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ON THE σ_2-SCALAR CURVATURE

YUXIN GE, CHANG-SHOU LIN & GUOFANG WANG

Abstract

In this paper, we establish an analytic foundation for a fully non-linear equation $\frac{\sigma_2}{\sigma_1} = f$ on manifolds with metrics of positive scalar curvature and apply it to give a (rough) classification of such manifolds. A crucial point is a simple observation that this equation is a degenerate elliptic equation without any condition on the sign of f and it is elliptic not only for $f > 0$ but also for $f < 0$. By defining a Yamabe constant $Y_{2,1}$ with respect to this equation, we show that a manifold with metrics of positive scalar curvature admits a conformal metric of positive scalar curvature and positive σ_2-scalar curvature if and only if $Y_{2,1} > 0$. We give a complete solution for the corresponding Yamabe problem. Namely, let g_0 be a positive scalar curvature metric, then in its conformal class there is a conformal metric with

$$\sigma_2(g) = \kappa\sigma_1(g),$$

for some constant κ. Using these analytic results, we give a rough classification of the space of manifolds with metrics of positive scalar curvature.

1. Introduction

Let (M, g_0) be a compact Riemannian manifold of dimension n with metric g_0 and $[g_0]$ the conformal class of g_0. Let Ric_g and R_g denote the Ricci tensor and scalar curvature of a metric g respectively. The Schouten tensor of the metric g is defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

The importance of the Schouten tensor in conformal geometry can be viewed in the following decomposition of the Riemann curvature tensor

$$Riem_g = W_g + S_g \owedge g,$$

where $\owedge$ is the Kulkani-Nomizu product and W_g is the Weyl tensor. Note that $g^{-1} \cdot W_g$ is invariant in a given conformal class. Therefore, in a conformal class the Schouten tensor is important.

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The objective of this paper has two-folds. First, we study a class of fully nonlinear quotient equations relating to the scalar curvature and the recently introduced σ_2-scalar curvature. Then, we apply the analysis established for this class of equations to study the space of metrics of positive scalar curvature. Let us first recall the definition of the σ_2-scalar curvature.

For a given $1 \leq k \leq n$, the *σ_k-scalar curvature* or *k-scalar curvature* is defined by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is locally defined by $(g^{-1} \cdot S_g)^i_j = \sum_k g^{ik}(S_g)_{kj}$ and σ_k is the kth elementary symmetric function. Here for an $n \times n$ symmetric matrix A we define $\sigma_k(A) = \sigma_k(\Lambda)$, where $\Lambda = (\lambda_1, \cdots, \lambda_n)$ is the set of eigenvalues of A. It is clear that $\sigma_1(g)$ is a constant multiple of the scalar curvature R_g. The σ_k-scalar curvature $\sigma_k(g)$, which was first considered by Viaclovsky [**60**], is a natural generalization of the scalar curvature.

In this paper, we focus on the σ_2-scalar curvature. One of the reasons to restrict ourself to σ_2-scalar curvature is that $\sigma_2(g)$ still has a variational structure. This is an observation of Viaclovsky [**60**] (For $k > 2$, $\sigma_k(g)$ has a variational structure if and only if the underlying manifold is locally conformally flat [**60**],[**6**]). The variational structure is very crucial for our paper.

Since [**60**] and [**10**], there has been an intensive study for a nonlinear Yamabe problem related to the σ_k-scalar curvature $\sigma_k(g)$, namely, finding a conformal metric g in a given conformal class $[g_0]$ satisfying

$$\sigma_k(g) = c, \tag{1}$$

where

$$g \in [g_0] \cap \Gamma_k^+. \tag{2}$$

Here Γ_k^+ is a convex open cone -the Garding cone- defined by

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, \forall j \leq k\}.$$

By $g \in \Gamma_k^+$ we mean that the Schouten tensor $S_g(x) \in \Gamma_k^+$ for any $x \in M$. $g \in \Gamma_1^+$ is equivalent to that g has positive scalar curvature. Note that the condition $g \in \Gamma_k^+$ guarantees that equation (1) is elliptic.

Equation (1) is a fully nonlinear conformal equation, but it admits many nice properties, which usually are only true for the semilinear equations. For example, the local a priori estimates were established in [**26**], a Liouville type theorem was given in [**45**]. The existence problem of (1) has been intensively studied. For the existence results see [**11**], [**18**], [**19**], [**22**], [**27**], [**29**], [**31**], [**33**], [**45**], [**46**], [**47**] [**54**], [**59**], [**62**]. See also a recent survey by Viaclovsky [**63**] and references therein or the lecture notes of Guan [**21**]. There are many interesting applications in geometry, especially in the 4-dimensional case, see for examples [**10**], [**12**], and [**64**]. See also [**22**], [**23**].

However, till now one can only deal with equations like

$$\sigma_k(g) = f \tag{3}$$

with condition $f \geq 0$ or even $f > 0$. This has something to do with the ellipticity of the corresponding equation. With this restriction we could not use equation (3) to study negative σ_2-scalar curvature. (cf. the work of Gursky and Viaclovsky [**34**] for metrics in Γ^k_- and a class of uniformly elliptic fully nonlinear equations [**29**].) In this paper, instead of $\sigma_2(g)$ we consider the quotient type equation

$$\frac{\sigma_2}{\sigma_1} = f. \tag{4}$$

When $f > 0$, it was studied recently in [**19**]. See also [**28**], [**31**], [**24**] and [**54**]. The crucial point of this paper is that we can also deal with the case with negative functions f for (4). In fact, we have

Observation. *The operator $\frac{\sigma_2}{\sigma_1}$ is elliptic in the cone $\Gamma_1^+ \setminus \mathcal{R}_1$ and concave in Γ_1^+.*

For the notation and the proof of this Observation, see Lemma 1. Recall that the ordinary Yamabe constant is defined

$$Y_1([g_0]) = Y_1(M, [g_0]) := \inf_{g\in\mathcal{C}_1([g_0])} \frac{\int \sigma_1(g) dvol(g)}{(vol(g))^{\frac{n-2}{n}}}.$$

We will call it the *first Yamabe constant.* For simplicity of notation, we denote $\Gamma_k^+ \cap [g_0]$ by $\mathcal{C}_k([g_0])$, or even just $\mathcal{C}_k$ if there is no confusion. A conformal class has positive first Yamabe constant if and only if this class contains a metric of positive scalar curvature. This is a simple, but important fact in the study of metrics of positive scalar curvature. As one of applications of our study of equation (4), we will prove a similar result for the σ_2-scalar curvature. From now on we consider the conformal class $[g]$ with $\mathcal{C}_1 \neq \emptyset$ and call a metric of positive scalar curvature a *psc metric.* Now we first define a "nonlinear eigenvalue" for $\sigma_2(g)$, more precisely for a fully nonlinear operator

$$\sigma_2\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right), \tag{5}$$

where $g = e^{-2u} g_0$. Define

$$\lambda(g_0,\sigma_2) = \lambda(M, g_0, \sigma_2) := \begin{cases} \inf\limits_{g\in\mathcal{C}_1([g_0])} \dfrac{\int \sigma_2(g) dvol(g)}{\int e^{4u} dvol(g)}, & \text{if } n > 4, \\ \displaystyle\int \sigma_2(g) dvol(g), & \text{if } n = 4, \\ \sup\limits_{g\in\mathcal{C}_1([g_0])} \dfrac{\int \sigma_2(g) dvol(g)}{\int e^{4u} dvol(g)}, & \text{if } n = 3. \end{cases} \tag{6}$$

One can also define a similar constant $\lambda(g_0, \sigma_k)$ for σ_k. When $k = 1$, one can check that $\lambda(g_0, \sigma_k)$ is the first eigenvalue of the conformal Laplacian. Hence we call the constant $\lambda(g_0, \sigma_2)$ *nonlinear eigenvalue* of operator (5). Such a nonlinear eigenvalue for fully nonlinear operator was first considered in [**49**]. See also [**65**]. In the context of fully nonlinear conformal operators, it was first considered in a preliminary version of [**27**].

