since

$$\begin{aligned}(1-\tan^2\alpha)^{\frac{k-1}{2}} &= \Big(1-\Big(1+4\Big(\alpha-\frac{\pi}{4}\Big)+o\Big(\alpha-\frac{\pi}{4}\Big)\Big)\Big)^{\frac{k-1}{2}}\\ &= (-4)^{\frac{k-1}{2}}\Big(\alpha-\frac{\pi}{4}\Big)^{\frac{k-1}{2}} + o\left(\Big(\alpha-\frac{\pi}{4}\Big)^{\frac{k-1}{2}}\right)\end{aligned}$$

and $\frac{k-1}{2} = \frac{n-3}{2} = m-1$, it follows that

$$((1-\tan^2\alpha)^{m-1})^{(m-1)}\Big(\frac{\pi}{4}\Big) = (-4)^{m-1}(m-1)!$$

implying the claim.

Now assume $n = 3$ so $k = 1$. Then $v_\epsilon(\alpha) = \eta\cos\alpha - \sin\alpha$, where again $\eta = \tan(\frac{\pi}{4}+\epsilon)$. Since $\lim_{\epsilon\to 0^-} v_\epsilon(\frac{\pi}{4}) = 0$ while $\lim_{\epsilon\to 0^-} v'_\epsilon(\frac{\pi}{4}) = \lim_{\epsilon\to 0^-} -\sin\alpha\eta - \cos\alpha = -\sqrt{2} \neq 0$, the claim follows. Q.E.D.

Remark 5.5. We note for the following that for odd values of n, both $\lim_{\epsilon\to 0^+} f(h_{k,\epsilon}(\alpha,C)g_{n,n-k})$ and $\lim_{\epsilon\to 0^-} f(h_{k,\epsilon}(\alpha,C)g_{n,n-k})$ are finite, where f is the unique G-invariant generalized Crofton measure supported on the light cone.

Proposition 5.6. *(Odd n) For odd n, no $n-2$-homogeneous valuation $\phi \in Val_{n-2}^{+}(\mathbb{R}^n)^G$ exists.*

Proof. Denote $k = n-2$. Assume first that ϕ is either pure cone-symmetric or cone-antisymmetric, according to $n \mod 4$, such that it is not supported on the light cone.

First, assume $n \equiv 1 \mod 4$, so $n \geq 5$ and $k \geq 3$. Then $\frac{n+1}{2}$ is odd, and $\phi = Cr(f_{n,k}^{-})$.

$$\phi(C_n) = \lim_{\epsilon \to 0^+} \phi(C_{n,\epsilon}) = \lim_{\epsilon \to 0^+} f^{-}_{-\frac{n+1}{2}}(h_k(\alpha; C_{n,\epsilon}) g_{n,n-k}).$$

Note that $h_k(\alpha; C_{n,\epsilon}) = C\eta \sin\alpha$ near $\alpha = \frac{\pi}{4}$, and so all derivatives at $\alpha = \frac{\pi}{4}$ of $h_k(\alpha; C_{n,\epsilon}) g_{n,n-k}$ converge to a finite limit as $\epsilon \to 0^+$. Write for an arbitrary function H on S^1,

$$\begin{aligned} N_{-}(\alpha; H) &= |\cos 2\alpha|^{-\frac{n+1}{2}} \\ &\cdot \left(H(\alpha) - H\left(\frac{\pi}{2} - \alpha\right) - 2\sum_{j=0}^{m} \frac{1}{(2j+1)!} H^{(2j+1)}\left(\frac{\pi}{4}\right)\left(\alpha - \frac{\pi}{4}\right)^{2j+1} \right) \end{aligned}$$

where $m = \frac{n-1}{4}$. Denote $H_\epsilon(\alpha) = h_k(\alpha; C_{n,\epsilon}) g_{n,n-k}(\alpha)$. We will show that the integral

$$I_{-}(H_\epsilon) = \int_0^{\frac{\pi}{4}} N_{-}(\alpha; H_\epsilon) d\alpha$$

which equals $\phi(C_{n,\epsilon})$ up to bounded summands, diverges as $\epsilon \to 0^+$. Then

$$\begin{aligned} I_{-}(H_\epsilon) = &\int_0^{\frac{\pi}{4}-\epsilon} \frac{h^{-}_{k,\epsilon}(\alpha) g_{n,n-k}(\alpha) - h^{+}_{k,\epsilon}(\alpha) g_{n,n-k}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}} d\alpha \\ &+ \int_0^{\frac{\pi}{4}} N_{-}(\alpha; h^{+}_{k,\epsilon}(\alpha) g_{n,n-k}(\alpha)) d\alpha. \end{aligned}$$

Now the second integral is bounded (uniformly in ϵ), for instance by $C|\int_0^{\frac{\pi}{4}} N_{-}(\alpha; h_k^{+}(\alpha) g_{n,n-k}(\alpha)) d\alpha|$.

