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ON A NEW CLASS OF HOLONOMY GROUPS IN PSEUDO-RIEMANNIAN GEOMETRY

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Abstract

We describe a new class of holonomy groups on pseudo-Riemannian manifolds. Namely, let g be a nondegenerate bilinear form on a vector space V, and $L\colon V \to V$ a g-symmetric operator. Then the identity component of the centraliser of L in $\mathrm{SO}(g)$ is a holonomy group for a suitable Levi-Civita connection.

1. Introduction and main result

Holonomy groups were introduced in the 1920s by Élie Cartan [**13**, **14**] for the study of Riemannian symmetric spaces and since then the classification of holonomy groups has remained one of the classical problems in differential geometry.

Definition 1. Let M be a smooth manifold endowed with an affine symmetric connection ∇. The *holonomy group of* ∇ is a subgroup $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(T_xM)$ that consists of the linear operators $A\colon T_xM \to T_xM$, being parallel transport transformations along closed loops γ with $\gamma(0) = \gamma(1) = x$.

Problem. Can a given subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ be realized as the holonomy group for an appropriate symmetric connection?

The fundamental results in this direction are due to Marcel Berger [**4**], who initiated the program of classification of Riemannian and irreducible holonomy groups that was completed by D. V. Alekseevskii [**1**], R. Bryant [**10**, **11**], D. Joyce [**20**, **21**], S. Merkulov, L. Schwachhöfer [**28**] and S.T. Yau [**36**]. Very good historical surveys can be found in [**12**, **30**].

The classification of Lorentzian holonomy groups has recently been obtained by T. Leistner [**25**] and A. Galaev [**17**]. However, in the general pseudo-Riemanian case, the complete description of holonomy groups is a very difficult problem which still remains open and even particular examples are of interest (see [**5**, **16**, **19**]). We refer to [**18**] for more information on recent developments in this field.

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In our paper, we deal with Levi-Civita connections only. In algebraic terms this means that we consider only subgroups of the (pseudo) orthogonal group $\mathrm{O}(g)$:

$$H \subset \mathrm{O}(g) = \{A \in \mathrm{GL}(V) \mid g(Au, Av) = g(u, v),\ u, v \in V\},$$

where g is a non-degenerate bilinear form on V.

The main result of our paper is

Theorem 1. *For every g-symmetric operator $L\colon V \to V$, the identity connected component G_L^0 of its centraliser in $\mathrm{O}(g)$,*

$$G_L = \{X \in \mathrm{O}(g) \mid XL = LX\},$$

is a holonomy group for a certain (pseudo)-Riemannian metric.

Notice that in the Riemannian case this theorem becomes trivial: L is diagonalisable and the connected component G_L^0 of its centraliser is isomorphic to the standard direct product $SO(k_1)\oplus\cdots\oplus\mathrm{SO}(k_m) \subset \mathrm{SO}(n)$, $\sum k_i \leq n$, which is, of course, a holonomy group. In the pseudo-Riemannian case, L may have non-trivial Jordan blocks and the structure of G_L^0 becomes more complicated.

The organization of the paper is as follows. First we recall in Section 2 the classical approach by Berger to studying holonomy groups. Like many other authors, we are going to use this approach in our paper. However, in our opinion, the most interesting part of the present work consists in two explicit matrix formulas (8) and (18) that, in essence, almost immediately lead to the solution. To the best of our knowledge, this kind of formula did not appear in the context of holonomy groups before and we would really appreciate any comments on this matter. They came to "holonomy groups" from "integrable systems on Lie algebras" via "projectively equivalent metrics," and we explain this passage in Section 3. The proof of Theorem 1 is given in Sections 4 (algebraic reduction), 5 (Berger test) and 6 (geometric realization).

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2. Some basic facts about holonomy groups: Ambrose–Singer theorem and Berger test

Let γ be a curve connecting two points $x, y \in M$ (we think of x as a fixed reference point while y is variable) and $P_\gamma\colon T_xM \to T_yM$ denotes the parallel transport transformation. The holonomy groups $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_y(\nabla)$ related to these points are obviously conjugate by means of P_γ, i.e.,

$$\mathrm{Hol}_y(\nabla) = P_\gamma \circ \mathrm{Hol}_x(\nabla) \circ P_\gamma^{-1}.$$

In particular, if M is connected, then the holonomy groups at different points are isomorphic.

Notice that P_γ allows us to "transfer" from x to y (or back from y to x) not only tangent vectors but also tensors of any type. For example, if $R\colon \Lambda^2(T_yM) \to \mathfrak{gl}(T_yM)$ is the curvature tensor of ∇ at the point y, then at the point x we can define the transported tensor $R_\gamma\colon \Lambda^2(T_xM) \to \mathfrak{gl}(T_xM)$ as

$$R_\gamma(u \wedge v) = P_\gamma^{-1} \circ R\bigl(P_\gamma(u) \wedge P_\gamma(v)\bigr) \circ P_\gamma, \quad u, v \in T_xM.$$

The famous Ambrose–Singer theorem [**2**] gives the following description of the Lie algebra $\mathfrak{hol}\,(\nabla)$ of the holonomy group $\mathrm{Hol}\,(\nabla) = \mathrm{Hol}_x\,(\nabla)$ in terms of the curvature tensor R:

$\mathfrak{hol}\,(\nabla)$ *is generated (as a vector space) by the operators of the form* $R_\gamma(u \wedge v)$.

This motivates the following construction.