Now we define the second conformal Yamabe constant for a conformal class with psc metrics.

$$Y_{2,1}([g_0]) = Y_{2,1}(M, [g_0]) :=$$

$$\begin{cases} \inf\limits_{g \in \mathcal{C}_1([g_0])} \dfrac{\int \sigma_2(g) dvol(g)}{\left(\int \sigma_1(g) dvol(g)\right)^{\frac{n-4}{n-2}}}, & \text{if } n > 4, \\ \displaystyle\int \sigma_2(g) dvol(g), & \text{if } n = 4, \\ \sup\limits_{g \in \mathcal{C}_1([g_0])} \displaystyle\int \sigma_2(g) dvol(g) \times \int \sigma_1(g) dvol(g), & \text{if } n = 3. \end{cases}$$

We emphasize that the infimum (or supremum) in the definition of λ_2 and $Y_{2,1}$ is taken over $\mathcal{C}_1$, not over $\mathcal{C}_2$. It is well-known that when $n = 4$, $\int_M \sigma_2(g) dvol(g)$ is a constant in a given conformal class. The relationship between the sign of $\lambda(g_0, \sigma_2)$ and $Y_{2,1}([g_0])$ will be discussed in the next section.

Now we state our main analytic results in this paper .

Theorem 1. *Assume $n \geq 3$. The constant $\lambda(g_0, \sigma_2)$ is achieved, provided $\lambda(g_0, \sigma_2) \geq 0$. More precisely, we have*

1) *If $\lambda(g_0, \sigma_2) > 0$, then $\mathcal{C}_2([g_0])$ is not empty and $\lambda(g_0, \sigma_2) > 0$ is achieved by a smooth metric $g = e^{-2u} g_0 \in \mathcal{C}_2$ satisfying*

$$\sigma_2(g) = \lambda e^{4u}. \tag{7}$$

2) *If $\lambda(g_0, \sigma_2) = 0$, then $\lambda(g_0, \sigma_2) = 0$ is achieved by a $C^{1,1}$ metric $g = e^{-2u} g_0 \in \overline{\mathcal{C}}_1$ satisfying*

$$\sigma_2(g) = 0.$$

Here $\overline{\mathcal{C}}_1$ is the closure of $\mathcal{C}_1$. Namely, $\overline{\mathcal{C}}_1$ is the space of metrics with non-negative scalar curvature. With the help of Theorem 1 and results in [**11**], [**33**] and [**19**] we can solve the Yamabe problem for $\frac{\sigma_2}{\sigma_1}$ completely.

Theorem 2. *Let (M^n, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_1^+$ and $n \geq 3$. The following holds*

1) *If* $Y_{2,1}([g_0]) > 0$*, then* $\mathcal{C}_2([g_0])$ *is not empty. Moreover there is a smooth metric* g *in* $\mathcal{C}_2$ *satisfying*

$$\frac{\sigma_2(g)}{\sigma_1(g)} = 1. \tag{8}$$

2) *If* $Y_{2,1}([g_0]) = 0$*, then there is a* $C^{1,1}$ *metric* g *in* $\overline{\mathcal{C}}_1$ *satisfying*

$$\sigma_2(g) = 0. \tag{9}$$

3) *If* $Y_{2,1}([g_0]) < 0$*, then there is a smooth metric* g *in* $\overline{\mathcal{C}}_1$ *satisfying*

$$\sigma_2(g) = -\sigma_1(g). \tag{10}$$

In fact, we can show that $Y_{2,1}$ is achieved when $n > 3$. This will be carried out elsewhere. We remark that in case 3) though the solution metric g is in $\overline{\mathcal{C}}_1$, it is smooth. One can show that any point $x \in M$ with $\sigma_2(g)(x) = 0$ (and hence $\sigma_1(g)(x) = 0$) is Ricci-flat, i.e., $Ric(x) = 0$. We believe that $g \in \mathcal{C}_1$.

A direct consequence of the main analytic results is

Corollary 1. *Let* (M^n, g_0) *be a closed Riemannian manifold of dimension* $n > 2$ *with positive Yamabe constant* $Y_1([g_0]) > 0$*. If* $Y_{2,1}([g_0]) > 0$*, then there is a metric* $g \in [g_0]$ *with* $g \in \Gamma_2^+$*, i.e., with*

$$R_g > 0 \text{ and } \sigma_2(g) > 0.$$

Note that $Y_{2,1}([g_0]) > 0$ if and only if $\lambda(g_0, \sigma_2) > 0$. See Lemma 3 below. When $n = 4$, this result was proved by Chang-Gursky-Yang [**10**]. Namely, for a closed 4-dimensional manifold M^4, if there is a metric g_0 with $Y_1([g_0]) > 0$ and

$$\int_{M^4} \sigma_2(g_0) dvol(g_0) > 0, \tag{11}$$

then there is a metric $g \in [g_0]$ with $R_g > 0$ and $\sigma_2(g) > 0$. Another direct proof was given in [**32**]. See also [**22**] for locally conformally flat manifolds. Many interesting applications to geometry were given in these papers, especially in [**12**]. This result of Chang-Gursky-Yang is in fact one of our main motivations of this paper. From their result, it is natural to ask if this is also true for 3 dimensional manifolds. Corollary 1 gives an affirmative answer. For convenience of the reader, we restate Corollary 1 for $n = 3$.

Corollary 2. *Let* (M^3, g_0) *be a closed manifold of dimension* 3 *with positive Yamabe constant* $Y_1([g_0]) > 0$ *and* $g_0 \in \Gamma_1^+$ *such that*

$$\int_{M^3} \sigma_2(g_0) dvol(g_0) > 0,$$

then there is a metric $g \in [g_0]$ *with* $g \in \Gamma_2^+$*, i.e., with*

$$R_g > 0 \text{ and } \sigma_2(g) > 0.$$

Together with the Hamilton's result [**35**] obtained by the Ricci flow, the Corollary gives

Corollary 3. *Let (M^3, g_0) be a compact 3-dimensional manifold. If $Y_1([g_0]) > 0$ and $Y_{2,1}([g_0]) > 0$, then M is diffeomorphic to a quotient of the sphere.*

The difference between $n = 4$ and $n = 3$ is follows: In $n = 4$, due to the conformal invariance of $\int_{M^4} \sigma_2(g) dvol(g)$, Condition (11) implies that $\int_{M^4} \sigma_2(g) dvol(g) > 0$ for all $g \in [g_0]$. This is not the case for $n = 3$. In fact one can show that for any 3-dimensional manifold (M^3, g_0) there is always a conformal metric $g \in [g_0]$ with $\int_{M^3} \sigma_2(g) dvol(g) < 0$ by using a method given in [**20**].

We remark that the method given in [**32**] and [**22**] can be used to give another proof of Corollary 1 at least for $n > 4$. In another direction, one may ask if the non-emptyness of Γ_2^+ implies the positivity of $Y_{2,1}$. This question is not easy to answer by only using the methods given in [**32**] and [**22**]. In [**19**], we showed that the non-emptyness of Γ_2^+ implies the positivity of

$$\inf_{g \in \mathcal{C}_2} \left(\frac{1}{\int \sigma_1(g) dvol(g))} \right)^{\frac{n-4}{n-2}} \int \sigma_2(g) dvol(g).$$

It is clear that $Y_{2,1}$ is less than or equals to the above constant. With the analysis established here we can show that the non-emptyness of Γ_2^+ implies the positivity of $Y_{2,1}$.