We will show that the first summand is unbounded. Calculate first that

(11)

$$\begin{aligned}
&\frac{d}{d\alpha}\Big(\frac{h^-_{k,\epsilon}(\alpha)-h^+_{k,\epsilon}(\alpha)}{\sin\alpha}\Big)\\
&\quad= 2\frac{d}{d\alpha}\left(\frac{2}{k-1}\cot\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}-\left(\int_{\eta\tan\alpha}^1(1-t^2)^{\frac{k-3}{2}}dt\right)\eta\right)\\
&\quad= \frac{\eta^2}{\cos^2\alpha}(1-\eta^2\tan^2\alpha)^{\frac{k-3}{2}}\\
&\qquad-(1-\eta^2\tan^2\alpha)^{\frac{k-3}{2}}\Big(\frac{2}{k-1}\frac{1-\eta^2\tan^2\alpha}{\sin^2\alpha}+\frac{\eta^2}{\cos^2\alpha}\Big)\\
&\quad= -\frac{2}{k-1}\frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}}{\sin^2\alpha}
\end{aligned}$$

which is negative. Since $h^+_{k,\epsilon}(\frac{\pi}{4}-\epsilon)=h^-_{k,\epsilon}(\frac{\pi}{4}-\epsilon)$, it follows that $h^-_{k,\epsilon}(\alpha)-h^+_{k,\epsilon}(\alpha)>0$ in $(0,\frac{\pi}{4}-\epsilon)$. Now

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}-\epsilon}\frac{h^-_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)-h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}}d\alpha\\
&\qquad\qquad\geq C_n+c_n\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon}\frac{h^-_{k,\epsilon}(\alpha)-h^+_{k,\epsilon}(\alpha)}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}d\alpha\\
&\qquad\qquad\quad\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon}\frac{h^-_{k,\epsilon}(\alpha)-h^+_{k,\epsilon}(\alpha)}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}d\alpha\\
&\qquad\qquad\qquad\geq c'_n\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon}\frac{1}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}\frac{h^-_{k,\epsilon}(\alpha)-h^+_{k,\epsilon}(\alpha)}{\sin\alpha}d\alpha.
\end{aligned}$$

Next we integrate by parts: we integrate $(\frac{\pi}{4}-\alpha)^{-\frac{n+1}{2}}$ and differentiate the other term. The boundary term is bounded uniformly in ϵ, and we already computed the derivative of $\frac{h^-_{k,\epsilon}(\alpha)-h^+_{k,\epsilon}(\alpha)}{\sin\alpha}$ in equation (11). The resulting integral thus equals

$$c_{n,k}\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon}\frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}d\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n-1}{2}}\sin^2\alpha}\geq c'_{n,k}\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon}\frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}d\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n-1}{2}}}.$$

Now $1-\eta^2\tan^2\alpha \geq \frac{1}{4}(\alpha_\epsilon-\alpha)$, so the integral is bounded from below by

$$c'_{n,k}\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon}\frac{(\alpha_\epsilon-\alpha)^{\frac{k-1}{2}}d\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n-1}{2}}} = c'_{n,k}\int_0^{\frac{\pi}{4}-\epsilon-\frac{\pi}{5}}\frac{t^{\frac{k-1}{2}}dt}{(\epsilon+t)^{\frac{n-1}{2}}}$$
$$\geq c'_{n,k}\int_0^{\frac{\pi}{100}}\frac{t^{\frac{k-1}{2}}dt}{(\epsilon+t)^{\frac{n-1}{2}}}.$$

Finally, the limit

$$\lim_{\epsilon\to 0^+}\int_0^{\frac{\pi}{100}}\frac{t^{\frac{k-1}{2}}dt}{(\epsilon+t)^{\frac{n-1}{2}}}=\infty$$

is infinite. Thus $I_-(H_\epsilon)$ is unbounded as $\epsilon\to 0^+$, i.e $\phi(C_{n,\epsilon})\to\infty$.

Now assume $n\equiv 3 \mod 4$ and $n\geq 7$, so $k\geq 5$ and ϕ corresponds to $f^+_{n,k}$. For an arbitrary function H on S^1, define $N_+(\alpha;H)$ by

$$N_+(\alpha;H)$$
$$=|\cos 2\alpha|^{-\frac{n+1}{2}}\left(H(\alpha)+H(\frac{\pi}{2}-\alpha)-2\sum_{j=0}^{m}\frac{1}{(2j)!}H^{(2j)}\left(\frac{\pi}{4}\right)(\alpha-\frac{\pi}{4})^{2j}\right)$$

where $m=\frac{n-3}{4}$. Exactly as before, the integral

$$I_+(H_\epsilon)=\int_0^{\frac{\pi}{4}}N_+(\alpha;H_\epsilon)d\alpha$$

is unbounded as $\epsilon\to 0^+$, i.e $\phi(C_{n,\epsilon})\to\infty$.