Definition 2. A map $R\colon \Lambda^2 V \to \mathfrak{gl}(V)$ is called a *formal curvature tensor* if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \quad \text{for all } u, v, w \in V. \tag{1}$$

Definition 3. Let $\mathfrak{h} \subset \mathfrak{gl}(V)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R\colon \Lambda^2 V \to \mathfrak{gl}(V)$ such that $\operatorname{Im} R \subset \mathfrak{h}$:

$$\mathcal{R}(\mathfrak{h}) = \{R\colon \Lambda^2 V \to \mathfrak{h} \mid R(u\wedge v)w + R(v\wedge w)u + R(w\wedge u)v = 0,\ u, v, w \in V\}.$$

We say that $\mathfrak{h}$ is a *Berger algebra* if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, i.e.,

$$\mathfrak{h} = \operatorname{span}\{R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}),\ u, v \in V\}.$$

Berger's test (sometimes referred to as Berger's criterion) is the following result that can, in fact, be viewed as a version of the Ambrose–Singer theorem:

Let ∇ be a symmetric affine connection on TM. Then the Lie algebra $\mathfrak{hol}\,(\nabla)$ *of its holonomy group* $\mathrm{Hol}\,(\nabla)$ *is Berger.*

Usually the solution of the classification problem for holonomy groups consists of two parts. First, one tries to describe all Lie subalgebras $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{R})$ of a certain type satisfying Berger's test (i.e., Berger algebras). This part is purely algebraic. The second (geometric) part is to find a suitable connection ∇ for a given Berger algebra $\mathfrak{h}$ that realizes $\mathfrak{h}$ as the holonomy Lie algebra, i.e., $\mathfrak{h} = \mathfrak{hol}\,(\nabla)$.

We follow the same scheme but use, in addition, some ideas from two other areas of mathematics: projectively equivalent metrics and integrable systems on Lie algebras. These ideas are explained in the next section. The reader who is interested only in the proof itself may proceed directly to Sections 4, 5, and 6, which are formally independent of this preliminary discussion.

3. Projectively equivalent metrics and sectional operators

The problem we are dealing with is closely related to the theory of projectively equivalent (pseudo)-Riemannian metrics [**3**, **7**, **22**, **26**, **32**].

Definition 4. Two metrics g and $\bar{g}$ on a manifold M are called *projectively equivalent* if they have the same geodesics considered as unparametrised curves.

As a particular case of projectively equivalent metrics g and $\bar{g}$, one can distinguish the following, which is closely related to our problem. Assume that g admits a covariantly constant g-symmetric $(1,1)$-tensor field L and introduce a new metric $\bar{g}$ by setting

$$\bar{g}(\xi,\eta) = g(L\xi,\eta).$$

Then the geodesics of g and $\bar{g}$ coincide as *parametrized* curves. In this case g and $\bar{g}$ are called *affinely equivalent*.

In the pseudo-Riemannian case, the classification of pairs (g, L), such that $\nabla L = 0$, is an interesting problem (e.g., [**33**]), which remained open until recently. A paper by C. Boubel [**8**], which has just appeared on the arXiv, seems to contain its complete solution.

The condition $\nabla L = 0$ can be interpreted in terms of the holonomy group $\mathrm{Hol}(\nabla)$. Namely, $\nabla L = 0$ implies that $\mathrm{Hol}(\nabla)$ is a subgroup of the centraliser of L in $\mathrm{O}(g)$. More precisely,

$$\mathrm{Hol}_{x_0}(\nabla) \subset G_{L(x_0)} = \{X \in \mathrm{O}(g) \mid XL(x_0)X^{-1} = L(x_0)\}.$$

Conversely, if $\mathrm{Hol}_{x_0}(\nabla) \subset G_{L(x_0)}$, then we can uniquely reconstruct L at any other point $x \in M$ from the initial value $L(x_0)$ by parallel transport. The independence of the choice of a path γ between x_0 and x is guaranteed by the above inclusion and L so obtained is automatically covariantly constant.

Since we are going to use Berger's approach, the role of the curvature tensor will be very important. Our proof will be based on one unexpected and remarkable relationship between the algebraic structure of the curvature tensor of projectively equivalent metrics and integrable Hamiltonian systems on Lie algebras.

To explain this relationship, we first notice that $\Lambda^2 V$ can be naturally identified with $\mathfrak{so}(g)$. Therefore, in the (pseudo)-Riemannian case, a curvature tensor at a fixed point can be understood as a linear map

$$R\colon \mathfrak{so}(g) \to \mathfrak{so}(g). \tag{2}$$

Some operators of this kind play an important role in the theory of integrable systems on semisimple Lie algebras.

Definition 5. We say that (2) is a *sectional operator* if R is self-adjoint w.r.t. the Killing form and satisfies the algebraic identity:

$$[R(X), L] = [X, M] \quad \text{for all } X \in \mathfrak{so}(n), \tag{3}$$

where L and M are some fixed symmetric matrices.

These operators first appeared in the famous paper by S. Manakov [**27**] on integrability of a multidimensional rigid body and then were studied by A. Mischenko and A. Fomenko in the framework of the argument shift method [**29**]. The terminology "sectional" was suggested by A. Fomenko and V. Trofimov [**15**] for a more general class of operators on Lie algebras with similar properties and originally was in no way related to "sectional curvature." However, such a relation exists and is, in fact, very close.

The following observation, which is, in fact, an algebraic interpretation of the so-called second Sinjukov equation [**32**] for projectively equivalent metrics, was made in [**6**].