Theorem 3. *Let (M, g) be a compact Riemannian manifold with positive scalar curvature. The second Yamabe constant is positive if and only if Γ_2^+ is non-empty.*

We remark again that the case $n = 4$ was given in [**10**]. As mentioned above, it is well-known that the positivity of the first Yamabe constant Y_1 is equivalent to the existence of a conformal metric of positive scalar curvature. Theorem 3 means that $Y_{2,1}$ has a similar property. Motivated by this result, we hope to use σ_2-scalar curvature to give a further classification of the manifolds admitting metric with positive scalar curvature. This is the second aim of this paper. For the further discussion let us first recall the following definition.

(1_+) Closed connected manifolds with a Riemannian metric whose scalar curvature is non-negative and not identically 0.
(1_0) Closed connected manifolds with a Riemannian metric with non-negative scalar curvature, but not in class (1_+).
(1_-) Closed connected manifolds not in classes (1_+) or (1_0).

There is a remarkable result of Kazdan and Warner obtained in 1975.

Theorem A (Trichotomy Theorem) ([**41**], [**42**]) *Let M^n be a closed connected manifold of dimension $n \geq 3$.*

1. *If M belongs to class (1_+), then every smooth function is the scalar curvature function for some Riemannian metric on M.*
2. *If M belongs to class (1_0), then a smooth function f is the scalar curvature function of some Riemannian metric on M if and only if $f(x) < 0$ for some point $x \in M$, or else $f = 0$. If the scalar curvature of some g vanishes identically, then g is Ricci flat.*
3. *If M belongs to class (1_-), then a smooth function f is the scalar curvature function of some Riemannian metric on M if and only if $f(x) < 0$ for some point $x \in M$.*

The analysis used in the proof of Theorem A is based on the analysis for the eigenvalue problem for the conformal Laplacian operator

$$-\Delta v + \frac{n-2}{4(n-1)} R_g v = \lambda v, \tag{12}$$

where Δ is the Laplacian with respect to g and v is related to u by $v = e^{-\frac{n-2}{2}u}$. And it is closely related to the famous Yamabe equation

$$-\Delta v + \frac{n-2}{4(n-1)} R_g v = k v^{\frac{n+2}{n-2}}, \tag{13}$$

where k is a constant. The existence of solutions for (13) is the so-called Yamabe problem, which was solved by Yamabe, Trudinger, Aubin and Scheon completely.

From Theorem A or an earlier result of Aubin [**2**] we know that a negative function can always be realized as a scalar function of a metric. See also the results of [**17**] and [**50**] for the existence of negative Ricci curvature. The class of (1_0) is very small and consisting of very special manifolds, thanks to a result of Futaki [**15**]. Theorem A also implies that class (1_+) is just the class of manifolds which admit a metric of positive scalar curvature. There are topological obstructions for the manifolds of positive scalar curvature, see [**48**] and [**38**]. This class attracts much attention of geometers for many years, especially after the work of Gromov-Lawson[**20**] and Schoen-Yau [**53**]. The most important problem in this field is the Gromov-Lawson-Rosenberg conjecture which was proved by Stolz [**55**] in the simply connected case. For this conjecture, see for instance [**51**] and [**57**].

Note that it is well-known that class (1_+) is equivalent to

$(1'_+)$ Closed connected manifolds with a Riemannian metric whose scalar curvature is positive.

Now using σ_2-scalar curvature we divide (1_+) further into 3 subclasses:

(2_+) Closed connected manifolds admitting a psc metric whose σ_2-scalar curvature is positive.
(2_0) Closed connected manifolds admitting a psc metric with non-negative σ_2-scalar curvature, but not in class (2_+).
(2_-) Closed connected manifolds in (1_+), but not in classes (2_+) or (2_0).

Remark 1. We believe that class (2_+) is equivalent to the class of closed connected manifolds admitting a psc metric whose σ_2-scalar curvature is non-negative and not identically 0. At the moment we could not prove this equivalence.

Analog to the Trichotomy Theorem of Kazdan-Warner for the scalar curvature and in view of the analysis established here, we propose the following

Conjecture (Trichotomy Theorem) Let M^n $(n > 2)$ be a closed connected manifold in class (1_+).

1. If M belongs to class (2_+), then for every smooth function f there is a psc metric g such that $f\sigma_1(g)$ is its σ_2-scalar curvature.
2. If M belongs to class (2_0), then for a smooth function f, M admits a psc metric g with $\sigma_2(g) = f\sigma_1(g)$ if and only if $f(x) < 0$ for some point $x \in M$, or else $f = 0$.
3. If M belongs to class (2_-), then for a smooth function f, M admits a psc metric g with $\sigma_2(g) = f\sigma_1(g)$ if and only if $f(x) < 0$ for some point $x \in M$.

Though we could not prove this conjecture at moment, we have the following results support this conjecture.

Theorem 4. *Any Riemannian manifold M^n $(n \geq 4)$ in the class (1_+) admits a metric with non-positive σ_2-scalar curvature and non-negative scalar curvature.*

The metric given in Theorem 4 satisfies $\int \sigma_2(g) dvol(g) < 0$ and has vanishing Ricci curvature at points with $\sigma_1(g)(x) = 0$.

Theorem 5. *Let M^n be a closed connected manifold of dimension $n > 3$.*

1. *If M belongs to class (2_+), then for every constant b there is a metric g (with non-negative scalar curvature when $b \leq 0$ and positive scalar curvature when $b > 0$) such that $b\sigma_1(g)$ is its σ_2-scalar curvature.*
2. *If M belongs to class (2_0), then for a constant b, M admits a metric g (with non-negative scalar curvature when $b \leq 0$ and positive scalar curvature when $b > 0$) such that $\sigma_2(g) = b\sigma_1(g)$ if and only if $b \leq 0$.*

3. *If M belongs to class (2_-), then for a constant b, M admits a metric g(with non-negative scalar curvature when $b \le 0$ and positive scalar curvature when $b > 0$) such that $\sigma_2(g) = b\sigma_1(g)$ if and only if $b < 0$.*

The metrics found in Theorem 5 may have points with vanishing σ_1-scalar curvature. At such points its Ricci tensor also vanishes. When $b > 0$, then the metric has positive scalar curvature. It is trivial that the sphere belongs to (2_+). By Corollary 3, in 3-dimension, only the sphere and its quotients belong to (2_+) and there are no manifolds in (2_0). In 4-dimension, many examples of manifolds with metrics in Γ_2^+ can be found in [**10**]. Those manifolds certainly belong to (2_+). One can prove that $\mathbb{S}^3 \times \mathbb{S}^1$ belongs to (2_0), namely there is no positive scalar metric on $\mathbb{S}^3 \times \mathbb{S}^1$ with positive σ_2-scalar curvature, see Section 7 below. Like class (1_0), class (2_0) should be very small. However, when $n > 4$, we have no example of manifolds in class (1_+) which do not belong to (2_+). This is somewhat strange. A possible candidate is the manifold $\mathbb{S}^6 \times H^3$, where H^3 is a compact quotient of the hyperbolic space $\mathbb{H}^3$. It is easy to check that its product metric g_P satisfies that $\sigma_1(g_P) > 0$ and $\sigma_2(g_P) = 0$. However, we believe that $\mathbb{S}^6 \times \mathbb{H}^3$ has a metric g with $\sigma_1(g) > 0$ and $\sigma_2(g) > 0$. See Example 2 in Section 7. Therefore we may ask

Problem. *Is there a topological obstruction for the existence of psc metric with positive σ_2-scalar curvature when $n > 4$?*

For the scalar curvature as mentioned above there are topological obstructions. What we ask is to find further conditions to distinguish manifolds between (2_+), (2_0) and (2_+) for higher dimensional manifolds. There is a similar and related problem proposed by Stolz in [**56**] for further obstructions to the existence of metrics with positive Ricci tensor. For some relationship between the positive Ricci tensor and positive σ_k-scalar curvature, see [**25**]. With our analysis, the problem to find a topological obstruction for σ_2-scalar curvature perhaps might be not very difficult.