Let us compute separately the case of $k=1$ and $n=3$. Then

$$I_-(H_\epsilon)=\int_0^{\frac{\pi}{4}-\epsilon}\frac{\cos\alpha g_{n,n-1}(\alpha)-\eta\sin\alpha g_{n,n-1}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}}d\alpha$$
$$+\int_0^{\frac{\pi}{4}}N_-(\alpha;\eta\sin\alpha g_{n,n-1}(\alpha))d\alpha$$

where

$$N_-(\alpha;H)=\frac{H(\alpha)-H(\frac{\pi}{2}-\alpha)-2H'(\frac{\pi}{4})(\alpha-\frac{\pi}{4})}{|\cos 2\alpha|^{\frac{n+1}{2}}}.$$

Now the second integral is bounded (uniformly in ϵ), for instance by $2|\int_0^{\frac{\pi}{4}}N_-(\alpha;\sin\alpha g_{n,n-1}(\alpha))d\alpha|$. The first integrand is non-negative, and since $g_{n,n-1}(\alpha)\geq c_n$ for $\alpha\in[\frac{\pi}{10},\frac{\pi}{4}]$ while $\cos 2\alpha\leq c|\alpha-\frac{\pi}{4}|$ in that interval, we get

$$\int_0^{\frac{\pi}{4}-\epsilon} \frac{\cos\alpha g_{n,n-1}(\alpha)-\eta\sin\alpha g_{n,n-1}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}}d\alpha \geq c\int_{\frac{\pi}{10}}^{\frac{\pi}{4}-\epsilon}\frac{\cos\alpha-\eta\sin\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}d\alpha \geq c\int_0^{\frac{\pi}{4}-\epsilon}\frac{\cos\alpha-\eta\sin\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}d\alpha.$$

The function $\cos\alpha-\eta\sin\alpha$ is decreasing and concave for $0\leq\alpha\leq\frac{\pi}{4}-\epsilon$, so $\cos\alpha-\eta\sin\alpha\geq 1-\frac{\alpha}{\frac{\pi}{4}-\epsilon}$ for $0\leq\alpha\leq\frac{\pi}{4}-\epsilon$. Therefore

$$\int_0^{\frac{\pi}{4}-\epsilon}\frac{\cos\alpha-\eta\sin\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}d\alpha \geq \frac{1}{\frac{\pi}{4}-\epsilon}\int_0^{\frac{\pi}{4}-\epsilon}(\frac{\pi}{4}-\alpha)^{-\frac{n-1}{2}}d\alpha+(1-\frac{\pi/4}{\pi/4-\epsilon})\int_0^{\frac{\pi}{4}-\epsilon}(\frac{\pi}{4}-\alpha)^{-\frac{n+1}{2}}d\alpha.$$

Recalling that $n=3$, this equals

$$-\frac{1}{\frac{\pi}{4}-\epsilon}\log\frac{\epsilon}{\frac{\pi}{4}}+\Big(1-\frac{\pi/4}{\frac{\pi}{4}-\epsilon}\Big)\frac{2}{3-1}\epsilon^{-\frac{3-1}{2}}=-\frac{1}{\frac{\pi}{4}-\epsilon}\log\epsilon+O(1).$$

Thus for all $k\geq 1$, $I_-(H_\epsilon)$ is unbounded as $\epsilon\to 0^+$.

Finally, consider a general $f=af^+_{n,k}+bf^-_{n,k}$, given by a linear combination of pure cone-symmetric and cone-antisymmetric sections, and assume it corresponds to a continuous valuation. Then by the preceding argument and Proposition 5.4, we must have both $a\neq 0$ and $b\neq 0$. When evaluated on H_ϵ, this would diverge as $\epsilon\to 0^+$, since the light cone-supported summand has a limit by Remark 5.5, while the other summand diverges as was just proved. Q.E.D.

Proposition 5.7. *(Even n, reduction to time-supported valuation) For even n, an $(n-2)$-homogeneous valuation $\phi\in Val^+_{n-2}(\mathbb{R}^n)^G$ on $\mathbb{R}^n$, if it exists, has generalized Crofton measure equal to a multiple of $f^T_{n,n-2}$.*

Proof. Denote $k=n-2$; assume ϕ corresponds to $f=af^T_{n,k}+bf^S_{n,k}$.

$$\phi(C_n)=\lim_{\epsilon\to 0^+}\phi(C_{n,\epsilon})=\lim_{\epsilon\to 0^+}f^T_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$$

Note that $h_{k,\epsilon}(\alpha)=C\eta\sin\alpha$ for $|\alpha|>\frac{\pi}{4}-\epsilon$, and so all derivatives at $\alpha=\frac{\pi}{4}$ of $h_{k,\epsilon}(\alpha)g_{n,n-k}$ converge to a finite limit as $\epsilon\to 0^+$, and likewise $\lim_{\epsilon\to 0^+}f^T_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$ is finite. We will show that $\lim_{\epsilon\to 0^+}f^S_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$ is infinite, implying $b=0$.