Theorem 2. *If g and $\bar g$ are projectively equivalent, then the curvature tensor of g considered as a linear map $R\colon \mathfrak{so}(g) \to \mathfrak{so}(g)$ is a sectional operator, i.e., satisfies identity* (3) *with L defined by $\bar g^{-1} g = \det L \cdot L$ and M being the Hessian of $2\mathrm{tr}\, L$, i.e., $M^i_j = 2\nabla^i \nabla_j \mathrm{tr}\, L$.*

There is an elegant explicit formula expressing $R(X)$ in terms of L and M. To get this formula, one first needs to notice that (3) immediately implies that M belongs to the center of the centraliser of L and, therefore, can be written as $M = p(L)$, where $p(t)$ is a certain polynomial. Then

$$R(X) = \frac{d}{dt}\Big|_{t=0} p(L + tX) \tag{4}$$

satisfies (3). To check this, it is sufficient to differentiate the identity $[p(L+tX), L+tX] = 0$ to get $[\frac{d}{dt}\big|_{t=0} p(L+tX), L] + [p(L), X] = 0$, i.e., $[R(X), L] + [M, X] = 0$ as needed.

In the case of affinely equivalent metrics (we are going to deal with this case only!), L is automatically covariantly constant and, therefore, $M = 0$. Thus, the curvature tensor R satisfies a simpler equation

$$[R(X), L] = 0,$$

which, of course, directly follows from $\nabla L = 0$ and seems to make all the discussion above irrelevant to our particular situation. However, formula (4) still defines a non-trivial operator if $p(t)$ is the minimal polynomial for L so that $p(L) = M = 0$.

The above discussion gives us a very good candidate for the role of a formal curvature tensor in our construction, namely, the operator $R(X)$ defined by (4) with $p(t)$ being the minimal polynomial of L. As we shall see below, this operator satisfies all the required conditions and plays a crucial role in the proof of Theorem 1 given in the next three sections.

4. Step one: Algebraic reduction

Let g be a non-degenerate bilinear form on V and $L\colon V \to V$ be a g-symmetric operator. First of all, we notice that it is sufficient to prove Theorem 1 for two special cases only:

- either L has a single real eigenvalue;
- or L has a pair of complex conjugate eigenvalues.

The reduction from the general case to one of these is standard. If L has several eigenvalues, then V splits into L-invariant and pairwise g-orthogonal subspaces

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s,$$

where V_i is either a generalized eigensubspace corresponding to a real eigenvalue λ_i, or a similar subspace corresponding to a pair of complex conjugate eigenvalues λ_i and $\bar{\lambda}_i$.

The group G_L^0 is compatible with this decomposition in the sense that G_L^0 is the direct product of the Lie groups $G_1, \dots, G_s$, each of which is naturally associated with V_i and is the connected component of the centraliser of $L|_{V_i}$ in $\mathrm{O}(g|_{V_i})$.

Thus G_L^0 is reducible and therefore G_L^0 is a holonomy group if and only if each G_i is a holonomy group. A similar reduction obviously takes place for the corresponding Lie algebras: the Lie algebra $\mathfrak{g}_L$ of G_L^0 splits into the direct sum $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$, and $\mathfrak{g}_L$ is Berger if and only if each $\mathfrak{g}_i$ is Berger, $i = 1, \dots, s$.

Thus, from now on we may assume that the g-symmetric operator L has either a single real eigenvalue or two complex conjugate eigenvalues. Below we concentrate on the case when $\lambda \in \mathbb{R}$ and all the necessary amendments related to the complex situation will be discussed in the appendix.

In the real case, we use the following well-known analog of the Jordan normal form theorem for g-symmetric operators in the case when g is pseudo-Euclidean (see, for example, [**23**, **35**]).

Proposition 1. *Let $L\colon V \to V$ be a g-symmetric operator with a single eigenvalue $\lambda \in \mathbb{R}$. Then by an appropriate choice of a basis in V, we can simultaneously reduce L and g to the following block diagonal matrix form:*

$$(5) \qquad L = \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_k \end{pmatrix}$$

where

$$L_i = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \quad \textit{and} \quad g_i = \pm \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

The blocks L_i *and* g_i *are of the same size* $n_i \times n_i$ *and* $n_1 \le n_2 \le \dots \le n_k$. *As a particular case, we admit* 1×1 *blocks* $L_i = \lambda$ *and* $g_i = \pm 1$.

In what follows, we shall assume that g_i has $+1$ on the antidiagonal. This assumption is not very important but allows us to simplify the formulae below.

The next statement gives an explicit matrix description for $\mathfrak{so}(g)$ and the Lie algebra $\mathfrak{g}_L$ of the group G_L^0 for L and g described in Proposition 1. The proof is straightforward and we omit it.

Proposition 2. *In the canonical basis from Proposition* 1, *the orthogonal Lie algebra* $\mathfrak{so}(g)$ *consists of block matrices of the form*

$$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kk} \end{pmatrix} \tag{6}$$

where X_{ij} *is an* $n_i \times n_j$ *block. The diagonal blocks* X_{ii}*'s are skew-symmetric with respect to their antidiagonal. The off-diagonal blocks* X_{ij} *and* X_{ji} *are related by* $X_{ji} = -g_j X_{ij}^\top g_i$.