The paper is organized as follows. In Section 2, we show the Observation and discuss the relationship between $\lambda(g_0, \sigma_2)$ and $Y_{2,1}$. We introduce a class of perturbed equations (28) and a Yamabe type flow (31) in Section 3 and establish local a priori estimates for these equations and flows in Section 4. In Section 5 we prove the global existence of the Yamabe type flow and Theorem 1 and Theorem 2. One of another crucial points of this paper, the Yamabe type flow preserves the positivity of the scalar curvature, will also be proved in this section. In Section 6, we show Theorem 3. We give the geometric applications in

Section 7 by proving Theorems 4 and 5. In the last section, we mention further applications in similar equations.

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After the paper has been circulated and submitted, the first and third authors were kindly informed by Paul Yang in a conference in CIRM in June 2007 that Catino and Djadli just announced a similar result to Corollary 2. The preprint appeared later in Arxiv [**9**].

We would like to thank the referees for their critical reading and useful suggestions.

2. Some preliminary facts

Let $\mathcal{S}_n$ be the space of $n\times n$ real symmetric matrices and F a smooth function in $\mathcal{S}_n$. By extending F to the whole space of $n\times n$ real matrices by $F(A) = F(\frac{1}{2}(A+A^t))$, we view F as a function of $n\times n$ variables w_{ij} and define

$$F^{ij} = \frac{\partial F}{\partial w_{ij}}.$$

The quotient function $\frac{\sigma_2}{\sigma_1}$ can be viewed as a function in $\mathcal{S}_n$ as follows. Let W be a symmetric matrix and $\Lambda_W = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ its eigenvalues. Then we define $\frac{\sigma_2}{\sigma_1}(W) = \frac{\sigma_2(\Lambda_W)}{\sigma_1(\Lambda_W)}$. A symmetric matrix is said to be in Γ_k^+ if $\Lambda_W \in \Gamma_k^+$. Let $\mathcal{R}_1$ be the subspace of $\mathcal{S}_n$ consisting of symmetric matrices of rank 1.

Lemma 1. *For $1 < k \le n$ set $F = \frac{\sigma_k}{\sigma_{k-1}}$. We have*

1) *the matrix $(F^{ij})(W)$ is semi-positive definite at $W \in \Gamma_{k-1}^+$ and is positive definite at $W \in \Gamma_{k-1}^+ \backslash \mathcal{R}_1$.*
2) *The function F is concave in the cone Γ_{k-1}^+. When $k=2$, for all $W \in \Gamma_1^+$ and for all $R = (r_{ij}) \in \mathcal{S}_n$, we have*

$$\sum_{ijkl} \frac{\partial^2}{\partial w_{ij}\partial w_{kl}} \left(\frac{\sigma_2(W)}{\sigma_1(W)}\right) r_{ij} r_{kl} \quad = \quad -\frac{\sum_{ij}(\sigma_1(W) r_{ij} - \sigma_1(R) w_{ij})^2}{\sigma_1^3(W)}. \tag{14}$$

Proof. For $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $i \in \{1, 2, \cdots, n\}$ let $\Lambda_i = \{1, 2, \cdots, \hat{i}, \cdots, n\}$ be the $(n-1)$-tuple obtained from Λ without the ith-component. A direct calculation gives

$$\frac{\partial F}{\partial \lambda_i} = \frac{\sigma_{k-1}(\Lambda_i)\sigma_{k-1}(\Lambda) - \sigma_k(\Lambda)\sigma_{k-2}(\Lambda_i)}{\sigma_{k-1}(\Lambda)^2}. \tag{15}$$

Since $\Lambda \in \Gamma_{k-1}^+$, $\Lambda_i \in \Gamma_{k-2}^+$. For the proof see for instance [**39**]. From two identities

$$\sigma_k(\Lambda) = \lambda_i \sigma_{k-1}(\Lambda_i) + \sigma_k(\Lambda_i) \text{ and } \sigma_{k-1}(\Lambda) = \lambda_i \sigma_{k-2}(\Lambda_i) + \sigma_{k-1}(\Lambda_i),$$

(15) becomes

$$\frac{\partial F}{\partial \lambda_i} = \frac{\sigma_{k-1}^2(\Lambda_i) - \sigma_k(\Lambda_i)\sigma_{k-2}(\Lambda_i)}{\sigma_{k-1}(\Lambda)^2}. \tag{16}$$

Note that for convenience we set $\sigma_0(\Lambda) = 1$. By the Newton-McLaughlin inequality

$$(k-1)(n-k)\sigma_{k-1}^2(\Lambda_i) \geq k(n-k+1)\sigma_k(\Lambda_i)\sigma_{k-2}(\Lambda_i),$$

we have

$$\frac{\partial F}{\partial \lambda_i} \geq 0$$

and equality implies that $\Lambda_i = \{0, 0, \cdots, 0\}$, i.e., $\Lambda = \{0, \cdots, 0, \lambda_i, 0, \cdots, 0\}$. This proves 1).

2) was proved in [**39**]. q.e.d.

Though Lemma 1 is a rather simple fact, as mentioned in the introduction it is one of crucial points of our paper. (We remark that this Lemma might be observed also by other mathematicians. For example in [**14**] there is a rather similar formula. We would like to thank Tobias Lamm who told us this reference.) It means that $\frac{\sigma_2}{\sigma_1}$ is a concave, degenerate elliptic operator in Γ_1^+. With this observation in mind, we consider a family of perturbed operators for a positive number $\nu \in \mathbb{R}^+$

$$\begin{aligned} F_\nu : \quad \Gamma_1^+ \quad &\to \quad \mathbb{R} \\ W \quad &\mapsto \quad F_\nu(W) = \frac{\sigma_2(W) - \nu}{\sigma_1(W)}. \end{aligned} \tag{17}$$

As a direct consequence of Lemma 1, we have the following:

Lemma 2. *The matrix $(F_\nu^{ij})(W)$ is positive definite for all $W \in \Gamma_1^+$ and the function F_ν is strictly concave in Γ_1^+. Moreover, for all $W \in \Gamma_1^+$ and for all $R = (r_{ij}) \in \mathcal{S}_n(\mathbb{R})$*

$$\sum_{ijkl} \frac{\partial^2 F_\nu(W)}{\partial w_{ij} \partial w_{kl}} r_{ij} r_{kl} \quad \leq \quad -\frac{2\nu}{\sigma_1^3(W)} \left(\sum_i r_{ii} \right)^2. \tag{18}$$

Lemma 2 means that for any positive number $\nu > 0$, the operator F_ν is elliptic and strictly concave.

Now we discuss the relationship between the sign of the invariants $Y_{2,1}([g_0])$ and that of $\lambda(g_0, \sigma_2)$ defined in the introduction.

