HALF-SPACE THEOREMS FOR MINIMAL SURFACES IN Nil₃ AND Sol₃

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Abstract

We prove some half-space theorems for minimal surfaces in the Heisenberg group Nil₃ and the Lie group Sol₃ endowed with their standard left-invariant Riemannian metrics. If $\mathcal{S}$ is a properly immersed minimal surface in Nil₃ that lies on one side of some entire minimal graph $\mathcal{G}$, then $\mathcal{S}$ is the image of $\mathcal{G}$ by a vertical translation. If $\mathcal{S}$ is a properly immersed minimal surface in Sol₃ that lies on one side of a special plane $\mathcal{E}^t$ (see the discussion just before Theorem 1.5 for the definition of a special plane in Sol₃), then $\mathcal{S}$ is the special plane $\mathcal{E}^u$ for some $u \in \mathbb{R}$.

1. Introduction

A classical theorem in the global theory of proper minimal surfaces in Euclidean 3-space is the *half-space theorem* by Hoffman and Meeks [**8**]: if $\mathcal{S}$ is a properly immersed minimal surface in $\mathbb{R}^3$ that lies on one side of some plane $\mathcal{P}$, then $\mathcal{S}$ is a plane parallel to $\mathcal{P}$. The proof uses the maximum principle and the fact that catenoids converge to a double cover of a punctured plane as the necksize goes to zero. As a consequence, they proved the *strong half-space theorem*: two properly immersed minimal surfaces in $\mathbb{R}^3$ that do not intersect must be parallel planes.

These theorems have been generalized to some other ambient simply connected homogeneous manifolds. Let us first observe that there is no half-space theorem in Euclidean spaces of dimensions $n \geqslant 4$, since there exist rotational proper minimal hypersurfaces contained in a slab.

In hyperbolic 3-space $\mathbb{H}^3$, one does not have a half-space theorem for minimal surfaces (indeed, for instance any smooth closed curve in the asymptotic boundary of $\mathbb{H}^3$ bounds a minimal surface), but one has half-space theorems for constant mean curvature (CMC) 1 surfaces [**13**], which can be obtained using rotational catenoid cousins (see also [**2**]). One of the reasons that halfspace theorems exist for CMC 1 surfaces in $\mathbb{H}^3$ is that the "critical" value for mean curvature in $\mathbb{H}^3$ is 1, i.e., there exist compact CMC H surfaces in $\mathbb{H}^3$ if and only if $|H| > 1$.

Similarly, there is no half-space theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, since catenoids (i.e., rotational minimal surfaces) are contained in a slab [**11**, **12**]. On the other hand, Hauswirth, Rosenberg, and Spruck proved a half-space theorem for CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 1.1 ([**7**]). *Let $\mathcal{S}$ be a properly immersed CMC $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$.*

- *If $\mathcal{S}$ is contained on the mean convex side of a horocylinder $\mathcal{C}$, then $\mathcal{S}$ is a horocylinder parallel to $\mathcal{C}$.*
- *If $\mathcal{S}$ is embedded and contains a horocylinder $\mathcal{C}$ on its mean convex side, then $\mathcal{S}$ is a horocylinder parallel to $\mathcal{C}$.*

(A horocylinder is a product $\gamma \times \mathbb{R}$ where $\gamma \subset \mathbb{H}^2$ is a horocycle.)

Since rotational CMC $\frac{1}{2}$ surfaces are not suitable to obtain this theorem, their proof uses a continuous family of compact annuli bounded by two circles in parallel horocylinders, one circle being fixed and the other one having a radius going to infinity. To do this, they use the Schauder fixed point theorem and elliptic PDE techniques.

The aim of this paper is to prove, using geometric arguments, some half-space theorems for minimal surfaces in two simply connected homogenous 3-manifolds, the Heisenberg group Nil$_3$ and the Lie group Sol$_3$, which are two manifolds admitting isometries with remarkable properties.

The 3-dimensional Heisenberg group Nil$_3$ admits a Riemannian submersion $\pi \colon$ Nil$_3 \to \mathbb{R}^2$. Translations along the fibers are isometries called a *vertical translations*.

The inverse image by π of a straight line in $\mathbb{R}^2$ is a minimal surface called *vertical plane*; two vertical planes are said to be *parallel* if their images by π are parallel straight lines. These vertical planes are minimal, stable, and isometric to $\mathbb{R}^2$ but not totally geodesic (in fact, there are no local totally geodesic surfaces in Nil$_3$).

Other examples of stable minimal surfaces in Nil$_3$ are *entire minimal graphs*, i.e., minimal surfaces $\mathcal{G}$ such that $\pi_{|\mathcal{G}} \colon \mathcal{G} \to \mathbb{R}^2$ is a diffeomorphism. There exist many entire minimal graphs, and they were classified by Fernandez and Mira [**6**]. Some examples (in the usual coordinates (x_1, x_2, x_3) described in Section 3.1) are given by $x_3 = 0$ (which is rotational) and $x_3 = \frac{x_1 x_2}{2}$ (which is invariant by a one-parameter family of translations). In this space, there exist entire minimal graphs of parabolic conformal type and of hyperbolic conformal type.

Using rotational catenoids, Abresch and Rosenberg obtained the following result.

Theorem 1.2 ([**1**]). *Let $\mathcal{S}$ be a properly immersed minimal surface in Nil$_3$. If $\mathcal{S}$ lies on one side of the surface of equation $x_3 = 0$, then $\mathcal{S}$ is the surface of equation $x_3 = c$ for some $c \in \mathbb{R}$.*

Let us observe that one has the analogous statement for the surfaces of equation $x_3 = ax_1 + bx_2 + c$ for any a, b, c since all these surfaces are congruent.

Daniel and Hauswirth proved a *vertical half-space theorem* for minimal surfaces in Nil$_3$.

Theorem 1.3 ([**4**]). *Let $\mathcal{S}$ be a properly immersed minimal surface in* Nil$_3$*. If $\mathcal{S}$ lies on one side of some vertical plane $\mathcal{P}$, then $\mathcal{S}$ is a vertical plane parallel to $\mathcal{P}$.*

To do this, they constructed a one-parameter family of *horizontal catenoids*, which are non-rotational properly embedded minimal annuli that converge to a double cover of a punctured vertical plane as the necksize goes to zero (they are semi-explicit and obtained by integrating a Weierstrass-type representation).

In this paper, we will give another proof of this theorem (requiring less computations) and we will prove the following result.

Theorem 1.4. *Let $\mathcal{S}$ be a properly immersed minimal surface in* Nil$_3$*. If $\mathcal{S}$ lies on one side of some entire minimal graph $\mathcal{G}$, then $\mathcal{S}$ is the image of $\mathcal{G}$ by a vertical translation.*

Theorem 1.2 is the particular case of Theorem 1.4 when $\mathcal{G}$ is the surface of equation $x_3 = 0$. It is natural to conjecture that two properly immersed minimal surfaces in Nil$_3$ that do not intersect are either two parallel vertical planes or an entire minimal graph and its image by a vertical translation (this would be the analogue of the strong half-space theorem of $\mathbb{R}^3$).