The Lie algebra $\mathfrak{g}_L$ *consists of block matrices of the form:*

$$\begin{pmatrix} 0 & M_{12} & \cdots & M_{1k} \\ M_{21} & 0 & & \vdots \\ \vdots & & \ddots & M_{k-1,k} \\ M_{k1} & \cdots & M_{k,k-1} & 0 \end{pmatrix} \ \textit{with } M_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \mu_1 & \mu_2 & \cdots & \mu_{n_i} \\ 0 & \cdots & 0 & 0 & \mu_1 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \mu_2 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \mu_1 \end{pmatrix}, \tag{7}$$

where $i < j$ *and* M_{ij}*'s are* $n_i \times n_j$ *matrices. If* $n_i = n_j$, *then* M_{ij} *is a square matrix and the first zero columns are absent. The blocks* M_{ij} *and* M_{ji} *are related in the same way as* X_{ij} *and* X_{ji}, *i.e.,* $M_{ji} = -g_j M_{ij}^\top g_i$.

The subspace $\mathfrak{m}_{ij} \subset \mathfrak{g}_L$ $(i < j)$ *that consists of two blocks* M_{ij} *and* M_{ji} *is a commutative subalgebra of dimension* n_i. *As a vector space,* $\mathfrak{g}_L$ *is the direct sum* $\sum_{i<j} \mathfrak{m}_{ij}$. *In particular,* $\dim \mathfrak{g}_L = \sum_{i=1}^k (k-i) n_i$.

5. Step two: Berger's test

We consider a non-degenerate bilinear form g on a finite-dimensional real vector space V and a g-symmetric linear operator $L\colon V \to V$, i.e.,

$$g(Lv, u) = g(v, Lu), \quad \text{for all } u, v \in V.$$

As before, we denote the Lie algebra of the orthogonal group associated with g by $\mathfrak{so}(g)$. Recall that this Lie algebra consists of g-skew-symmetric operators:

$$\mathfrak{so}(g) = \{X\colon V \to V \mid g(Xv, u) = -g(v, Xu), \quad u, v \in V\}.$$

Consider the Lie algebra $\mathfrak{g}_L$ of the group G_L^0:

$$\mathfrak{g}_L = \{X \in \mathfrak{so}(g) \mid XL - LX = 0\}.$$

We are going to verify in this section that $\mathfrak{g}_L$ is a Berger algebra.

In what follows, we use the following natural identification of $\Lambda^2 V$ and $\mathfrak{so}(g)$: $v \wedge u = v \otimes g(u) - u \otimes g(v)$. Here the bilinear form g is understood as an isomorphism $g\colon V \to V^*$ between vectors and covectors. Taking into account this identification, we define the linear mapping $R\colon \mathfrak{so}(g) \simeq \Lambda^2 V \to \mathfrak{gl}(V)$ by:

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX), \tag{8}$$

where $p_{\min}(t)$ is the minimal polynomial of L.

Proposition 3. *Let $L\colon V \to V$ be a g-symmetric operator. Then (8) defines a formal curvature tensor $R\colon \Lambda^2 V \simeq \mathfrak{so}(g) \to \mathfrak{g}_L$ for the Lie algebra $\mathfrak{g}_L$. In other words, R satisfies the Bianchi identity and its image is contained in $\mathfrak{g}_L$.*

The proof consists of the following two lemmas.

Lemma 1. *The image of R is contained in $\mathfrak{g}_L$.*

Proof. First we check that $R(X) \in \mathfrak{so}(g)$, i.e., $R(X)^* = -R(X)$, where $*$ denotes "g–adjoint":

$$g(A^* u, v) = g(u, Av), \quad u, v \in V.$$

Since $L^* = L$, $X^* = -X$, $(p_{\min}(L + tX))^* = p_{\min}(L^* + tX^*)$ and "$\frac{d}{dt}$" and "$*$" commute, we have

$$\begin{aligned} R(X)^* &= \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX)^* = \frac{d}{dt}\Big|_{t=0} p_{\min}(L^* + tX^*) \\ &= \frac{d}{dt}\Big|_{t=0} p_{\min}(L - tX) = -\frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX) = -R(X), \end{aligned}$$

as needed. Thus, $R(X) \in \mathfrak{so}(g)$. Notice that this fact holds true for any polynomial $p(t)$, not necessarily minimal.

To prove that $R(X)$ commutes with L, we consider the obvious identity $[p_{\min}(L + tX), L + tX] = 0$ and differentiate it at $t = 0$:

$$[\frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX), L] + [p_{\min}(L), X] = 0.$$

Clearly, $p_{\min}(L) = 0$ as it is a minimal polynomial, whence $[R(X), L] = 0$, as required. Thus, $R(X) \in \mathfrak{g}_L$. q.e.d.

Lemma 2. *R satisfies the Bianchi identity* (1).

Proof. It is easy to see that our operator $R\colon \Lambda^2 V \simeq \mathfrak{so}(g) \to \mathfrak{gl}(V)$ can be written as $R(X) = \sum_k C_k X D_k$, where C_k and D_k are some g-symmetric operators (in our case these operators are some powers of L). Thus, it is sufficient to check the Bianchi identity for operators of the form $X \mapsto CXD$. For $X = u \wedge v$ we have

$$C(u \wedge v)Dw = Cu \cdot g(v, Dw) - Cv \cdot g(u, Dw)$$

If we cyclically permute u, v, and w and sum up the expressions so obtained, taking into account that both C and D are g-symmetric, we obtain zero, as required. q.e.d.

To prove that $\mathfrak{g}_L$ is a Berger algebra, it remains to compute the image of (8) and compare it with $\mathfrak{g}_L$. We are going to do it by means of matrix linear algebra and, from now on, we consider the reduced case with a single real eigenvalue $\lambda \in \mathbb{R}$ described in Proposition 1. Replacing L by $L - \lambda{\cdot}\mathrm{Id}$, we can assume without loss of generality that $\lambda = 0$, i.e., L is nilpotent.