Lemma 3. *The eigenvalue $\lambda(g_0, \sigma_2)$ is a finite number. Moreover, we have*

(1) *if $n \geq 4$, then $\lambda(g_0, \sigma_2) > 0$ (resp. $= 0$, < 0) if and only if $Y_{2,1}([g_0]) > 0$ (resp. $= 0$, < 0);*
(2) *if $n = 3$, then $\lambda(g_0, \sigma_2) > 0$ (resp. ≤ 0) if and only if $Y_{2,1}([g_0]) > 0$ (resp. ≤ 0).*

Proof. The following formula was given in the proof of the Sobolev inequality in [**18**]: for any $g = e^{-2u} g_0 \in [g_0]$ we have

(19)
$$\begin{aligned} & 2\int \sigma_2(g) dvol(g) \\ = \ & \frac{n-4}{2}\int \sigma_1(g)|\nabla u|^2_{g_0} e^{2u} dvol(g) + \frac{n-4}{4}\int |\nabla u|^4_{g_0} e^{4u} dvol(g) \\ & + \int \sum_{i,j} T^{ij} S(g_0)_{ij} dvol(g) + \int \sum_{i,j} S(g_0)^{ij} u_i u_j dvol(g) \\ & + \frac{1}{2}\int \sigma_1(g_0)|\nabla u|^2_{g_0} e^{4u} dvol(g), \end{aligned}$$

where $T^{ij} = \sigma_1(W) g^{ij} - W^{ij}$ is the first Newton transformation associated with W. From this formula, we have

(20)
$$\begin{aligned} & 2\int \sigma_2(g) dvol(g) \\ = \ & \frac{n-4}{2}\int \sigma_1(g)|\nabla u|^2_{g_0} e^{2u} dvol(g) + \frac{n-4}{4}\int |\nabla u|^4_{g_0} e^{4u} dvol(g) \\ & + \int e^{2u}\sigma_1(g)\sigma_1(g_0) dvol(g) - \int e^{4u}|S(g_0)|^2_{g_0} dvol(g) \\ & + (4-n)\int \sum_{i,j} S(g_0)^{ij} u_i u_j dvol(g) + \int \sigma_1(g_0)|\nabla u|^2_{g_0} e^{4u} dvol(g) \\ & + \int e^{4u}\langle \nabla u, \nabla \sigma_1(g_0)\rangle_{g_0} dvol(g). \end{aligned}$$

We first consider the case $n \geq 5$. By (20), there exists some $c > 0$ such that

(21)
$$\int \sigma_2(g)dvol(g) \quad \geq \quad \frac{n-4}{16}\int |\nabla u|^4_{g_0} e^{4u} dvol(g) - c\int e^{4u} dvol(g),$$

provided that $g \in \Gamma_1^+$. Here we have used the fact

$$\begin{aligned} & \left|\int \sum_{i,j} S(g_0)^{ij} u_i u_j dvol(g)\right| \\ \leq \; & \frac{1}{8}\int |\nabla u|^4_{g_0} e^{4u} dvol(g) + 2\sup_M |S(g_0)|^2_{g_0}\int e^{4u} dvol(g) \end{aligned}$$

and

$$\left|\int e^{4u}\langle \nabla u, \nabla\sigma_1(g_0)\rangle_{g_0}\right| \leq \int \sigma_1(g_0)|\nabla u|^2_{g_0} e^{4u} dvol(g) + c\int e^{4u} dvol(g).$$

As a consequence, we conclude

$$\lambda(g_0, \sigma_2) \geq -c.$$

Similarly, in case $n = 3$, we have for all $g \in \mathcal{C}_1([g_0])$

$$\begin{aligned} & \int \sigma_2(g)dvol(g) \\ = \; & -\frac{1}{4}\int \sigma_1(g)|\nabla u|^2_{g_0} e^{2u} dvolg - \frac{1}{8}\int |\nabla u|^4_{g_0} e^{4u} dvol(g) \\ & -\frac{1}{4}\int e^{4u}|\nabla u|^2_{g_0}\sigma_1(g_0)dvol(g) \\ & +\frac{1}{2}\int e^{4u}(\sigma_1^2(g_0) - |S(g_0)|^2_{g_0})dvol(g) \\ & +\frac{1}{2}\int \sum_{i,j} S(g_0)^{ij} u_i u_j dvol(g) \\ \leq \; & -\frac{1}{16}\int |\nabla u|^4_{g_0} e^{4u} dvol(g) + c\int e^{4u} dvol(g), \end{aligned} \tag{22}$$

which yields $\lambda(g_0, \sigma_2) \leq c$. Thus, we finish the proof of the first part of Lemma.

Now we begin to prove the second part. It is easy to see that for all $g \in \mathcal{C}_1([g_0])$

(23)
$$\begin{aligned}&\int_M \sigma_1(g)dvol(g)\\ = & \int\left(\frac{n-2}{2}|\nabla u|^2+\sigma_1(g_0)\right)e^{2u}dvol(g)\geq Y_1([g_0])(Vol(g))^{\frac{n-2}{n}}.\end{aligned}$$

Now we consider the case $n\geq 5$. Suppose $\lambda(g_0,\sigma_2)>0$, i.e.,
$$\int_M \sigma_2(g)dvol(g)\geq \lambda(g_0,\sigma_2)\int_M e^{4u}dvol(g)$$
for any $g=e^{-2u}g_0\in\Gamma_1^+$. From (21), (23) and Hölder's inequality, we deduce
$$\begin{aligned}\int\sigma_2(g)dvol(g) &\geq c\left(\int|\nabla u|^4_{g_0}e^{4u}dvol(g)+\int e^{4u}dvol(g)\right)\\ &\geq c\left(\int_M\sigma_1(g)dvol(g)\right)^2(Vol(g))^{-1} \qquad (24)\\ &\geq cY_1([g_0])^{\frac{n}{n-2}}\left(\int_M\sigma_1(g)dvol(g)\right)^{\frac{n-4}{n-2}},\end{aligned}$$
which implies $Y_{2,1}([g_0])>0$. Conversely, using Hölder's inequality and (23), we have
(25)
$$\begin{aligned}\int\sigma_2(g)dvol(g) &\geq c\left(\int_M\sigma_1(g)dvol(g)\right)^{\frac{n-4}{n-2}}\\ &\geq cY_1([g_0])^{\frac{n-4}{n-2}}(Vol(g))^{\frac{n-4}{n}}\geq c\int e^{4u}dvol(g).\end{aligned}$$
This gives the desired result. Clearly, we have $\lambda(g_0,\sigma_2)<0$ if and only if $Y_{2,1}([g_0])<0$. Consequently, $\lambda(g_0,\sigma_2)=0$ if and only if $Y_{2,1}([g_0])=0$. In the cases $n=4$ and $n=3$, the result is trivial.

q.e.d.

Remark 2. In the case $n=3$, if $\lambda(g_0,\sigma_2)=\lambda<0$, then $Y_{2,1}([g_0])<0$. To see this, for any $g\in\mathcal{C}_1([g_0])$, it follows from Hölder's inequality and (23) that there holds
$$\begin{aligned}&\int\sigma_1(g)dvol(g)\int\sigma_2(g)dvol(g)\\ \leq\ &\lambda\int\sigma_1(g)dvol(g)\int e^{4u}dvol(g) \qquad (26)\\ \leq\ &c\lambda\int e^{-u}dvol(g_0)\int e^{u}dvol(g_0)\leq c\lambda<0.\end{aligned}$$

Remark 3. In the case $n \geq 5$, the invariant $Y_{2,1}([g_0])$ is finite real number. To see this, for any $g \in \mathcal{C}_1([g_0])$ it follows from the Hölder's inequality, (21) and (23) that
(27)
$$\begin{aligned} \frac{\int \sigma_2(g)dvol(g)}{\left(\int \sigma_1(g)dvol(g)\right)^{\frac{n-4}{n-2}}} &\geq -c\frac{\int e^{4u}dvol(g)}{\left(\int \sigma_1(g)dvol(g)\right)^{\frac{n-4}{n-2}}} \\ &\geq -c\frac{(Vol(g))^{\frac{n-4}{n}}}{\left(\int \sigma_1(g)dvol(g)\right)^{\frac{n-4}{n-2}}} \geq -c > -\infty. \end{aligned}$$

In the case $n = 3$, we do not know if $Y_{2,1}([g_0]) < +\infty$ or not, although it is always true that $Y_{2,1}([g_0]) > 0$ if and only if $\lambda(g_0, \sigma_2) > 0$, and that $\lambda(g_0, \sigma_2)$ is finite. However, we believe that it is true.