The Lie group Sol$_3$ admits a Riemannian submersion $\psi\colon \mathrm{Sol}_3 \to \mathbb{R}$ such that, for any $s \in \mathbb{R}$, the surface

$$\mathcal{E}^s := \psi^{-1}(s)$$

is minimal, stable, and isometric to $\mathbb{R}^2$ but not totally geodesic. We will call these surfaces *special planes*. Other remarkable minimal foliations in Sol$_3$ are the foliations $(\mathcal{H}_j^t)_{t\in\mathbb{R}}$ ($j = 1, 2$) where $\mathcal{H}_j^t$ is defined by $x_j = t$ in the usual coordinates (x_1, x_2, s) defined in Section 2.1. The leaves are totally geodesic, stable, and have intrinsic curvature -1 (in fact, they are the only totally geodesic surfaces in Sol$_3$). They are symmetry planes in Sol$_3$, which permits Alexandrov reflection. Two surfaces $\mathcal{H}_j^t$ and $\mathcal{H}_k^u$ are congruent.

In this paper, we will prove the following theorem.

Theorem 1.5. *Let $\mathcal{S}$ be a properly immersed minimal surface in* Sol$_3$*. If $\mathcal{S}$ lies on one side of some special plane $\mathcal{E}^t$ ($t \in \mathbb{R}$), then $\mathcal{S}$ is the special plane $\mathcal{E}^u$ for some $u \in \mathbb{R}$.*

Remark 1.6. There is no half-space theorem for the surfaces $\mathcal{H}_1^t$ and $\mathcal{H}_2^t$; indeed the equation $x_1 = ae^{-s}$ for $a \neq 0$ defines a properly embedded minimal surface lying on one side of $\mathcal{H}_1^0$.

We now outline the proofs of the theorems. We assume that there exist a properly immersed minimal surface $\mathcal{S}$ lying on one side of the surface Σ with respect to which we want to prove the half-space theorem (i.e., Σ is an entire minimal graph in Nil_3, a vertical plane in Nil_3, or a special plane in Sol_3). We assume that $\mathcal{S}$ and Σ are not congruent. We consider the image Σ^ε of Σ by a "small" translation, a fixed circle ("small") in Σ^ε and a circle of varying radius ("large") in Σ. We construct a least area annulus bounded by the two circles (using the Douglas criterion) and prove using curvature estimates that a subsequence of these annuli converges to a properly embedded annulus bounded by the small circle as the radius of the large circle goes to infinity.

Then we prove that the distance between the limit annulus and the surface Σ is positive. To do this, we distinguish two cases.

1) When Σ is a vertical plane in Nil_3 or a special plane in Sol_3 (Theorems 1.3 and 1.5), we prove, using curvature estimates, that the limit annulus is quasi-isometric to Σ and hence parabolic, and we find a suitable bounded subharmonic function on the limit annulus.

2) When Σ is an entire minimal graph in Nil_3 (Theorem 1.4), we cannot use such an argument since some entire minimal graphs are hyperbolic. Instead, we use a nodal domain argument to prove that the limit annulus is a graph, and then we use a generalization by Leandro and Rosenberg [9] of a theorem by Collin and Krust [3] about graphs with prescribed mean curvature. This theorem states that if u and v are two solutions to the same prescribed mean curvature graph equation over a domain $\Omega \subset \mathbb{R}^2$ that coincide on $\partial\Omega$, then either $u - v$ is unbounded or $u \equiv v$ on Ω.

This implies that some annulus bounded by two circles must intersect $\mathcal{S}$, and we conclude by translating this annulus until reaching an interior last point of contact, contradicting the maximum principle.

The paper is organized as follows. Section 2 is devoted to preliminaries about Sol_3 and to the proof of Theorem 1.5. Section 3 contains preliminaries about Nil_3 and the proofs of Theorems 1.3 and 1.4.

2. A half-space theorem in Sol$_3$

2.1. The Lie group Sol$_3$. The Lie group Sol$_3$ can be viewed as $\mathbb{R}^3$ endowed with the Riemannian metric

$$e^{2s}\mathrm{d}x_1^2 + e^{-2s}\mathrm{d}x_2^2 + \mathrm{d}s^2$$

where (x_1, x_2, s) denote the canonical coordinates of $\mathbb{R}^3$.

In these coordinates, the Riemannian submersion is given by

$$\psi \colon (x_1, x_2, s) \mapsto s,$$

and so the special plane $\mathcal{E}^t$ is simply defined by the equation $s = t$.

We consider the left-invariant orthonormal frame (E_1, E_2, E_3) defined by

$$E_1 = e^{-s}\frac{\partial}{\partial x_1}, \quad E_2 = e^s\frac{\partial}{\partial x_2}, \quad E_3 = \frac{\partial}{\partial s}.$$

We call it the *canonical frame*. The expression of the Riemannian connection $\widehat{\nabla}$ of Sol$_3$ in this frame is the following:

$$(1) \quad \begin{array}{lll} \widehat{\nabla}_{E_1}E_1 = -E_3, & \widehat{\nabla}_{E_2}E_1 = 0, & \widehat{\nabla}_{E_3}E_1 = 0, \\ \widehat{\nabla}_{E_1}E_2 = 0, & \widehat{\nabla}_{E_2}E_2 = E_3, & \widehat{\nabla}_{E_3}E_2 = 0, \\ \widehat{\nabla}_{E_1}E_3 = E_1, & \widehat{\nabla}_{E_2}E_3 = -E_2, & \widehat{\nabla}_{E_3}E_3 = 0. \end{array}$$

The isometry group of Sol$_3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$(x_1, x_2, s) \mapsto (x_1 + c, x_2, s), \quad (x_1, x_2, s) \mapsto (x_1, x_2 + c, s),$$

$$(x_1, x_2, s) \mapsto (e^{-c}x_1, e^c x_2, s + c).$$

The corresponding Killing fields are

$$F_1 = \frac{\partial}{\partial x_1}, \quad F_2 = \frac{\partial}{\partial x_2}, \quad F_3 = -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial s}.$$

We will call *translations* isometries belonging to the identity component of the identity (they are in fact left multiplications for the Lie group structure).

The isotropy group of the origin $(0, 0, 0)$ is isomorphic to the dihedral group D$_4$ and is generated by the following two orientation-reversing isometries:

$$(2) \quad \sigma \colon (x_1, x_2, s) \mapsto (x_2, -x_1, -s), \quad \tau \colon (x_1, x_2, s) \mapsto (-x_1, x_2, s).$$

The reflection with respect to the surface $x_2 = 0$ is given by $\sigma^2\tau$.

For more details, we refer to [**5**] and references therein.

2.2. Proof of Theorem 1.5. Before proving the theorem, we will need some preliminary results.

Lemma 2.1. *Let Σ be a minimal surface (possibly with boundary) in* Sol_3 *such that*

$$0 < s \leqslant 2$$

on Σ. Then the function

$$\varphi := \frac{1}{s}$$

is subharmonic on Σ.