Denote $H_\epsilon(\alpha) = h_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)$. Write for an arbitrary function H on S^1,

$$N(\alpha; H) = \frac{H(\alpha) - (H(\frac{\pi}{4}) + \frac{1}{1!}H^{(1)}(\frac{\pi}{4})(\alpha - \frac{\pi}{4}) + \cdots + \frac{1}{m!}H^{(m)}(\frac{\pi}{4})(\alpha - \frac{\pi}{4})^m)}{|\cos 2\alpha|^{\frac{n+1}{2}}}$$

where $m = \frac{n-2}{2}$. The integral

$$I(H_\epsilon) = \int_0^{\frac{\pi}{4}} N(\alpha; H_\epsilon) d\alpha$$

equals $f^S_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$ up to summands corresponding to derivatives of $h_{k,\epsilon}(\alpha)g_{n,n-k}$ at the light cone, of order up to m. Those derivatives are uniformly bounded as $\epsilon \to 0^+$, since $h_{k,\epsilon}(\alpha) \to h_k(\alpha)$ in the $C^m(S^1)$ topology by the remark following Proposition 3.6.

We will show that $I(H_\epsilon)$ diverges as $\epsilon \to 0^+$. Write

$$I(H_\epsilon) = \int_0^{\frac{\pi}{4}-\epsilon} \frac{h^-_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha) - h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}} d\alpha + \int_0^{\frac{\pi}{4}} N(\alpha; h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)) d\alpha.$$

Now the second integral is bounded (uniformly in ϵ), for instance by $C|\int_0^{\frac{\pi}{4}} N(\alpha; h^+_k(\alpha)g_{n,n-k}(\alpha))d\alpha|$, and the first summand is unbounded, exactly as in the case of odd n before. This concludes the proof. Q.E.D.

Proposition 5.8. *(Non-existence of time-supported valuation with $k = n-2$) For n even, $Cr(f^T_{n,n-2})$ is not a continuous valuation.*

Proof. Denote $k = n-2$, $m = \frac{k}{2} = \frac{n}{2} - 1$, and assume $\phi = Cr(f^T_{n,n-2})$ is a continuous valuation. As before $H_\epsilon(\alpha) = h_{k,\epsilon}g_{n,n-k}(\alpha)$. By Remark 3.6, $h_{k,\epsilon}(\alpha) \to h_k(\alpha)$ as $\epsilon \to 0$ in $C^m(S^1)$.

Introduce the notations

$$J_j(\alpha; H, \alpha_0) = H(\alpha_0) + \frac{1}{1!}H^{(1)}(\alpha_0)(\alpha - \alpha_0) + \cdots + \frac{1}{j!}H^{(j)}(\alpha_0)(\alpha - \alpha_0)^j$$

$$N(\alpha; H, j) = \frac{H(\alpha) - J_j(\alpha; H, \frac{\pi}{4})}{|\cos 2\alpha|^{j+\frac{3}{2}}}$$

and

$$I(u) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} N(\alpha; u, m) d\alpha$$

Observe that $H_\epsilon \to H$ in $C^m(S^1)$ as well, so all the derivatives satisfy $H_\epsilon^{(j)}(\frac{\pi}{4}) \to H^{(j)}(\frac{\pi}{4})$ for $j \le m$ as $\epsilon \to 0$. We will show that

$$\lim_{\epsilon \to 0^+} f^T_{-\frac{n+1}{2}}(H_\epsilon) \neq \lim_{\epsilon \to 0^-} f^T_{-\frac{n+1}{2}}(H_\epsilon).$$

Equivalently, due to C^m convergence, we will show that $I(H_\epsilon)$ has different one-sided limits.