3. Yamabe type flows

Now we want to consider the existence of the following equation

$$F_\nu(g) = \frac{\sigma_2(g) - \nu e^{4u}}{\sigma_1} = \text{constant}, \tag{28}$$

with $g = e^{-2u}g_0$ and $\nu > 0$ a positive number (we could consider $\nu : M \to \mathbb{R}^+$ a positive function, but in this paper we will choose ν as a small positive constant). Following [**27**], [**18**] and [**19**] we will introduce a suitable Yamabe type flow to study equation (28).

For any $\nu \in (0, +\infty)$ and for $g = e^{-2u}g_0$, consider the following perturbed functional

$$\mathcal{E}_\nu(g) := \begin{cases} \dfrac{2}{n-4}\displaystyle\int_M (\sigma_2(g) - \nu e^{4u})dvol(g), & \text{if } n \neq 4, \\ -\displaystyle\int_0^1 \int_M (\sigma_2(g_t) - 2\nu e^{4tu})udvol(g_t)dt, & \text{if } n = 4, \end{cases}$$

where $g_t = e^{-2tu}g_0$. When $\nu = 0$, the functional was considered in [**60**], [**10**] and [**7**]. Set

$$\mathcal{F}_1(g) = \int_M \sigma_1(g)dvol(g) \quad \text{ and } \quad \mathcal{F}_2(g) = \int_M \sigma_2(g)dvol(g)$$

From the variational formula given in [**60**], [**10**] and [**7**], we have

$$\frac{d}{dt}\mathcal{E}_\nu(g) = \int (\sigma_2(g) - \nu e^{4u})g^{-1} \cdot \frac{d}{dt}gdvol(g) \tag{29}$$

and

$$\frac{d}{dt}\mathcal{F}_1(g) = \frac{n-2}{2}\int \sigma_1(g)g^{-1} \cdot \frac{d}{dt}gdvol(g). \tag{30}$$

Now we introduce a Yamabe type flow, which non-increases $\mathcal{E}_\nu$ and preserves $\mathcal{F}_1$.

$$\frac{du}{dt} = -\frac{1}{2}g^{-1}\frac{d}{dt}g := e^{-2u}\frac{\sigma_2(g) - \nu e^{4u}}{\sigma_1(g)} - r_\nu(g)e^{-2u} + s_\nu(g), \tag{31}$$

where $r_\nu(g)$ and $s_\nu(g)$ are space constants, given by

$$r_\nu(g) := \frac{\mathcal{F}_2(g) - \int_M \nu e^{4u} dvol_g}{\mathcal{F}_1(g)} \tag{32}$$

and

$$\int_M \sigma_1(g)\left\{e^{-2u}\frac{\sigma_2(g) - \nu e^{4u}}{\sigma_1(g)} - r_\nu(g)e^{-2u} + s_\nu(g)\right\} dvol(g) = 0. \tag{33}$$

Lemma 4. *Flow (31) preserves $\mathcal{F}_1$ and non-increases $\mathcal{E}_\nu$. Hence when $n \geq 4$, then r_ν is non-increasing along the flow, and when $n = 3$, then r_ν is non-decreasing along the flow.*

Proof. By the definition of $s_\nu(g)$ and (30), flow (31) preserves $\mathcal{F}_1$. By the definition of s_ν and r_ν, we can compute as follows

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_\nu(g) &= \int_M (\sigma_2(g) - \nu e^{4u})g^{-1} \cdot \frac{d}{dt} g dvol(g) \\ &= -2\int e^{2u}\sigma_1(g)\left(e^{-2u}\frac{\sigma_2(g) - \nu e^{4u}}{\sigma_1(g)} - r_\nu(g)e^{-2u}\right)^2 dvol(g). \end{aligned} \tag{34}$$

Therefore, the desired result yields. q.e.d.

4. Local estimates

In this section, we will establish a priori estimates for flow (31) and equations (8), (9) and (10). Local estimates for this class of fully nonlinear conformal equations were first given in [**26**]. Since then there are many extensions. See for instance [**13**] and the survey paper [**63**]. It is important to note that the a priori estimates established below do not depend on the perturbation $\nu > 0$.

Given $\nu > 0$, assume $g_0 \in \mathcal{C}_1([g_0])$. By Lemma 2, (31) is parabolic. By the standard implicit function theorem we have the short-time existence result. Let $T^* \in (0, \infty]$ so that $[0, T^*)$ is the maximum interval for the existence of the flow $g(t) \in \Gamma_1^+$.

Theorem 6. *Assume that $n \geq 3$, $\nu > 0$ and $g_0 \in \Gamma_1^+$. Let u be a solution of (31) in a geodesic ball $B_R \times [0, T]$ for $T < T^*$ and $R < \tau_0$, the injectivity radius of M.*

(1) *Assume that $\forall t \in [0, T]$*

$$r_\nu(t) \leq 0.$$

Then there is a constant C depending only on (B_R, g_0) (independent of ν and T) such that for any $(x,t) \in B_{R/2} \times [0,T]$

$$|\nabla u|^2 + |\nabla^2 u| \le C. \tag{35}$$

(2) *Assume that $\forall t \in [0,T]$*

$$r_\nu(t) > 0.$$

Then there is a constant C depending only on (B_R, g_0) (independent of ν and T) such that for any $(x,t) \in B_{R/2} \times [0,T]$

$$|\nabla u|^2 + |\nabla^2 u| \le C \left(1 + \sup_{t\in[0,T]} r_\nu(t) \times e^{-2\inf_{(x,t)\in B_R\times[0,T]} u(x,t)}\right). \tag{36}$$

In particular, if we assume $n \ge 4$, we have

$$|\nabla u|^2 + |\nabla^2 u| \le C \left(1 + e^{-2\inf_{(x,t)\in B_R\times[0,T]} u(x,t)}\right). \tag{37}$$

Proof. In the proof, C is a constant independent of T and ν, which may vary from line to line. Let $W = (w_{ij})$ be an $n \times n$ matrix with $w_{ij} = \nabla^2_{ij} u + u_i u_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}$. Here u_i and u_{ij} are the first and second derivatives of u with respect to the background metric g_0. Recall $F_\nu(W) = \dfrac{\sigma_2(W) - \nu}{\sigma_1(W)}$. Set

$$\begin{aligned}(F_\nu^{ij}(W)) \quad &:= \quad \left(\frac{\partial F_\nu}{\partial w_{ij}}(W)\right)\\ &= \quad \left(\frac{\sigma_1(W)T^{ij} - \sigma_2(W)\delta^{ij} + \nu\delta^{ij}}{\sigma_1^2(W)}\right)\end{aligned} \tag{38}$$

where $(T^{ij}) = (\sigma_1(W)\delta^{ij} - w^{ij})$ is the first Newton transformation associated with W, and δ^{ij} is the Kronecker symbol. In view of Lemma 2 we know that (F_ν^{ij}) is positive definite and F_ν is concave in Γ_1^+. For the simplicity of notation, we now drop the index ν, if there is no confusion. We try to show the local estimates for first and second order derivatives together. Let $S(TM)$ denote the unit tangent bundle of M with respect to the background metric g_0. We define a function $\tilde{G}: \ S(TM) \times [0,T] \ \to \mathbb{R}$

$$\tilde{G}(e,t) = (\nabla^2 u + |\nabla u|^2 g_0)(e,e) \tag{39}$$

Without loss of generality, we assume $R = 1$. Let $\rho \in C_0^\infty(B_1)$ be a cut-off function defined as in [**26**] such that

$$\begin{aligned} \rho &\geq 0, && \text{in } B_1, \\ \rho &= 1, && \text{in } B_{1/2}, \\ |\nabla \rho(x)| &\leq 2b_0 \rho^{1/2}(x), && \text{in } B_1, \\ |\nabla^2 \rho| &\leq b_0, && \text{in } B_1. \end{aligned} \tag{40}$$