Proof. We view Σ as a conformal minimal immersion $X = (x_1, x_2, s) \colon \Sigma \to \mathrm{Sol}_3$ from a Riemann surface Σ. Let z be a conformal coordinate. We set

$$A_1 = e^s x_{1z}, \quad A_2 = e^{-s} x_{2z}, \quad A_3 = s_z,$$

so that

$$X_z = A_1 E_1 + A_2 E_2 + A_3 E_3.$$

The conformality of X means that

(3) $$A_1^2 + A_2^2 + A_3^2 = 0.$$

Since X is minimal, we have $\widehat{\nabla}_{X_{\bar{z}}} X_z = 0$, and so

(4) $$A_{3\bar{z}} = \langle E_3, X_z \rangle_{\bar{z}} = \langle \widehat{\nabla}_{X_{\bar{z}}} E_3, X_z \rangle = |A_1|^2 - |A_2|^2,$$

by (1). Then by (4) we have

$$\varphi_{z\bar{z}} = -\left(\frac{A_3}{s^2}\right)_{\bar{z}} = \frac{2|A_3|^2 - s(|A_1|^2 - |A_2|^2)}{s^3}.$$

On the other hand, by (3) we have

$$\begin{aligned} |A_3|^4 &= |A_1^2 + A_2^2|^2 = |A_1|^4 + |A_2|^4 + 2\,\mathrm{Re}(A_1^2 \bar{A}_2^2) \\ &\geqslant |A_1|^4 + |A_2|^4 - 2|A_1|^2|A_2|^2 = (|A_1|^2 - |A_2|^2)^2. \end{aligned}$$

Consequently, we get $-|A_3|^2 \leqslant |A_1|^2 - |A_2|^2 \leqslant |A_3|^2$, and so, since $0 < s \leqslant 2$, we conclude that $\varphi_{z\bar{z}} \geqslant 0$.   q.e.d.

Lemma 2.2. *There exist positive constants a, b, and d such that, for all stable minimal surfaces Σ (possibly with boundary), for all $p \in \Sigma$ such that $\mathrm{dist}(p, \partial\Sigma) > d$, if $|\langle N(p), E_3 \rangle| \leqslant a$ where $N(p)$ denotes a unit normal vector to Σ at p, then there exist points q_1 and q_2 in Σ such that $s(q_1) - s(q_2) \geqslant b$.*

Proof. We recall that stable minimal surfaces admit uniform curvature estimates away from their boundary, and so there exist positive constants d and δ such that, for any stable minimal surface Σ (possibly with boundary), for each $p \in \Sigma$ such that $\mathrm{dist}(p, \partial\Sigma) > d$, there is a piece $S(p)$ of Σ around p that is a graph (in exponential coordinates) over the disk in $\mathrm{T}_p\Sigma$ of radius 2δ centered at the origin of

$T_p\Sigma$. Moreover these graphs have uniformly bounded second fundamental form. If $|\langle N(p), E_3 \rangle|$ is smaller than some constant $a > 0$, such a piece $S(p)$ necessarily has points q_1 and q_2 such that $s(q_1) - s(p) \geqslant \frac{b}{2}$ and $s(p) - s(q_2) \geqslant \frac{b}{2}$ for some constant $b > 0$, and these constants a and b are independent of Σ (see Figure 1): otherwise, one could produce a sequence of such pieces with unbounded second fundamental form. Moreover, these constants are also independent from the point p since E_3 and the differences of the s function are invariant by translations. q.e.d.
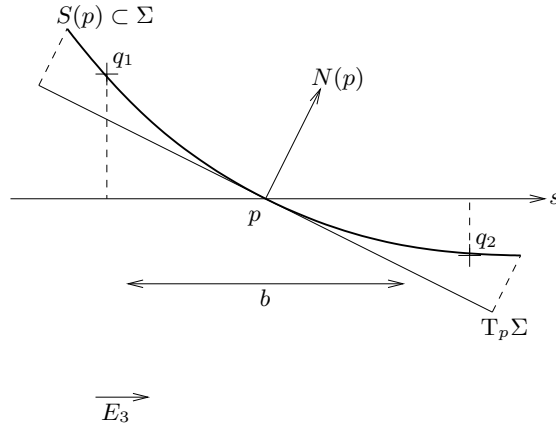
Figure 1. A piece of a stable minimal surface.

From now on, $\mathcal{S}$ denotes a properly immersed minimal surface in Sol_3. We assume that $\mathcal{S}$ lies on one side of some special plane $\mathcal{E}^t$ ($t \in \mathbb{R}$).

Up to isometries in Sol_3, we can assume $t = 0$, $s \geqslant 1$ on $\mathcal{S}$ and

$$\inf\{u \in \mathbb{R} \mid \mathcal{S} \cap \mathcal{E}^u \neq \emptyset\} = 1.$$

If $\mathcal{S} \cap \mathcal{E}^1 \neq \emptyset$, then the maximum principle implies that $\mathcal{S} = \mathcal{E}^1$. So from now on we assume that

$$\mathcal{S} \cap \mathcal{E}^1 = \emptyset.$$

For $R > 0$ and $h > 0$ we let

$$K_R := \{(x_1, x_2, s) \in \mathrm{Sol}_3 \mid x_1^2 + x_2^2 \leqslant R^2\},$$

$$C_R := \{(x_1, x_2, s) \in \mathrm{Sol}_3 \mid x_1^2 + x_2^2 = R^2\},$$

$$D_R^h := K_R \cap \mathcal{E}^h,$$

$$\Gamma_R^h := C_R \cap \mathcal{E}^h,$$

$$Q_R^h := \bigcup_{t \in [1, 1+h]} D_R^t,$$

$$q_R^h := \bigcup_{t \in [1, 1+h]} \Gamma_R^t.$$

We now fix real numbers $r > 0$ and $\varepsilon > 0$ so that $\varepsilon < 1$, $\varepsilon < \frac{b}{2}$ where b is defined in Lemma 2.2,

(5) $$\mathrm{Area}(q_r^\varepsilon) < \mathrm{Area}(D_r^1) + \mathrm{Area}(D_r^{1+\varepsilon})$$

and

(6) $$\mathcal{S} \cap Q_{er}^\varepsilon = \emptyset,$$

which is possible since $\mathcal{S}$ is proper in Sol_3.

Claim 2.3. If $R > r$, then there exists a least area annulus $\mathcal{A}_R$ bounded by $\Gamma_r^{1+\varepsilon}$ and Γ_R^1. Moreover, this annulus lies between the special planes $\mathcal{E}^{1+\varepsilon}$ and $\mathcal{E}^1$ and it is embedded.

Proof. The solutions to the Plateau problem for $\Gamma_r^{1+\varepsilon}$ and Γ_R^1 are respectively $D_r^{1+\varepsilon}$ and D_R^1; indeed, the unique compact minimal surface bounded by an embedded closed curve in the special plane $\mathcal{E}^h$ ($h \in \mathbb{R}$) is the part of this special plane bounded by this curve, by the maximum principle (since we have the minimal foliation $(\mathcal{E}^u)_{u \in \mathbb{R}}$). The total area of these two disks is $\mathrm{Area}(D_R^1) + \mathrm{Area}(D_r^{1+\varepsilon})$.
Let
$$M := (D_R^1 \setminus D_r^1) \cup q_r^\varepsilon.$$
By (5), the area of M is smaller than $\mathrm{Area}(D_R^1) + \mathrm{Area}(D_r^{1+\varepsilon})$.
Consequently, the Douglas criterion implies the existence of a least area annulus $\mathcal{A}_R$ bounded by $\Gamma_r^{1+\varepsilon}$ and Γ_R^1. This annulus is embedded by the Geometric Dehn's Lemma in [**10**].   q.e.d.