Denote

$$u_\epsilon(\alpha) = \frac{h_{k,\epsilon}(\alpha)}{\sin\alpha}.$$

Recall that $u_\epsilon(\alpha) = A_k\eta$ for $\alpha \geq \frac{\pi}{4} - \epsilon$, while for $0 \leq \alpha \leq \frac{\pi}{4} - \epsilon$ we have

$$u_\epsilon(\alpha) = A_k\eta - 2\eta\int_{\eta\tan\alpha}^1 (1-t^2)^{\frac{k-3}{2}}dt + \frac{2}{k-1}\cot\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}$$

where $A_k = \int_{-\pi/2}^{\pi/2}\cos^{k-2}\phi d\phi$. Therefore,

$$\lim_{\epsilon\to 0^+} I(u_\epsilon) = \lim_{\epsilon\to 0^+} I(A_k\eta) = 0.$$

Now write $H_\epsilon = t(\alpha)u_\epsilon(\alpha)$ where $t(\alpha) = g_{n,n-k}(\alpha)\sin\alpha$. According to Lemma A.1, we may write

$$\begin{aligned} &H(\alpha) - J_j\Big(\alpha; H, \frac{\pi}{4}\Big) \\ &= t\Big(\frac{\pi}{4}\Big)\Big(u_\epsilon(x) - J_m\Big(\alpha; u_\epsilon, \frac{\pi}{4}\Big)\Big) + u_\epsilon(\alpha)R_{m+1}(\alpha) + O(C_\epsilon|\alpha - \frac{\pi}{4}|^{m+1}\Big) \end{aligned}$$

where $R_{m+1}(\alpha) = t(\alpha) - J_m(\alpha; t, \frac{\pi}{4})$, and the constant C_ϵ in the error term is bounded by

$$C_m \sup_{1\leq j\leq m} |u_\epsilon^{(j)}|$$

with

$$C_m = m\sup_{0\leq j\leq m+1} |(g_{n,n-k}(\alpha)\sin\alpha)^{(j)}|$$

where everywhere $\alpha \in [0, \frac{\pi}{2}]$. By the convergence of $u_\epsilon(\alpha) \to A_k$ in $C^m[\frac{\pi}{4}, \frac{\pi}{2}]$, we conclude that $C_\epsilon \to 0$ as $\epsilon \to 0$.

Since $|R_{m+1}(\alpha)| \leq C|\alpha - \frac{\pi}{4}|^{m+1}$, and u_ϵ converges in $C[\frac{\pi}{4}, \frac{\pi}{2}]$, the integral

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{u_\epsilon(\alpha)R_{m+1}(\alpha)}{|\cos 2\alpha|^{m+\frac{3}{2}}}d\alpha$$

has a limit as $\epsilon \to 0$. Also, the integral

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{O(C_\epsilon|\alpha - \frac{\pi}{4}|^{m+1})}{|\cos 2\alpha|^{m+\frac{3}{2}}}d\alpha$$

converges to 0 as $\epsilon \to 0$. We conclude that $I(H_\epsilon) - t(\frac{\pi}{4})I(u_\epsilon)$ converges, and thus it suffices to show that the functional $I(u_\epsilon)$ has different one-sided limits. We will verify that

$$\lim_{\epsilon\to 0^-} I(u_\epsilon) \neq 0.$$

From now on, $\epsilon < 0$. We will use the approximations

$$\eta = \tan\left(\frac{\pi}{4} + \epsilon\right) = 1 + 2\epsilon + O(\epsilon^2)$$

$$1-\eta^4=-8\epsilon+O(\epsilon^2)$$

$$(1-\eta^2)^{\frac{1}{2}}=(-4\epsilon+O(\epsilon^2))^{\frac{1}{2}}=2|\epsilon|^{\frac{1}{2}}+O(|\epsilon|).$$

Then for $\alpha<\frac{\pi}{4}-\epsilon$,

$$u'_\epsilon(\alpha)=-\frac{2}{k-1}(1-\eta^2\tan^2\alpha)^{m-\frac{1}{2}}\frac{1}{\sin^2\alpha}.$$

It follows by induction that for $\alpha\in(\frac{\pi}{4}+\epsilon,\frac{\pi}{4}-\epsilon)$ and $j\geq 1$,

$$\begin{aligned}u_\epsilon^{(j)}(\alpha)&=(-1)^j\frac{2}{k-1}\frac{1}{\sin^2\alpha}\frac{1}{2^{j-1}}\frac{(2m-1)!!}{(2m-2j+1)!!}\\&\cdot\eta^{2j-2}(1-\eta^2\tan^2\alpha)^{m+\frac{1}{2}-j}\left(\frac{2\tan\alpha}{\cos^2\alpha}\right)^{j-1}+O\Big((1-\eta^2\tan^2\alpha)^{m+\frac{3}{2}-j}\Big)\\&=(-1)^j\frac{2}{k-1}\frac{(2m-1)!!}{(2m-2j+1)!!}\eta^{2j-2}\frac{\sin^{j-3}\alpha}{\cos^{3j-3}\alpha}(1-\eta^2\tan^2\alpha)^{m+\frac{1}{2}-j}\\&\qquad+O\Big((1-\eta^2\tan^2\alpha)^{m+\frac{3}{2}-j}\Big).\end{aligned}$$