Proposition 2 implies that in the case of a single Jordan block the algebra $\mathfrak{g}_L$ is trivial and thus we begin with the first non-trivial case when L consists of two Jordan blocks L_1 and L_2.

Proposition 4. *Let $L\colon V \to V$ be a g-symmetric nilpotent operator that consists of two Jordan blocks. Then the image of the formal curvature tensor $R\colon \Lambda^2 V \simeq \mathfrak{so}(g) \to \mathfrak{g}_L$ defined by* (8) *coincides with $\mathfrak{g}_L$. In particular, $\mathfrak{g}_L$ is Berger.*

Proof. Consider $L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, where L_1 and L_2 are standard nilpotent Jordan blocks of size m and n, respectively, $m \le n$, as in Proposition 1. The minimal polynomial for L is $p_{\min}(t) = t^n$ so that $R(X) = \frac{d}{dt}\big|_{t=0}(L + tX)^n = L^{n-1}X + L^{n-2}XL + \cdots + XL^{n-1}$.

If we represent $X \in \mathfrak{so}(g)$ as a block matrix (see Proposition 2), then we immediately see that R acts independently of each block, i.e.,

$$R(X) = R\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} R_{11}(X_{11}) & R_{12}(X_{12}) \\ R_{21}(X_{21}) & R_{22}(X_{22}) \end{pmatrix} = \begin{pmatrix} 0 & M_{12} \\ M_{21} & 0 \end{pmatrix}. \tag{9}$$

where $M_{12} = R_{12}(X_{12}) = L_1^{n-1}X_{12} + L_1^{n-2}X_{12}L_2 + \cdots + X_{12}L_2^{n-1}$.

Since $R(X) \in \mathfrak{g}_L$, the matrices M_{12} and M_{21} have the form described in Proposition 2 and a straightforward computation shows that the entries μ_α of M_{12} (see (7)), $\alpha = 1, \dots, m$, are related to the entries of

$$X_{12} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{k2} & \dots & x_{mn} \end{pmatrix}$$

as $\mu_\alpha = \sum_{j=1}^{\alpha} x_{m-\alpha+j,j}$.

Clearly, there are no relations between μ_α's and therefore the image of R coincides with $\mathfrak{m}_{12} = \mathfrak{g}_L$, which completes the proof. q.e.d.

Thus, formula (8) solves the problem in the case of two blocks. Now, let us consider the case of k Jordan blocks, $k > 2$. In this case, the image of the formal curvature tensor defined by (8) can be smaller than $\mathfrak{g}_L$ and formula (8) needs to be modified.

We start with the following obvious remark. Let $V' \subset V$ be a subspace of V such that $g' = g|_{V'}$ is non-degenerate. Consider the standard embedding $\mathfrak{so}(g') \to \mathfrak{so}(g)$ induced by the inclusion $V' \subset V$. If $R' \colon \mathfrak{so}(g') \to \mathfrak{so}(g')$ is a formal curvature tensor, then its trivial extension $R \colon \mathfrak{so}(g) \to \mathfrak{so}(g)$ defined by

$$R \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} R'(X) & 0 \\ 0 & 0 \end{pmatrix}$$

is a formal curvature tensor too. In particular, if $\mathfrak{h} \subset \mathfrak{so}(g')$ is a Berger subalgebra, then $\mathfrak{h}$ as a subalgebra of $\mathfrak{so}(g)$ will be also Berger.

This remark allows us to construct a "big" formal curvature tensor as the sum of "small" curvature tensors related to different pairs of Jordan blocks and in this way to reduce the general case to the situation treated in Proposition 4.

Consider the operator $\widehat{R}_{12} \colon \mathfrak{so}(g) \to \mathfrak{so}(g)$ defined by:

$$\widehat{R}_{12} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kk} \end{pmatrix} = \begin{pmatrix} 0 & R_{12}(X_{12}) & \cdots & 0 \\ R_{21}(X_{21}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{10}$$

where $R_{12}(X_{12})$ and $R_{21}(X_{21})$ are defined as in Proposition 4 (see (9)) and all the other blocks in the right-hand side vanish. Then applying the above remark to the subspace $V' \subset V$ related to the first two blocks L_1 and L_2, we see that $\widehat{R}_{12}$ is a formal curvature tensor and its image coincides with the Abelian subalgebra $\mathfrak{m}_{12} \subset \mathfrak{g}_L$ (see Proposition 2). In particular, $\mathfrak{m}_{12} \subset \mathfrak{so}(g)$ is a Berger algebra.

To construct the "big" formal curvature operator $R \colon \mathfrak{so}(g) \to \mathfrak{g}_L$ we simply do the same for each pair of blocks, namely we set:

$$R \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kk} \end{pmatrix} = \begin{pmatrix} 0 & R_{12}(X_{12}) & \cdots & R_{1k}(X_{1k}) \\ R_{21}(X_{21}) & 0 & \cdots & R_{2k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ R_{k1}(X_{k1}) & R_{k2}(X_{k2}) & \cdots & 0 \end{pmatrix} \tag{11}$$

In other words, R acts independently on each block X_{ij} (compare with the proof of Proposition 4) and each of its components

$$R_{ij} \colon X_{ij} \mapsto R_{ij}(X_{ij})$$

is defined in the same way as in Proposition 4 provided we ignore all the blocks of L except for L_i and L_j. In other words, instead of the minimal polynomial of L, we use that of $\begin{pmatrix} L_i & 0 \\ 0 & L_j \end{pmatrix}$; that is, $p(t) = t^{n_{ij}}$ with $n_{ij} = \max\{n_i, n_j\}$, where n_i, n_j are the sizes of the nilpotent Jordan blocks L_i and L_j.