Here $b_0 > 1$ is a constant. Since $e^{-2u}g_0 \in \Gamma_1^+$, to bound $|\nabla u|$ and $|\nabla^2 u|$ we only need to bound $(\nabla^2 u + |\nabla u|^2 g_0)(e,e)$ from above for all $e \in S(TM)$ and for all $t \in [0,T]$. For this purpose, denote $G(e,t) = \rho(x)\tilde{G}(e,t)$. Assume $(e_1, t_0) \in S(T_{x_0}M) \times [0,T]$ such that

$$G(e_1, t_0) = \max_{S(TM)\times[0,T]} G(e,t), \tag{41}$$

$$t_0 > 0, \tag{42}$$

$$G(e_1, t_0) > n \max_{B_1} \sigma_1(g_0). \tag{43}$$

Let $(e_1, \cdots, e_n)$ be a orthonormal basis at point (x_0, t_0). It follows from the fact $W \in \Gamma_1^+$

$$\begin{aligned} nG(e_1,t_0) &\geq \rho(\Delta u + n|\nabla u|^2) \geq \rho\left(n|\nabla u|^2 + \frac{n-2}{2}|\nabla u|^2 - \sigma_1(g_0)\right), \\ &\geq \frac{3n-2}{2}\rho|\nabla u|^2 - \frac{1}{n}G(e_1,t_0), \end{aligned}$$

so that

$$G(e_1,t_0) \geq \frac{\frac{3n-2}{2}}{n+\frac{1}{n}}\rho|\nabla u|^2 \geq \frac{21}{20}\rho|\nabla u|^2.$$

Consequently, we obtain

$$\nabla_{11}^2 u(x_0,t_0) \geq \frac{1}{20}|\nabla u|^2(x_0,t_0). \tag{44}$$

Set for any $i \neq j = 1, \cdots, n$

$$e' = \frac{1}{\sqrt{2}}(e_i \pm e_j).$$

We have

$$G(e',t_0) = \frac{1}{2}(G(e_i,t_0) + G(e_j,t_0)) \pm \rho\nabla_{ij}^2 u(x_0,t_0). \tag{45}$$

Thus, there holds

$$\rho|\nabla_{ij}^2 u(x_0,t_0)| \leq G(e_1,t_0) - \frac{1}{2}(G(e_i,t_0) + G(e_j,t_0)). \tag{46}$$

On the other hand, we have $\forall i = 1, \cdots, n$

$$(n-1)G(e_1,t_0) + G(e_i,t_0) \geq \rho(\Delta u + n|\nabla u|^2) \geq \tag{47}$$

$$\rho\left(\frac{3n-2}{2}|\nabla u|^2-\sigma_1(g_0)\right),$$

which implies

$$G(e_i,t_0)\geq\rho\left(\frac{3n-2}{2}|\nabla u|^2-\sigma_1(g_0)\right)-(n-1)G(e_1,t_0). \tag{48}$$

Together with (46), we deduce

(49)

$$\rho|\nabla^2_{ij}u(x_0,t_0)|\leq nG(e_1,t_0)-\frac{3n-2}{2}\rho|\nabla u|^2+\rho\sigma_1(g_0)\leq(n+1)G(e_1,t_0).$$

(Indeed, at any point (x,t), the estimate $\rho|\nabla^2_{ij}u|\leq(n+1)G(e_1,t_0)$ holds). Now choose the normal coordinates around x_0 such that at point x_0

$$\frac{\partial}{\partial x_1}=e_1$$

and consider the function G on $M\times[0,T]$ defined by

$$G(x,t):=\rho(x)(u_{11}+|\nabla u|^2)(x,t).$$

Clearly, (x_0,t_0) is a maximum point of $G(x,t)$ on $M\times[0,T]$. At (x_0,t_0), we have

$$0\leq G_t=\rho\left(u_{11t}+2\sum_l u_lu_{lt}\right), \tag{50}$$

$$0=G_j=\frac{\rho_j}{\rho}G+\rho\left(u_{11j}+2\sum_{l\geq1}u_lu_{lj}\right),\qquad\text{for any } j, \tag{51}$$

(52) $\quad 0\geq$

$$(G_{ij})=\left(\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho(u_{11ij}+\sum_{l\geq1}(2u_{li}u_{lj}+2u_lu_{lij}))\right).$$

Recall that (F^{ij}) is definite positive. Hence, we have

$$\begin{aligned}0&\geq\sum_{i,j\geq1}F^{ij}G_{ij}-G_t\\&\geq\sum_{i,j\geq1}F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G\\&\quad+\rho\textstyle\sum_{i,j\geq1}F^{ij}\left(u_{11ij}+\sum_{l\geq1}(2u_{li}u_{lj}+2u_lu_{lij})\right)\\&\quad-\rho\left(u_{11t}+2\displaystyle\sum_{l\geq1}u_lu_{lt}\right).\end{aligned} \tag{53}$$

First, from the definition of ρ, we have

$$\sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G \geq -C\sum_{i,j\geq 1}|F^{ij}|\frac{1}{\rho}G, \tag{54}$$

and

$$\begin{aligned} \sum_{i,j\geq 1}|F^{ij}| &\geq \sum_i F^{ii} \\ &= \left(n-1-\frac{n\sigma_2(W)}{\sigma_1^2(W)}\right)+\frac{n\nu}{\sigma_1^2(W)} \geq C\sum_{i,j\geq 1}|F^{ij}| \end{aligned} \tag{55}$$

since W is positive definite . Using the facts that

$$u_{kij} = u_{ijk} + \sum_m R_{mikj}u_m, \tag{56}$$

$$u_{kkij} = \tag{57}$$

$$u_{ijkk}+\sum_m(2R_{mikj}u_{mk}-Ric_{mj}u_{mi}-Ric_{mi}u_{mj}-Ric_{mi,j}u_m+R_{mikj,k}u_m)$$

and

$$\left(\sum_l u_l^2\right)_{11} = 2\sum_l(u_{11l}u_l+u_{1l}^2)+O(|\nabla u|^2), \tag{58}$$

we have

(59)

$$\begin{aligned} &\sum_{i,j\geq 1}F^{ij}u_{11ij} \\ \geq &\sum_{i,j\geq 1}F^{ij}\left(w_{ij11}-(u_{11})_iu_j-u_i(u_{11})_j+\sum_{l\geq 1}(u_{1l}^2+u_{11l}u_l)(g_0)_{ij}\right) \\ &-2\sum_{i,j\geq 1}F^{ij}u_{i1}u_{j1}-C(1+|\nabla^2u|+|\nabla u|^2)\sum_{i,j\geq 1}|F^{ij}| \end{aligned}$$

and

(60)

$$\begin{aligned} &\sum_{i,j,l}F^{ij}u_lu_{lij} \\ \geq &\sum_{i,j,l}F^{ij}u_lw_{ijl}-\sum_{i,j,l}F^{ij}(u_lu_{il}u_j+u_lu_iu_{jl}) \\ &+\frac{1}{2}\sum_{i,j}F^{ij}\langle\nabla u,\nabla(|\nabla u|^2)\rangle(g_0)_{ij}-C(1+|\nabla u|^2)\sum_{i,j\geq 1}|F^{ij}|. \end{aligned}$$