By Lemma 2.2, if $\Omega \subset \mathcal{A}_R$ and $\mathrm{dist}(\Omega, \partial \mathcal{A}_R) > d$, then $|\langle N, E_3 \rangle| > a$ where N denotes the unit normal to $\mathcal{A}_R$ (otherwise there would exist two points $q_1, q_2 \in \mathcal{A}_R$ such that $s(q_1) - s(q_2) \geqslant b > 2\varepsilon$, which contradicts the fact that $\mathcal{A}_R$ lies between the special planes $\mathcal{E}^{1+\varepsilon}$ and $\mathcal{E}^1$). In particular, Ω is transverse to E_3.
We now introduce a constant $\rho > r$ such that

(7) $$\mathrm{dist}(\Gamma_r^{1+\varepsilon}, q_\rho^\varepsilon) > d.$$

If R is large enough, this implies that $\mathcal{A}_R$ intersects transversely q_ρ^ε in a smooth curve $\widetilde{\Gamma}_R$. We denote by $\widetilde{\mathcal{A}}_R$ the part of $\mathcal{A}_R$ lying outside Q_ρ^ε: it is an annulus bounded by $\widetilde{\Gamma}_R$ and Γ_R^1.

Claim 2.4. Let (R_n) be an increasing sequence of positive real numbers such that $R_n \to +\infty$ as $n \to +\infty$. Then, up to a subsequence, the annuli $\widetilde{\mathcal{A}}_{R_n}$ converge to a properly embedded minimal annulus $\widetilde{\mathcal{A}}_\infty$ whose boundary is a closed curve $\widetilde{\Gamma}_\infty \subset q_\rho^\varepsilon$ and such that

$$\inf_{\widetilde{\mathcal{A}}_\infty} s > 1.$$

Proof. The annuli $\mathcal{A}_{R_n}$ are stable, so they admit uniform curvature bounds for their points lying at a distance $\geqslant d$ from their boundary. Hence, by (7), up to a subsequence, the annuli $\widetilde{\mathcal{A}}_{R_n}$ converge to a properly embedded minimal surface $\widetilde{\mathcal{A}}_\infty$ whose boundary is a curve $\widetilde{\Gamma}_\infty$ lying between the special planes $\mathcal{E}^{1+\varepsilon}$ and $\mathcal{E}^1$ and in q_ρ^ε (properness of the limit annulus follows, easily deduced from the fact that outside a fixed sized regular neighborhood of its boundary, each $\mathcal{A}_{R_n}$ is transverse to E_3 and graphical over its s-projection to $\mathcal{E}^1$). Moreover, since $\widetilde{\mathcal{A}}_{R_m}$ lies above $\widetilde{\mathcal{A}}_{R_n}$ if $m > n$, the curve $\widetilde{\Gamma}_\infty$ lies above all the curves $\widetilde{\Gamma}_{R_n}$; in particular, we have

$$h := \min_{\widetilde{\Gamma}_\infty} s > 1.$$

Let N be a unit normal vector field to $\widetilde{\mathcal{A}}_\infty$. Then we have $|\langle N, E_3 \rangle| \geqslant a$, so $\widetilde{\mathcal{A}}_\infty$ can be written as a graph $s = f(x_1, x_2)$ over

$$L := \{(x_1, x_2, 1) \in \mathcal{E}^1 \mid x_1^2 + x_2^2 \geqslant \rho^2\},$$

where f is a function with bounded gradient. The vector field E_3 is orthogonal to $\mathcal{E}^1$, and by construction, the annulus $\widetilde{\mathcal{A}}_\infty$ lies between the special planes $\mathcal{E}^1$ and $\mathcal{E}^{1+\varepsilon}$. This implies that the map $(x_1, x_2, 1) \mapsto (x_1, x_2, f(x_1, x_2))$ is a quasi-isometry between L and $\widetilde{\mathcal{A}}_\infty$. Since $\mathcal{E}^1$ is flat, this implies that $\widetilde{\mathcal{A}}_\infty$ is parabolic (it has quadratic area growth).

On $\widetilde{\mathcal{A}}_\infty$ we have $1 \leqslant s \leqslant 1 + \varepsilon \leqslant 2$, so by Lemma 2.1 the function $\varphi := 1/s$ is subharmonic on $\widetilde{\mathcal{A}}_\infty$. Moreover, φ is bounded on $\widetilde{\mathcal{A}}_\infty$, so since $\widetilde{\mathcal{A}}_\infty$ has parabolic conformal type, we have

$$\varphi \leqslant \sup_{\partial \widetilde{\mathcal{A}}_\infty} \varphi = \frac{1}{h}$$

on $\widetilde{\mathcal{A}}_\infty$. Consequently, we have $s \geqslant h > 1$ on $\widetilde{\mathcal{A}}_\infty$.   q.e.d.

We can now conclude the proof of the theorem.

Proof of Theorem 1.5. Because of Claim 2.4, there exists $m \in \mathbb{N}$ such that $\mathcal{S} \cap \widetilde{\mathcal{A}}_{R_m} \neq \emptyset$, and so

$$\mathcal{S} \cap \mathcal{A}_{R_m} \neq \emptyset.$$

For $c \in \mathbb{R}$, we let $T^c \colon \mathrm{Sol}_3 \to \mathrm{Sol}_3$ denote the isometry $(x_1, x_2, s) \mapsto (e^{-c}x_1, e^c x_2, s + c)$. We consider the annuli $T^{-c}(\mathcal{A}_{R_m})$ for $c \geqslant 0$. We notice that $\mathcal{S} \cap T^{-c}(\mathcal{A}_{R_m}) = \emptyset$ when $c > \varepsilon$. Then there exists a largest c for which

$$\mathcal{S} \cap T^{-c}(\mathcal{A}_{R_m}) \neq \emptyset.$$

We claim that no point of intersection lies on the boundary of $T^{-c}(\mathcal{A}_{R_m})$. Indeed, this boundary consists of $T^{-c}(\Gamma_r^{1+\varepsilon})$ and $T^{-c}(\Gamma_{R_m}^1)$. The curve

$T^{-c}(\Gamma_r^{1+\varepsilon})$ is defined by

$$\begin{cases} e^{-2c}x_1^2 + e^{2c}x_2^2 &= r^2, \\ s &= 1 + \varepsilon - c, \end{cases}$$

so, since $0 \leqslant c \leqslant \varepsilon < 1$, it is contained in Q_{er}^ε and hence cannot intersect $\mathcal{S}$ by (6). On the other hand, $T^{-c}(\Gamma_{R_m}^1)$ lies below $\mathcal{E}^1$ and hence cannot intersect $\mathcal{S}$ either.