If we introduce the operators $\widehat{R}_{ij} \colon \mathfrak{so}(g) \to \mathfrak{so}(g)$ by generalizing (10) for arbitrary indices $i < j$, we can rewrite (11) as

$$R_{\text{formal}} = R = \sum_{i<j} \widehat{R}_{ij}. \tag{12}$$

The following statement completes Berger's test for $\mathfrak{g}_L$.

Proposition 5. *Let L be an arbitrary nilpotent g-symmetric operator. Then the operator $R_{\text{formal}} = R$ defined by* (11) (*or equivalently by* (12)) *is a formal curvature tensor. Moreover,* $\operatorname{Im} R = \mathfrak{g}_L$ *and, therefore, $\mathfrak{g}_L$ is a Berger algebra.*

Proof. Since each $\widehat{R}_{ij}$ is a formal curvature tensor, so is R by linearity. By Proposition 4, the image of $\widehat{R}_{ij}$ is the subalgebra $\mathfrak{m}_{ij}$. Since the operators $\widehat{R}_{ij}$ do not interact among themselves, we immediately obtain $\operatorname{Im} R = \sum_{i<j} \operatorname{Im} \widehat{R}_{ij} = \sum_{i<j} \mathfrak{m}_{ij} = \mathfrak{g}_L$, as required. q.e.d.

This proposition implies that $\mathfrak{g}_L$ is Berger whenever L has a single real eigenvalue $\lambda \in \mathbb{R}$. In the case of a pair of complex eigenvalues λ and $\bar{\lambda}$, the proof just needs the few additional comments given in the appendix. Taking into account the reduction in Section 4, we come to the following final conclusion.

Theorem 3. *Let $L \colon V \to V$ be a g-symmetric operator. Then $\mathfrak{g}_L = \{X \in \mathfrak{so}(g) \mid XL = LX\}$ is a Berger algebra.*

6. Step three: Geometric realization

Now, for a given operator $L \colon T_{x_0}M \to T_{x_0}M$, we need to find a pseudo-Riemannian metric g on a small neighbourhood $U(x_0) \subset M$ and a $(1,1)$-tensor field $L(x)$ (with the initial condition $L(x_0) = L$) such that

1) $\nabla L(x) = 0$;
2) $\mathfrak{hol}\,(\nabla) = \mathfrak{g}_L$.

Notice that the first condition guarantees that $\mathfrak{hol}\,(\nabla) \subset \mathfrak{g}_L$. On the other hand, $\operatorname{Im} R(x_0) \subset \mathfrak{hol}\,(\nabla)$, where $x_0 \in M$ is a fixed point and R is the curvature tensor of g. Thus, taking into account Theorem 3, the second condition can be replaced by

2′) $R(x_0)$ coincides with the formal curvature tensor R_{formal} from Proposition 5.

Thus, our goal in this section is to construct (at least one example of) $L(x)$ and $g(x)$ satisfying conditions 1 and $2'$. Apart from formula (8) (whose modification (18) leads to the desired example), the construction below is based on two well-known geometric facts. The first one is

Proposition 6. *For every metric g there exists a local coordinate system such that $\frac{\partial g_{ij}}{\partial x^\alpha}(0) = 0$ for all i, j, α. In particular, in this coordinate system, $\Gamma^k_{ij}(0) = 0$ and the components of the curvature tensor at $x_0 = 0$ are defined as some combinations of second derivatives of g.*

The second result states that covariantly constant $(1,1)$-tensor fields L are actually very simple. To the best of our knowledge, this theorem was first proved by A. P. Shirokov [**31**] (see also [**9**, **24**, **34**]).

Theorem 4. *If L satisfies $\nabla L = 0$ for a symmetric connection ∇, then there exists a local coordinate system $x^1, \dots, x^n$ in which L is constant.*

In this coordinate system the equation $\nabla L = 0$, for a g-symmetric L, can be rewritten in a very simple way:

$$\left(\frac{\partial g_{ip}}{\partial x^\beta} - \frac{\partial g_{i\beta}}{\partial x^p}\right) L^\beta_k = \left(\frac{\partial g_{i\beta}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\beta}\right) L^\beta_p \,. \tag{13}$$

This equation is linear and if we expand g in a power series of x, then (13) must hold for each term of this expansion. Moreover, if we consider the constant and second order terms only, then they will give us a particular (local) solution.

This suggests the idea to set $L(x) = \text{const}$ and then try to find the desired metric $g(x)$ in the form "constant" + "quadratic", i.e.,

$$g_{ij}(x) = g^0_{ij} + \sum \mathcal{B}_{ij,pq} x^p x^q, \tag{14}$$

where $\mathcal{B}$ satisfies obvious symmetry relations, namely, $\mathcal{B}_{ij,pq} = \mathcal{B}_{ji,pq}$ and $\mathcal{B}_{ij,pq} = \mathcal{B}_{ij,qp}$.

Before discussing the explicit formula for $\mathcal{B}$, we give some general remarks about the metrics (14).