In particular, for $1\leq j\leq m$ and $\alpha\in(\frac{\pi}{4}+\epsilon,\frac{\pi}{4}-\epsilon)$, $|1-\eta^2\tan^2\alpha|=O(|\epsilon|)$ so

$$|u_\epsilon^{(j)}(\alpha)|=O(|\varepsilon|^{m+\frac{1}{2}-j}). \tag{12}$$

It therefore also holds that

$$|u_\epsilon(\frac{\pi}{4})-A_k\eta|=|u_\epsilon(\frac{\pi}{4})-u_\varepsilon(\frac{\pi}{4}-\epsilon)|=O(|\varepsilon|^{m+\frac{1}{2}}). \tag{13}$$

Write

$$\begin{aligned}I(u_\epsilon)=&\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{A_k\eta-J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha\\&+\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u_\epsilon(\alpha)-J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha\end{aligned} \tag{14}$$

where

$$w(\alpha)=\frac{|\alpha-\frac{\pi}{4}|^{m+\frac{3}{2}}}{|\cos 2\alpha|^{m+\frac{3}{2}}}$$

is a C^∞ function, strictly positive in $[0, \frac{\pi}{2}]$. Now integrate by parts—we integrate the denominator and differentiate the numerator:

$$\begin{aligned}\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{A_k\eta - J_m(\alpha; u_\epsilon, \frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}} w(\alpha)d\alpha &= -\frac{1}{m+\frac{1}{2}} \frac{A_k\eta - J_m(\frac{\pi}{2}; u_\epsilon, \frac{\pi}{4})}{(\frac{\pi}{4})^{m+\frac{1}{2}}} w\Big(\frac{\pi}{2}\Big) \\ &+ \frac{1}{m+\frac{1}{2}} \frac{A_k\eta - J_m(\frac{\pi}{4}-\epsilon; u_\epsilon, \frac{\pi}{4})}{|\epsilon|^{m+\frac{1}{2}}} w\Big(\frac{\pi}{4}-\epsilon\Big) \\ &+ \frac{1}{m+\frac{1}{2}} \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{J_{m-1}(\alpha; -u'_\epsilon, \frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}} w(\alpha)d\alpha \\ &+ \frac{1}{m+\frac{1}{2}} \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{A_k\eta - J_m(\alpha; u_\epsilon, \frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}} w'(\alpha)d\alpha.\end{aligned}$$

The first summand is $o(1)$ as $\epsilon \to 0^-$, since $u_\epsilon \to A_k \in C^m[\frac{\pi}{4}, \frac{\pi}{2}]$, so $J_m(\frac{\pi}{2}; u_\epsilon, \frac{\pi}{4}) \to J_m(\frac{\pi}{2}; A_k, \frac{\pi}{4}) = A_k$ as $\epsilon \to 0^-$. Let us verify that the last summand is also $o(1)$. Indeed,

$$\begin{aligned}&\left| \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{A_k\eta - J_m(\alpha; u_\epsilon, \frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}} w'(\alpha)d\alpha \right| \\ &\qquad \le C \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \left(\frac{|A_k\eta - u_\epsilon(\frac{\pi}{4})|}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}} + \frac{1}{j!} \sum_{j=1}^{m} \frac{|u_\epsilon^{(j)}(\frac{\pi}{4})|}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}-j}} \right) d\alpha.\end{aligned}$$

This can be integrated explicitly. The terms corresponding to $\frac{\pi}{2}$ are all $o(1)$ again since $u_\epsilon \to A_k \in C^m[\frac{\pi}{4}, \frac{\pi}{2}]$, while the terms corresponding to $\frac{\pi}{4} - \epsilon$ are all $O(|\epsilon|)$ by estimates (12) and (13). Therefore,

$$\begin{aligned}&\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{A_k\eta - J_m(\alpha; u_\epsilon, \frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}} w(\alpha)d\alpha \\ &\qquad = \frac{1}{m+\frac{1}{2}} \left(\frac{A_k\eta - J_m(\frac{\pi}{4}-\epsilon; u_\epsilon, \frac{\pi}{4})}{|\epsilon|^{m+\frac{1}{2}}} w(\frac{\pi}{4}-\epsilon) \right. \\ &\qquad\qquad \left. + \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{J_{m-1}(\alpha; -u'_\epsilon, \frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}} w(\alpha)d\alpha \right) + o(1).\end{aligned}$$

Similarly, we may integrate by parts the second summand of $I(u_\epsilon)$ in equation (14) as follows:

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} &\frac{u_\epsilon(\alpha)-J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha\\ &= -\frac{1}{m+\frac{1}{2}}\frac{A_k\eta-J_m(\frac{\pi}{4}-\epsilon;u_\epsilon,\frac{\pi}{4})}{|\epsilon|^{m+\frac{1}{2}}}w(\frac{\pi}{4}-\epsilon)\\ &\quad+\frac{1}{m+\frac{1}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u_\epsilon'(\alpha)+J_{m-1}(\alpha;-u_\epsilon',\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha.\end{aligned}$$