Consequently, there exists an intersection point of $\mathcal{S}$ and $T^{-c}(\mathcal{A}_{R_m})$ lying in the interior of $T^{-c}(\mathcal{A}_{R_m})$. But since c is maximal, $\mathcal{S}$ lies on one side of $T^{-c}(\mathcal{A}_{R_m})$; this contradicts the maximum principle (see Figure 2).   q.e.d.

3. Half-space theorems in Nil_3

3.1. The Lie group Nil_3. The 3-dimensional Heisenberg group Nil_3 can be viewed as $\mathbb{R}^3$ endowed with the metric

$$\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \left(\frac{1}{2}(x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2) + \mathrm{d}x_3 \right)^2.$$

The projection $\pi \colon \mathrm{Nil}_3 \to \mathbb{R}^2, (x_1, x_2, x_3) \mapsto (x_1, x_2)$ is a Riemannian submersion.

We consider the left-invariant orthonormal frame (E_1, E_2, E_3) defined by

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}, \quad E_3 = \xi = \frac{\partial}{\partial x_3}.$$

We call it the *canonical frame*. The expression of the Riemannian connection $\widehat{\nabla}$ of Nil_3 in this frame is the following:

$$(8) \quad \begin{aligned} &\widehat{\nabla}_{E_1}E_1 = 0, &&\widehat{\nabla}_{E_2}E_1 = -\frac{1}{2}E_3, &&\widehat{\nabla}_{E_3}E_1 = -\frac{1}{2}E_2, \\ &\widehat{\nabla}_{E_1}E_2 = \frac{1}{2}E_3, &&\widehat{\nabla}_{E_2}E_2 = 0, &&\widehat{\nabla}_{E_3}E_2 = \frac{1}{2}E_1, \\ &\widehat{\nabla}_{E_1}E_3 = -\frac{1}{2}E_2, &&\widehat{\nabla}_{E_2}E_3 = \frac{1}{2}E_1, &&\widehat{\nabla}_{E_3}E_3 = 0. \end{aligned}$$

A vector is said to be vertical if it is proportional to ξ, and horizontal if it is orthogonal to ξ. A surface is said to be a (local) ξ-*graph* if it is transverse to ξ. We will call the inverse image by π of a straight line in $\mathbb{R}^2$ a *vertical plane*.

The isometry group of Nil_3 has dimension 4. The connected component of the identity is generated by the following four families of isometries:

$$(x_1, x_2, x_3) \mapsto \left(x_1 + c, x_2, x_3 + \frac{cx_2}{2} \right),$$
$$(x_1, x_2, x_3) \mapsto \left(x_1, x_2 + c, x_3 - \frac{cx_1}{2} \right),$$
$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + c),$$
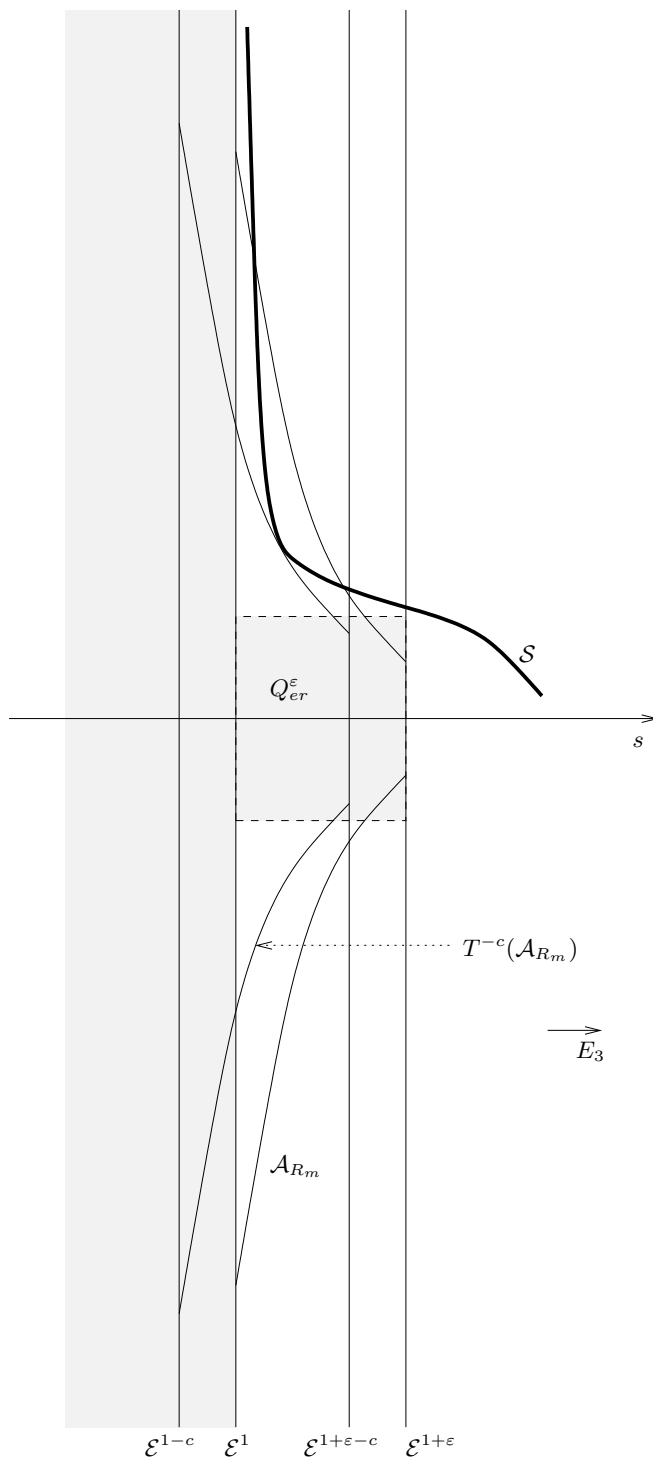
Figure 2. Translating the annulus $\mathcal{A}_{R_m}$.

$$(x_1, x_2, x_3) \mapsto ((\cos\theta)x_1 - (\sin\theta)x_2, (\sin\theta)x_1 + (\cos\theta)x_2, x_3).$$

The corresponding Killing fields are

$$F_1 = \frac{\partial}{\partial x_1} + \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad F_2 = \frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial x_3}, \quad F_3 = \xi = \frac{\partial}{\partial x_3},$$

$$F_4 = (-(\sin\theta)x_1 - (\cos\theta)x_2)\frac{\partial}{\partial x_1} + ((\cos\theta)x_1 - (\sin\theta)x_2)\frac{\partial}{\partial x_2}.$$

We will call *translations* isometries generated by the first three of these families (they are in fact left multiplications for the Lie group structure).

The entire isometry group of Nil_3 is generated by the aforementioned isometries and the orientation-reversing isometry

$$(x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3).$$

For more details, we refer to [**4**] and references therein.