- The condition $\nabla L = 0$ amounts to the following equation for $\mathcal{B}$:

$$(\mathcal{B}_{ip,\beta q} - \mathcal{B}_{i\beta,pq}) L^\beta_k = (\mathcal{B}_{\beta i,kq} - \mathcal{B}_{ik,\beta q}) L^\beta_p \,. \tag{15}$$

- The condition that L is g-symmetric reads:

$$\mathcal{B}_{ij,pq} L^i_l = \mathcal{B}_{il,pq} L^i_j \,. \tag{16}$$

- The curvature tensor of g at the origin $x = 0$ takes the following form:

$$R^i_{k\,\alpha\beta} = g^{is}(\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k}). \tag{17}$$

Thus, the realization problem admits the following purely algebraic version: find $\mathcal{B}$ satisfying (15), (16) and such that (17) coincides with $R_{\rm formal}$ from Proposition 5.

We will be looking for $\mathcal{B}$ in the form $\mathcal{B}=\sum_\alpha \mathcal{C}_\alpha\otimes\mathcal{D}_\alpha$, i.e., $\mathcal{B}_{ij,pq}=\sum_\alpha(\mathcal{C}_\alpha)_{ij}\cdot(D_\alpha)_{pq}$, where $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$ are symmetric bilinear forms. Along with $\mathcal{B}$, we consider the $(2,2)$-tensor $B=\sum_\alpha C_\alpha\otimes D_\alpha$, where C_α and D_α are g^0-symmetric operators associated with the forms $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$. In other words, $\mathcal{B}_{ij,pq}=g^0_{is}g^0_{pt}B^{s,t}_{j,q}$. It is convenient to think of B as a linear map

$$B\colon \mathfrak{gl}(V)\to\mathfrak{gl}(V)\quad\text{defined by } B(X)=\sum C_\alpha X D_\alpha\,.$$

In terms of this map, the conditions (15), (16), (17) for the corresponding metric $g=g^0+\mathcal{B}(x,x)$ can be rewritten as

$$[B(X),L]+[B(X),L]^*=0\quad\text{for any } X\in\mathfrak{gl}(V), \tag{15$'$}$$

$$[C_\alpha,L]=0, \tag{16$'$}$$

$$R(X)=-B(X)+B(X)^*,\quad X\in\mathfrak{so}(g^0). \tag{17$'$}$$

As the reader may notice, we prefer to work with operators rather than forms. The reason is easy to explain: operators form an associative algebra and we use this property throughout the paper.

The latter formula (17$'$), in fact, shows how to reconstruct B from $R(X)$: we need to "replace" X by $\otimes$, i.e., $B=-\frac{1}{2}R(\otimes)$. Namely, we consider the following formal expression:

$$B=-\frac{1}{2}\cdot\frac{d}{dt}\Big|_{t=0}p_{\min}(L+t\cdot\otimes), \tag{18}$$

where $p_{\min}(t)$ is the minimal polynomial of L. This formula, obtained from the right-hand side of (8), looks a bit strange but, in fact, defines a tensor B of type $(2,2)$ whose meaning is very simple. If the minimal polynomial of L is $p_{\min}(t)=\sum_{m=0}^n a_m t^m$, then

$$B=-\frac{1}{2}\cdot\sum_{m=0}^n a_m\sum_{j=0}^{m-1}L^{m-1-j}\otimes L^j. \tag{19}$$

Proposition 7. *Assume that L is a g^0-symmetric operator and consider it as $(1,1)$-tensor field whose components are all constant in coordinates x. Define the quadratic metric $g(x)=g^0+\mathcal{B}(x,x)$ with $\mathcal{B}_{ij,pq}=g^0_{is}g^0_{pt}B^{s,t}_{j,q}$, where B is constructed from L by* (18) (*or, equivalently, by* (19)). *Then*

1) *L is g-symmetric;*

2) *$\nabla L=0$, where ∇ is the Levi-Civita connection for g;*

3) *the curvature tensor for* g *at the origin is defined by* (8), *i.e.*,

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX) \quad \textit{for } X \in \mathfrak{so}(g^0).$$

Proof. Since B is of the form $\sum_\alpha C_\alpha \otimes D_\alpha$, where C_α and D_α are some powers of L, we can use formulas (15′), (16′), and (17′).

Item (1) is equivalent to (16′) and hence is obvious.

Next, to check (2) it suffices, according to (15′), to show that

$$[B(X), L] = 0, \quad \text{where } B(X) = -\frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t \cdot X),$$

which has been already done in Lemma 1.

Finally, we compute the curvature tensor R at the origin by using (17′). Namely, for $X \in \mathfrak{so}(g^0)$ we have:

$$R(X) = -B(X) + B(X)^* = -2B(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX),$$

as stated. Here we use Proposition 3 which says, in particular, that $B(X)$ belongs to $\mathfrak{g}_L \subset \mathfrak{so}(g^0)$, i.e., $B(X) = -B(X)^*$. q.e.d.

This proposition together with Proposition 4 solves the realization problem in the most important "two Jordan blocks" case. To get the realization in the general case, we proceed in the same way as we did for the algebraic part. Namely, we split L into Jordan blocks and for each pair of Jordan blocks L_i, L_j define a formal curvature tensor $\widehat{R}_{ij}$ (see Section 5 for details). Then by using (18) we can realize this formal curvature tensor by an appropriate quadratic metric $g(x) = g^0 + \widehat{B}_{ij}(x, x)$ satisfying $\nabla L = 0$. We omit the details because this construction is straightforward and just repeats its algebraic counterpart discussed in Section 5. Now, if we set

$$g(x) = g^0 + \mathcal{B}(x, x), \quad \text{with } B = \sum_{i<j} \widehat{B}_{ij},$$

then, by linearity, this metric still satisfies $\nabla L = 0$ and its curvature tensor coincides with $R_{\text{formal}} = \sum_{i<j} \widehat{R}_{ij}$ from Proposition 4. This completes the realization part of the proof of Theorem 1.