The $\frac{\pi}{4}$-boundary term vanishes since u_ϵ is C^∞ near $\frac{\pi}{4}$. Thus

$$\begin{aligned}I(u_\epsilon)=\frac{1}{m+\frac{1}{2}}\Bigg(&\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{J_{m-1}(\alpha;-u_\epsilon',\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha\\ &+\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u_\epsilon'(\alpha)-J_{m-1}(\alpha;u_\epsilon',\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha\Bigg)+o(1)\end{aligned}$$

so we should show that the expression in the brackets does not vanish as $\epsilon\to 0^-$. Repeatedly applying integration by parts as we did for equation (14), we end up having to show that

$$J(\epsilon)=\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{-u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}}w(\alpha)d\alpha+\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u_\epsilon^{(m)}(\alpha)-u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}}w(\alpha)d\alpha$$

does not converge to 0 as $\epsilon\to 0^-$.

Recall that

$$\begin{aligned}u_\epsilon^{(m)}(\alpha)=(-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha}\\ +O\Big((1-\eta^2\tan^2\alpha)^{3/2}\Big).\end{aligned}$$

In particular,

$$\begin{aligned}u_\epsilon^{(m)}(\frac{\pi}{4})&=(-1)^m\frac{2}{k-1}(2m-1)!!(1-\eta^2)^{\frac{1}{2}}\eta^{2m-2}2^m+O(|\epsilon|^{3/2})\\ &=(-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}2^{m+1}|\epsilon|^{1/2}+O(|\epsilon|^{3/2}).\end{aligned}$$

We will also need the finer estimate

$$\begin{aligned}&u_\epsilon^{(m)}(\alpha)-u_\epsilon^{(m)}(\frac{\pi}{4})=\\ &(-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}\Big((1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha}-(1-\eta^2)^{\frac{1}{2}}2^m\Big)\\ &\qquad+O\Big(\alpha-\frac{\pi}{4}\Big)\end{aligned}$$

which is obtained by writing

$$u_\epsilon^{(m)}(\alpha) = (-1)^m \frac{2}{k-1}(2m-1)!!\eta^{2m-2}(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} \\ + s_\epsilon(\alpha)(1-\eta^2\tan^2\alpha)^{3/2}$$

where $s_\epsilon(\alpha) \in C^1(\frac{\pi}{5},\frac{\pi}{3})$ is uniformly bounded in $C^1(\frac{\pi}{5},\frac{\pi}{3})$. Then the error term in $u_\epsilon^{(m)}(\alpha) - u_\epsilon^{(m)}(\frac{\pi}{4})$ is easily seen to equal

$$O\Big((1-\eta^2)^{3/2} - (1-\eta^2\tan^2\alpha)^{3/2}\Big) + O\Big(\alpha - \frac{\pi}{4}\Big)$$

and since $(1-\eta^2\tan^2\alpha)^{3/2}$ is $C^1(\frac{\pi}{5},\frac{\pi}{3})$ and uniformly bounded, one has

$$(1-\eta^2)^{3/2} - (1-\eta^2\tan^2\alpha)^{3/2} = O\Big(\alpha - \frac{\pi}{4}\Big).$$

Integrating the first summand of $J(\epsilon)$ by parts, we get that

$$\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{-u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}} w(\alpha)d\alpha = -u_\epsilon^{(m)}(\frac{\pi}{4})\frac{2}{|\epsilon|^{\frac{1}{2}}}w(\frac{\pi}{4}) + o(1) \\ = (-1)^{m+1}\frac{2}{k-1}(2m-1)!!\eta^{2m-2}w(\frac{\pi}{4})\Big(2^{m+2}+o(1)\Big)$$

while

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{u_\epsilon^{(m)}(\alpha) - u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}} w(\alpha)d\alpha = (-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2} \\ \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{\Big((1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} - (1-\eta^2)^{\frac{1}{2}}2^m\Big)}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}} w(\alpha)d\alpha + o(1) \\ = (-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}w(\frac{\pi}{4}) \\ \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} - (1-\eta^2)^{\frac{1}{2}}2^m}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}} d\alpha + o(1).$$

So it remains to show that

$$-2^{m+2} + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} - (1-\eta^2)^{\frac{1}{2}}2^m}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}} d\alpha \nrightarrow 0.$$

Since

$$\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} = 2^m + O(\alpha - \frac{\pi}{4})$$

this boils down to

$$-4 + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{1}{2}} - (1-\eta^2)^{\frac{1}{2}}}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}} d\alpha \nrightarrow 0.$$

And indeed, the integral is non-positive. This concludes the proof. Q.E.D.