3.2. A new proof of Theorem 1.3. We can prove Theorem 1.3 in a very similar way to that of Theorem 1.5. For our purpose, it will be useful to introduce the following coordinates in Nil_3:

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + \frac{x_1 x_2}{2}.$$

In these coordinates we have

$$E_1 = \frac{\partial}{\partial y_1}, \quad E_2 = \frac{\partial}{\partial y_2} + y_1\frac{\partial}{\partial y_3}, \quad E_3 = \xi = \frac{\partial}{\partial y_3}.$$

What is important is that E_1 is the partial derivative with respect to a coordinate whose level sets are precisely the vertical planes to which E_1 is orthogonal. The proof of Theorem 1.3 will be analogous to that of Theorem 1.5, replacing the s coordinate in Sol_3 by the y_1 coordinate in Nil_3. The main difference lies in the analogue of Lemma 2.1, which is the following.

Lemma 3.1. *Let Σ be a minimal surface (possibly with boundary) in Nil_3 such that*

$$0 < y_1 \leqslant 4$$

on Σ. Then the function

$$\varphi := \frac{1}{y_1}$$

is subharmonic on Σ.

Proof. We view Σ as a conformal minimal immersion $X = (y_1, y_2, y_3)\colon \Sigma \to \mathrm{Nil}_3$ from a Riemann surface Σ. Let z be a conformal coordinate. We set

$$A_1 = y_{1z}, \quad A_2 = y_{2z}, \quad A_3 = y_{3z} - y_1 y_{2z},$$

so that

$$X_z = A_1 E_1 + A_2 E_2 + A_3 E_3.$$

The conformality of X means that

(9)
$$A_1^2 + A_2^2 + A_3^2 = 0.$$

Since X is minimal, we have $\widehat{\nabla}_{X_{\bar{z}}} X_z = 0$, and so

(10) $$A_{1\bar{z}} = \langle E_1, X_z \rangle_{\bar{z}} = \langle \widehat{\nabla}_{X_{\bar{z}}} E_1, X_z \rangle = -\frac{1}{2}(\bar{A}_2 A_3 + A_2 \bar{A}_3)$$

by (8). Then by (10) we have

$$\varphi_{z\bar{z}} = -\left(\frac{A_1}{y_1^2}\right)_{\bar{z}} = \frac{4|A_1|^2 + y_1(\bar{A}_2 A_3 + A_2 \bar{A}_3)}{2y_1^3}.$$

On the other hand, by (9) we have

$$|A_1|^4 = |A_2^2 + A_3^2|^2 = |A_2|^4 + |A_3|^4 + \bar{A}_2^2 A_3^2 + A_2^2 \bar{A}_3^2$$

and

$$(\bar{A}_2 A_3 + A_2 \bar{A}_3)^2 = \bar{A}_2^2 A_3^2 + A_2^2 \bar{A}_3^2 + 2|A_2|^2|A_3|^2.$$

Consequently, we get $-|A_1|^2 \leqslant \bar{A}_2 A_3 + A_2 \bar{A}_3 \leqslant |A_1|^2$, and so, since $0 < y_1 \leqslant 4$, we conclude that $\varphi_{z\bar{z}} \geqslant 0$.   q.e.d.

Lemma 3.2. *There exist positive constants a, b, and d such that, for all stable minimal surface Σ (possibly with boundary), for all $p \in \Sigma$ such that $\mathrm{dist}(p, \partial\Sigma) > d$, if $|\langle N(p), E_1 \rangle| \leqslant a$ where $N(p)$ denotes a unit normal vector to Σ at p, then there exist points q_1 and q_2 in Σ such that $y_1(q_1) - y_1(q_2) \geqslant b$.*

Proof. The proof is analogous to that of Lemma 2.2.   q.e.d.

From now on, $\mathcal{S}$ denotes a properly immersed minimal surface in Nil$_3$ and $\mathcal{P}$ a vertical plane. We assume that $\mathcal{S}$ lies on one side of $\mathcal{P}$.

For $h \in \mathbb{R}$, we let $\mathcal{P}^h$ denote the plane of equation $y_1 = h$. Up to isometries in Nil$_3$, we can assume $\mathcal{P} = \mathcal{P}^1$, $y_1 \geqslant 1$ on $\mathcal{S}$ and

$$\inf\{u \in \mathbb{R} \mid \mathcal{S} \cap \mathcal{P}^u \neq \emptyset\} = 1.$$

If $\mathcal{S} \cap \mathcal{P}^1 \neq \emptyset$, then the maximum principle implies that $\mathcal{S} = \mathcal{P}^1$. So from now on we assume that

$$\mathcal{S} \cap \mathcal{P}^1 = \emptyset.$$

For $R > 0$ and $h > 0$ we let

$$K_R := \{(y_1, y_2, y_3) \in \mathrm{Nil}_3 \mid y_2^2 + y_3^2 \leqslant R^2\},$$
$$C_R := \{(y_1, y_2, y_3) \in \mathrm{Nil}_3 \mid y_2^2 + y_3^2 = R^2\},$$
$$D_R^h := K_R \cap \mathcal{P}^h,$$
$$\Gamma_R^h := C_R \cap \mathcal{P}^h,$$
$$Q_R^h := \bigcup_{t \in [1, 1+h]} D_R^t,$$
$$q_R^h := \bigcup_{t \in [1, 1+h]} \Gamma_R^t.$$

We now fix real numbers $r > 0$ and $\varepsilon > 0$ so that $\varepsilon < 1$, $\varepsilon < \frac{b}{2}$ where b is defined in Lemma 3.2,

$$\text{(11)} \qquad \text{Area}(q_r^\varepsilon) < \text{Area}(D_r^1) + \text{Area}(D_r^{1+\varepsilon})$$

and

$$\text{(12)} \qquad \mathcal{S} \cap Q_{2r}^\varepsilon = \emptyset,$$

which is possible since $\mathcal{S}$ is proper in Nil_3.

The following facts can be proved in the same way as Claims 2.3 and 2.4.

Claim 3.3. If $R > r$, then there exists a least area annulus $\mathcal{A}_R$ bounded by $\Gamma_r^{1+\varepsilon}$ and Γ_R^1. Moreover, this annulus lies between the planes $\mathcal{P}^{1+\varepsilon}$ and $\mathcal{P}^1$ and it is embedded.

By Lemma 3.2, if $\Omega \subset \mathcal{A}_R$ and $\text{dist}(\Omega, \partial \mathcal{A}_R) > d$, then $|\langle N, E_1 \rangle| > a$ where N denotes the unit normal to $\mathcal{A}_R$ (otherwise there would exist two points $q_1, q_2 \in \mathcal{A}_R$ such that $y_1(q_1) - y_1(q_2) \geqslant b > 2\varepsilon$, which contradicts the fact that $\mathcal{A}_R$ lies between the planes $\mathcal{P}^{1+\varepsilon}$ and $\mathcal{P}^1$). In particular, Ω is transverse to E_1.

We now introduce a constant $\rho > r$ such that

$$\text{(13)} \qquad \text{dist}(\Gamma_r^{1+\varepsilon}, q_\rho^\varepsilon) > d.$$

If R is large enough, this implies that $\mathcal{A}_R$ transversely intersects q_ρ^ε in a smooth curve $\widetilde{\Gamma}_R$. We denote by $\widetilde{\mathcal{A}}_R$ the part of $\mathcal{A}_R$ lying outside Q_ρ^ε: it is an annulus bounded by $\widetilde{\Gamma}_R$ and $\Gamma_R^{1+\varepsilon}$.