7. Appendix: The case of a pair of complex conjugate eigenvalues

Let $L \colon V \to V$ be a g-symmetric operator with two complex conjugate eigenvalues λ and $\bar{\lambda}$. In this case an analog of Proposition 1 can be formulated in complex terms.

The point is that on the vector space V there is a canonical complex structure J that can be uniquely defined by the following condition: the i and $-i$ eigenspaces of J in $V^{\mathbb{C}}$ coincide with λ and $\bar{\lambda}$ generalized eigenspaces of L, respectively.

The complex structure J obviously commutes with L and is g-symmetric. This immediately implies that if we consider V as a complex vector space with respect to J, then $L\colon V \to V$ is a complex operator and g can be considered as the imaginary part of the following complex bilinear form $g^{\mathbb{C}}\colon V \times V \to \mathbb{C}$:

$$g^{\mathbb{C}}(u,v) = g(Ju,v) + ig(u,v).$$

It is easy to see that L is still g-symmetric with respect to $g^{\mathbb{C}}$.

Thus, instead of looking for a real canonical form for L and g, it is much more convenient to use a complex canonical form for L and $g^{\mathbb{C}}$. As a complex operator, L has a single eigenvalue λ and therefore we are led to the situation described in Proposition 1. Replacing $\mathbb{R}$ by $\mathbb{C}$ does not change the conclusion: there exists a complex coordinate system such that L and $g^{\mathbb{C}}$ are given by exactly the same matrices as L and g are in Proposition 1.

In this canonical complex coordinate system, the statement of Proposition 2 remains unchanged if we replace the real Lie algebra $\mathfrak{so}(g)$ by the complex Lie algebra $\mathfrak{so}\big(g^{\mathbb{C}}\big)$ (the entries of all matrices in (6) and (7) are now, of course, complex). These two Lie algebras are different, but we have the obvious inclusion $\mathfrak{so}\big(g^{\mathbb{C}}\big) \subset \mathfrak{so}(g)$. It is also important that $\mathfrak{g}_L$ turns out to be a complex Lie algebra, i.e., $\mathfrak{g}_L \subset \mathfrak{so}\big(g^{\mathbb{C}}\big)$.

To show that $\mathfrak{g}_L$ is still Berger in this case, we first need to verify the conclusion of Propositions 4; i.e., to check that the image of the operator (8) coincides with $\mathfrak{g}_L$.

Proposition 4 is purely algebraic, so it remains true for a complex operator L and a complex bilinear form $\mathfrak{g}^{\mathbb{C}}$, if we define $R\colon \mathfrak{so}(g^{\mathbb{C}}) \to \mathfrak{g}_L$ by (8) with $p_{\min}(t) = (t-\lambda)^n$.

But we have two new issues. First of all, R should be defined on a larger Lie algebra, namely on $\mathfrak{so}(g)$. Second, instead of $(t-\lambda)^n$ we should consider the real minimal polynomial $p_{\min}(t) = (t-\lambda)^n(t-\bar\lambda)^n$ (otherwise, R won't be real!).

The first issue is not much trouble at all: we can restrict R on the subalgebra $\mathfrak{so}(g^{\mathbb{C}}) \subset \mathfrak{so}(g)$ and if the image still coincides with $\mathfrak{g}_L$, then the same will be true for the original operator (we use the fact that the image of R belongs to $\mathfrak{g}_L$ automatically, Lemma 1).

To sort out the second problem, we simply compute R for $p_{\min}(t) = (t-\lambda)^n(t-\bar\lambda)^n$, thinking of L and $X \in \mathfrak{so}(g^{\mathbb{C}})$ as complex operators and using the fact that $(L-\lambda)^n = 0$:

$$\begin{aligned}
R(X) &= \frac{d}{dt}\Big|_{t=0}\Big((L-\lambda+tX)^n \cdot (L-\bar\lambda+tX)^n\Big) \\
&= \left(\frac{d}{dt}\Big|_{t=0}(L-\lambda+tX)^k\right)\cdot(L-\bar\lambda)^n + (L-\lambda)^n \cdot \frac{d}{dt}\Big|_{t=0}(L-\bar\lambda+tX)^n \\
&= \left(\frac{d}{dt}\Big|_{t=0}(L-\lambda+tX)^n\right)\cdot(L-\bar\lambda)^n.
\end{aligned}$$

The operator in the first bracket is the same as in Proposition 4. In particular, its image coincides with $\mathfrak{g}_L$, as needed. After this we multiply the result by the non-degenerate matrix $(L-\bar{\lambda})^k$. This operation cannot change the dimension of the image, and since we know that $\operatorname{Im} R$ is contained in $\mathfrak{g}_L$ automatically (Lemma 1), we conclude that $\operatorname{Im} R = \mathfrak{g}_L$.

The proof of Proposition 5 does not use any specific property of the "small" operators $\widehat{R}_{ij}$. We only need the image of $\widehat{R}_{ij}$ to coincide with the subalgebra $\mathfrak{m}_{ij} \subset \mathfrak{g}_L$. But this is exactly the statement of Proposition 4, which still holds true in the case of two complex blocks.

Thus, if L has two complex conjugate eigenvalues λ and $\bar{\lambda}$, the Lie algebra $\mathfrak{g}_L$ is still Berger.

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