6. Applications

Recently in [**22**], some negative results on continuity properties of classical constructions in the theory of valuations were proved. We will now explain how some of those results can be seen immediately from the classification of Lorentz-invariant valuations.

6.1. The image of the Klain imbedding is not closed. Denote by $\phi_{n,k}^{\pm} \in Val_k^{ev,-\infty}(V)^G$ the two independent generalized valuations that we found. The generalized Klain sections $Kl(\phi_{n,k}^{\pm}) \in \Gamma(K^{n,k})$ for $1 \leq k \leq n-2$ are in fact continuous sections of the Klain bundle that do not correspond to a continuous valuation. They do belong to the closure (in the C^0 topology) of the image of the Klain imbedding on continuous valuations.

6.2. The Fourier transform does not extend to continuous valuations. The Fourier transform on smooth even valuations extends to the space of generalized smooth valuations by self-adjointness (see [**8**]): For $\phi \in Val_k^{ev,-\infty}(V)$, we define $\mathbb{F}\phi \in Val_{n-k}^{ev,-\infty}(V^*) \otimes D(V)$ by letting for all $\psi \in Val_k^{ev,\infty}(V^*)$

$$\langle \mathbb{F}\phi, \psi \rangle = \langle \phi, \mathbb{F}\psi \rangle.$$

It is a $GL(V)$-equivariant involution (in the sense that $(\mathbb{F}_{V^*} \otimes Id) \circ \mathbb{F}_V = Id$). Restricting to $G = SO^+(n-1,1)$, we get a G-equivariant involution

$$\mathbb{F} : Val_k^{ev,-\infty}(V) \to Val_{n-k}^{ev,-\infty}(V)$$

which restricts to the usual (G-equivariant) Fourier transform on smooth even valuations.

Let $\phi_{n,n-1}^{\pm} \in Val_{n-1}^{ev}(V)^G$ be the cone-symmetric and cone-anti-symmetric continuous valuations that we found. It follows by equivariance that

$$\mathbb{F}(\phi_{n,n-1}^{\pm}) \in Val_1^{ev,-\infty}(V)^G$$

Since $Val_1^{ev,-\infty}(V)^G$ contains no non-trivial continuous valuations when $n \geq 3$, it follows that the Fourier transform does not extend by continuity to continuous valuations for $n \geq 3$.

Appendix A. A technical lemma

We denote by $J^m(x; f, a)$ the Taylor polynomial of order m for the function f around a.

Lemma A.1. *For $w \in C^\infty(\mathbb{R})$ and $h \in C^m(\mathbb{R})$, it holds in any fixed compact interval I around 0 that*

$$\begin{aligned} &w(x)h(x) - J_m(x; wh, 0) \\ &\qquad = w(0)(h(x) - J_m(x; h, 0)) + h(x)R_{m+1}(x) + O(|x|^{m+1}) \end{aligned}$$

as $x \to 0$, where $R_{m+1}(x) = w(x) - J_m(x; w, 0)$. More precisely, if $|h^{(j)}(x)| \leq H_j$ for all $x \in I$ and $0 \leq j \leq m$ and $|w^{(j)}(x)| \leq W$ for all $x \in I$ and $j \leq m+1$, then $O(|x|^{m+1}) \leq C_{m,I}(H_m + \cdots + H_1)W|x|^{m+1}$.

Proof. Write $J_m(f)$ for $J_m(x; f, 0)$. Then

$$h = J_m(h) + e_1(x)$$

$$w - w(0) = J_m(w - w(0)) + e_2(x)$$

where

$$|e_1(x)| \leq c_{m,I} H_m |x|^m$$

$$|e_2(x)| \leq c'_{m,I} W |x|^{m+1}$$

so

$$\begin{aligned} wh &= (w - w(0))h + w(0)h = J_m(w - w(0))h + w(0)h + hR_{m+1} \\ &= J_m(w - w(0))(J_m(h) + O(H_m|x|^m)) + w(0)h + hR_{m+1} \\ &= J_m(w - w(0))J_m(h) + w(0)h + O\Big(H_m W|x|^{m+1}\Big) + hR_{m+1} \end{aligned}$$

where the last equality holds since $J_m(w - w(0)) = O(W_1|x|)$. Note that

$$\begin{aligned} J_m(w - w(0))J_m(h) &= J_m((w - w(0))h) + O\Big((H_m + \cdots + H_1)W|x|^{m+1}\Big) \\ &= J_m(wh) - w(0)J_m(h) + O\Big((H_m + \cdots + H_1)W|x|^{m+1}\Big) \end{aligned}$$

so

$$wh = J_m(wh) - w(0)J_m(h) + w(0)h + O((H_m + \cdots + H_1)W|x|^{m+1}) + hR_{m+1}$$

as claimed. Q.E.D.

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