Claim 3.4. Let (R_n) be an increasing sequence of positive real numbers such that $R_n \to +\infty$ as $n \to +\infty$. Then, up to a subsequence, the annuli $\widetilde{\mathcal{A}}_{R_n}$ converge to a properly embedded minimal annulus $\widetilde{\mathcal{A}}_\infty$ whose boundary is a closed curve $\widetilde{\Gamma}_\infty \subset q_\rho^\varepsilon$ and such that

$$\inf_{\widetilde{\mathcal{A}}_\infty} y_1 > 1.$$

We can now conclude the proof of the theorem.

Proof of Theorem 1.3. This is similar to the conclusion to the proof of Theorem 1.5. There exists $m \in \mathbb{N}$ such that

$$\mathcal{S} \cap \mathcal{A}_{R_m} \neq \emptyset.$$

For $c \in \mathbb{R}$, we let $\Phi^c \colon \text{Nil}_3 \to \text{Nil}_3$ denote the isometry $(y_1, y_2, y_3) \mapsto (y_1 + c, y_2, y_3 + cy_2)$. We consider the largest $c \geqslant 0$ such that $\mathcal{S} \cap \Phi^{-c}(\mathcal{A}_{R_m}) \neq \emptyset$. We have $c \leqslant \varepsilon < 1$, so using (12) we prove that no intersection points lies on the boundary, and obtain a contradiction with the maximum principle. q.e.d.

3.3. Proof of Theorem 1.4. In this section, $\mathcal{S}$ denotes a properly immersed minimal surface in Sol$_3$ and $\mathcal{G}$ an entire minimal ξ-graph. We assume that $\mathcal{S}$ lies on one side of $\mathcal{G}$. For $c \in \mathbb{R}$, we let $T^c \colon \text{Nil}_3 \to \text{Nil}_3$ denote the vertical translation $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + c)$.

Without loss of generality, we can assume that $\mathcal{S}$ lies above $\mathcal{G}$. Let

$$c_0 := \inf\{c \in \mathbb{R} \mid \mathcal{S} \cap T^c(\mathcal{G}) \neq \emptyset\}.$$

If $\mathcal{S} \cap T^{c_0}(\mathcal{G}) \neq \emptyset$, then the maximum principle implies that $\mathcal{S} = T^{c_0}(\mathcal{G})$. So from now on we assume that

$$\mathcal{S} \cap T^{c_0}(\mathcal{G}) = \emptyset.$$

Moreover, applying a vertical translation to $\mathcal{S}$, we can assume that

$$c_0 = 0.$$

For $R > 0$ and $h > 0$ we let

$$K_R := \{(x_1, x_2, x_3) \in \text{Nil}_3 \mid x_1^2 + x_2^2 \leqslant R^2\},$$

$$C_R := \{(x_1, x_2, x_3) \in \text{Nil}_3 \mid x_1^2 + x_2^2 = R^2\},$$

$$D_R^h := K_R \cap T^h(\mathcal{G}),$$

$$\Gamma_R^h := C_R \cap T^h(\mathcal{G}),$$

$$Q_R^h := \bigcup_{t \in [0,h]} D_R^t,$$

$$q_R^h := \bigcup_{t \in [0,h]} \Gamma_R^t.$$

We now fix a real number $r \in (0,1)$. The part of $\mathcal{G}$ bounded by Γ_r^0 and Γ_1^0 is the unique compact minimal surface with this boundary, hence a small perturbation of its boundary also bounds a unique least area minimal surface that is near it; in particular, for a small $\varepsilon > 0$, there exists a least area annulus $\mathcal{A}$ bounded by Γ_r^ε and Γ_1^0 so that

$$(14) \qquad \qquad \text{Area}(\mathcal{A}) < \text{Area}(D_r^\varepsilon) + \text{Area}(D_1^0)$$

and this annulus is a ξ-graph. We fix such an ε and we moreover assume that ε is small enough so that

$$(15) \qquad \qquad \mathcal{S} \cap Q_1^\varepsilon = \emptyset,$$

which is possible since $\mathcal{S}$ is proper in Nil$_3$.

Claim 3.5. *If $R > r$, then there exists a least area annulus $\mathcal{A}_R$ bounded by Γ_r^ε and Γ_R^0. Moreover, this annulus lies between the graphs $T^\varepsilon(\mathcal{G})$ and $\mathcal{G}$, above $\mathcal{A}$, and it is embedded.*

Proof. Let W be the closure of the noncompact domain bounded by $T^\varepsilon(\mathcal{G})$ from above and by $\mathcal{G}$ and $\mathcal{A}$ from below. Note that Γ_r^ε and Γ_R^0 are homotopic in W and individually homotopically nontrivial in W. By the Geometric Dehn's Lemma in [**10**], there exists an embedded minimal annulus $\mathcal{A}_R \subset W$ bounded by Γ_r^ε and Γ_R^0 such that $\mathcal{A}_R$ has least area in W.   q.e.d.

For $R > r$, we set

$$U_R := \{(x_1, x_2) \in \mathbb{R}^2 \mid r^2 < x_1^2 + x_2^2 < R^2\}.$$

We also set

$$U_\infty := \{(x_1, x_2) \in \mathbb{R}^2 \mid r^2 < x_1^2 + x_2^2\}.$$

Claim 3.6. If $R > 1$, then the annulus $\mathcal{A}_R$ is a ξ-graph of a function $u_R : \overline{U_R} \to \mathbb{R}$.

Proof. We first prove that $\mathcal{A}_R$ is a ξ-graph near Γ_r^ε. The annulus $\mathcal{A}_R$ lies below $T^\varepsilon(\mathcal{G})$; moreover, since $R > 1$, $\mathcal{A}_R$ is situated above $\mathcal{A}$, which is a ξ-graph. As $\mathcal{A}_R$ is analytic and without branch points up to the boundary [**10**], this implies that $\mathcal{A}_R$ is a ξ-graph near Γ_r^ε (see Figure 3). We now let N be the unit normal vector field to $\mathcal{A}_R$ so that N points upward near Γ_r^ε.
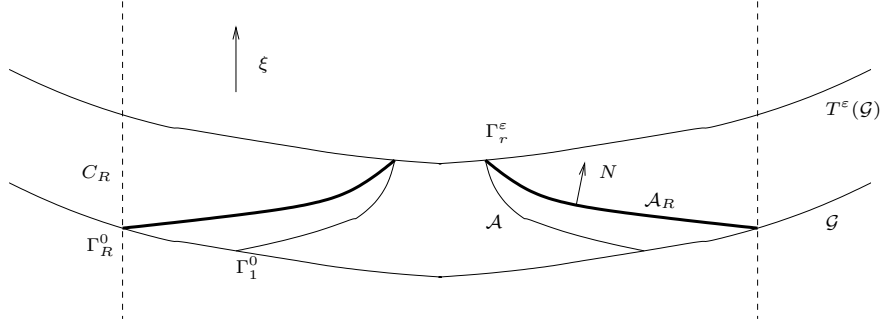
Figure 3. The annulus $\mathcal{A}_R$.

We now prove that $\mathcal{A}_R$ is a ξ-graph near Γ_R^0. This comes from the fact that the mean curvature vector of C_R does not vanish and points inside K_R, so $\mathcal{A}_R$ lies inside K_R and cannot be tangent to C_R along Γ_R^0. Moreover, since $\mathcal{A}_R$ is embedded, this implies that N also points upward near Γ_R^0.

We now prove that the whole $\mathcal{A}_R$ is a ξ-graph. Assume this is not the case. Then the Jacobi function $\nu := \langle N, \xi \rangle$ admits a nodal domain Ω on which $\nu < 0$. Denoting by λ_1 the first eigenvalue of the Jacobi operator, this means that $\lambda_1(\bar{\Omega}) = 0$. On the other hand, $\bar{\Omega}$ is contained in the

interior of $\mathcal{A}_R$ because $\nu > 0$ near the boundary of $\mathcal{A}_R$. From this we conclude that

$$\lambda_1(\mathcal{A}_R) < \lambda_1(\bar{\Omega}) = 0,$$

which contradicts the fact that $\mathcal{A}_R$ is stable.   q.e.d.

Claim 3.7. Let (R_n) be an increasing sequence of positive real numbers such that $R_n \to +\infty$ as $n \to +\infty$. Then, up to a subsequence, the functions u_{R_n} converge (in the C^2 topology on compact sets) to a smooth function $u_\infty \colon U_\infty \to \mathbb{R}$. Moreover, this function u_∞ extends to a continuous function

$$u_\infty \colon \overline{U_\infty} \to \mathbb{R}$$

by setting

$$\forall p \in \partial U_\infty, \quad u_\infty(p) := v(p),$$

where $v \colon \mathbb{R}^2 \to \mathbb{R}$ is the function whose ξ-graph is $T^\varepsilon(\mathcal{G})$.

Proof. The annuli $\mathcal{A}_{R_n}$ are stable, so one has uniform curvature bounds over any compact subset of U_∞. Hence, up to a subsequence, the annuli $\mathring{\mathcal{A}}_{R_n} := \mathcal{A}_{R_n} \setminus \partial \mathcal{A}_{R_n}$ converge to a properly embedded open minimal surface $\mathring{\mathcal{A}}_\infty$ lying between the entire graphs $T^\varepsilon(\mathcal{G})$ and $\mathcal{G}$.

Let N be a unit normal vector field to $\mathring{\mathcal{A}}_\infty$ and $\nu := \langle N, \xi \rangle$. Since $\mathring{\mathcal{A}}_\infty$ is the limit of the annuli $\mathring{\mathcal{A}}_{R_n}$, which are ξ-graphs by Claim 3.6, we have either $\nu \geqslant 0$ or $\nu \leqslant 0$. Up to a change of orientation, we can assume that $\nu \geqslant 0$. If ν vanishes at some interior point, then, since $\nu \geqslant 0$ and ν satisfies an elliptic equation of the form $\Delta \nu + V\nu = 0$ for some potential V, the maximum principle implies that $\nu \equiv 0$. This means that $\mathring{\mathcal{A}}_\infty$ is part of a vertical surface, hence a vertical plane since $\mathring{\mathcal{A}}_\infty$ is minimal; this is a contradiction.

Consequently, $\mathring{\mathcal{A}}_\infty$ is the ξ-graph of a function $u_\infty \colon U_\infty \to \mathbb{R}$. Finally, the barriers $T^\varepsilon(\mathcal{G})$ and $\mathcal{A}$ imply that u_∞ extends to a continuous function

$$u_\infty \colon \overline{U_\infty} \to \mathbb{R}$$

by setting $u_\infty(p) := v(p)$ for all $p \in \partial U_\infty$.   q.e.d.

Claim 3.8. We have

$$u_\infty \equiv v.$$

In other words, the annulus $\mathring{\mathcal{A}}_\infty$ is the part of the entire graph $T^\varepsilon(\mathcal{G})$ lying outside Γ_r^ε.

Proof. The functions u_∞ and v satisfy the minimal graph equation on U_∞, $u_\infty \equiv v$ on ∂U_∞, and $-\varepsilon < u_\infty - v \leqslant 0$ since $\mathring{\mathcal{A}}_\infty$ lies between the graphs $T^\varepsilon(\mathcal{G})$ and $\mathcal{G}$. Then Theorem 5.1 in [**9**] implies that $u_\infty \equiv v$ (observe that in the theorems of [**3**, **9**] the functions only need to be continuous along the boundary of the domain).   q.e.d.

We can now conclude the proof of the theorem.

Proof of Theorem 1.4. Since the annuli $\mathcal{A}_{R_n}$ converge, as $n \to +\infty$, to the part of $T^\varepsilon(\mathcal{G})$ lying outside Γ_r^ε, there exists $m \in \mathbb{N}$ such that

$$\mathcal{S} \cap \mathcal{A}_{R_m} \neq \emptyset.$$

We consider the annuli $T^{-c}(\mathcal{A}_{R_m})$ for $c \geqslant 0$. We notice that $\mathcal{S} \cap T^{-c}(\mathcal{A}_{R_m}) = \emptyset$ when $c > \varepsilon$. Then there exists a largest c for which

$$\mathcal{S} \cap T^{-c}(\mathcal{A}_{R_m}) \neq \emptyset.$$

We claim that no point of intersection lies on the boundary of $T^{-c}(\mathcal{A}_{R_m})$. Indeed, this boundary consists of $\Gamma_r^{\varepsilon-c}$, which is contained in Q_1^ε and hence cannot intersect $\mathcal{S}$ by (15), and of $\Gamma_{R_m}^{-c}$, which lies below $\mathcal{G}$ and hence cannot intersect $\mathcal{S}$ either.

Consequently, there exists an intersection point of $\mathcal{S}$ and $T^{-c}(\mathcal{A}_{R_m})$ lying in the interior of $T^{-c}(\mathcal{A}_{R_m})$. But since c is maximal, $\mathcal{S}$ lies on one side of $T^{-c}(\mathcal{A}_{R_m})$; this contradicts the maximum principle.   q.e.d.

Remark 3.9. Since entire minimal graphs in Nil$_3$ and entire CMC $1/2$ graphs in $\mathbb{H}^2 \times \mathbb{R}$ are sister surfaces [**4**], it would be interesting to prove a half-space theorem in $\mathbb{H}^2 \times \mathbb{R}$ for CMC $1/2$ surfaces with respect to an entire CMC $1/2$ graph. However, in this setting there is no known Collin-Krust type theorem (and actually such a theorem fails for *minimal* graphs in $\mathbb{H}^2 \times \mathbb{R}$). Nevertheless, we propose the following conjecture.

Conjecture 3.10. *Let $\mathcal{G}$ be an entire CMC $1/2$ graph in $\mathbb{H}^2 \times \mathbb{R}$. If $\mathcal{S}$ is a properly immersed CMC $1/2$ surface in $\mathbb{H}^2 \times \mathbb{R}$ that is contained in the mean convex side of $\mathcal{G}$, then $\mathcal{S}$ is the image of $\mathcal{G}$ by a vertical translation.*

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