Combining (59) and (60), we deduce

(61)
$$\begin{aligned}
&\sum_{i,j\geq 1} F^{ij}\left(u_{11ij}+2\sum_{l\geq 1}(u_{li}u_{lj}+u_l u_{lij})\right)\\
\geq\ & \sum_{i,j\geq 1} F^{ij}\left(w_{ij11}+2\sum_{l\geq 1} w_{ijl}u_l\right)\\
&+2\textstyle\sum_{i,j\geq 1} F^{ij}\sum_{l\geq 2} u_{li}u_{lj}+\sum_{i,j,l\geq 1} u_{1l}^2 F^{ij}(g_0)_{ij}\\
&-\sum_{i,j} F^{ij}\Big[(u_{11}+|\nabla u|^2)_i u_j + u_i(u_{11}+|\nabla u|^2)_j\\
&-\langle \nabla u, \nabla(u_{11}+|\nabla u|^2)\rangle (g_0)_{ij}\Big] - C(1+|\nabla^2 u|+|\nabla u|^2)\sum_{i,j\geq 1}|F^{ij}|\\
\geq\ & \sum_{i,j} F^{ij}(w_{ij11}+2\sum_l w_{ijl}u_l)+u_{11}^2\sum_{i,j}F^{ij}(g_0)_{ij}\\
&+\sum_{i,j}F^{ij}\left(\rho_i u_j+\rho_j u_i-\langle\nabla\rho,\nabla u\rangle(g_0)_{ij}\right)\frac{G}{\rho^2}\\
&-C(1+|\nabla^2 u|+|\nabla u|^2)\textstyle\sum_{i,j\geq 1}|F^{ij}|.
\end{aligned}$$

Now, we want to estimate $\sum_{i,j,l} F^{ij}w_{ijl}u_l$ and $\sum_{i,j} F^{ij}w_{ij11}$ respectively. Using Lemma 1, there holds

$$\sum_{i,j,l} F^{ij} w_{ijl}u_l \quad = \quad \sum_l F_l u_l, \tag{62}$$

and

$$\sum_{i,j} F^{ij} w_{ij11} \quad = \quad F_{11} - \sum_{i,j,k,m}\frac{\partial^2 F}{\partial w_{ij}\partial w_{km}} w_{ij1}w_{km1} \geq F_{11}. \tag{63}$$

Therefore, these estimates give

$$\sum_{i,j\geq 1} F^{ij}\left(w_{ij11}+2\sum_{l\geq 1} w_{ijl}u_l\right) \geq F_{11}+2\sum_l F_l u_l. \tag{64}$$

Recall from (31) that

$$F = u_t + r_\nu(g)e^{-2u} - s_\nu(g), \tag{65}$$

so that

$$F_{11} = u_{11t} + r_\nu(g)e^{-2u}(-2u_{11}+4u_1^2), \tag{66}$$

$$F_l = u_{lt} + r_\nu(g)e^{-2u}(-2u_l), \ \forall l = 1, \cdots, n. \tag{67}$$

Gathering (44), (49), (53), (54), (55), (61) (64), (66) and (67), we obtain

$$\begin{aligned} 0 \ \geq \ & -C\left(\sum_{i,j}\left|F^{ij}\right|\right)\frac{G}{\rho} + \rho\left(\sum_i F^{ii}\right)u_{11}^2 \\ & -C\rho\left(\textstyle\sum_{i,j}\left|F^{ij}\right|\right)(1+|\nabla u|^2+|\nabla^2 u|) \\ & +\sum_{i,j}F^{ij}\left(\rho_i u_j + \rho_j u_i - \langle\nabla\rho,\nabla u\rangle(g_0)_{ij}\right)\frac{G}{\rho} \\ & +\rho r_\nu(g)e^{-2u}\left(-2u_{11} - 4\textstyle\sum_{l=2}^n u_l^2\right) \\ \geq \ & -C\left(\sum_i F^{ii}\right)\frac{G}{\rho} + \rho\left(\sum_i F^{ii}\right)u_{11}^2 \\ & -C\left(\sum_i F^{ii}\right)\frac{G|\nabla u|}{\sqrt{\rho}} - C\left(\sum_i F^{ii}\right)(\rho+G)\,, \end{aligned} \tag{68}$$

since $r_\nu(g) \leq 0$ for any $t \in [0,T]$. As a consequence, there holds

$$C\rho^2 \geq -CG - CG\sqrt{G} + \rho^2 u_{11}^2. \tag{69}$$

Recall (44) holds at point (x_0, t_0) so that

$$G(x_0,t_0) = \rho(u_{11} + |\nabla u|^2)(x_0,t_0) \leq 21(\rho u_{11})(x_0,t_0). \tag{70}$$

Together with (69), we deduce

$$C\rho^2 \geq -CG - CG\sqrt{G} + \frac{G^2}{21^2}, \tag{71}$$

which implies

$$C \geq -CG + \frac{G^2}{2\times 21^2}. \tag{72}$$

This yields

$$G \leq C. \tag{73}$$

Here C is a constant independent of T and ν. Therefore, we have finished the proof of the first part of the Theorem. The second part yields from (68) and Lemma 4. q.e.d.

The same proof gives the local estimates for the elliptic equation.

Corollary 4. *Assume $n \geq 3$, $\nu > 0$ and $g_0 \in \Gamma_1^+$. Let B_R be a geodesic ball for $R < \tau_0$, the injectivity radius of M. Assume that*

$e^{-2u}g_0 \in \Gamma_1^+$ is a solution of the following equation in B_R

$$\frac{\sigma_2(g) - \nu e^{4u}}{\sigma_1(g)} = \kappa, \tag{74}$$

for some constant κ.

(1) *If $\kappa \le 0$, then there is a constant C depending only on (B_R, g_0) (independent of ν) such that for any $x \in B_{R/2}$*

$$|\nabla u|^2 + |\nabla^2 u| \le C. \tag{75}$$

(2) *If $\kappa > 0$, then there is a constant C depending only on (B_R, g_0) and κ (independent of ν) such that for any $x \in B_{R/2}$*

$$|\nabla u|^2 + |\nabla^2 u| \le C(1 + e^{-2\inf_{x\in B_R} u(x)}). \tag{76}$$

Corollary 5. *Under the same hypotheses as in Theorem 6 with $r_\nu \le 0$ there is a constant C depending only on g_0 (independent of ν) such that for any $t \in [0, T^*)$*

$$\|\nabla u\|_{C(M)} + \|\nabla^2 u\|_{C(M)} \le C. \tag{77}$$

Corollary 6. *Under the same hypotheses as in Theorem 6 with $r_\nu \le 0$, then there is a constant C depending only on g_0 (independent of ν) such that for any $t \in [0, T^*)$*

$$\|u\|_{C^2(M)} \le C. \tag{78}$$

Proof. First, for any $t \in [0, T^*)$ we have

$$\mathcal{F}_1(g(t)) \equiv \mathcal{F}_1(g(0)). \tag{79}$$

Hence the following Sobolev inequality

$$\int_M \sigma_1(g) dvol(g) \ge Y_1([g_0])(Vol(g))^{\frac{n-2}{n}}, \ \forall g \in \Gamma_1^+,$$

implies that the volume $Vol(g(t))$ along the flow is bounded from above, i.e.,

$$\int_M e^{-nu} dvol(g_0) < C, \tag{80}$$

for some constant $C > 0$. On the other hand, applying Corollary 5, we have

$$\int_M e^{2u} dvol(g) \ge C \int_M \sigma_1(g) dvol(g) = C\mathcal{F}_1(g(0)) > 0.$$

This, together with (80), Hölder's inequality and Corollary 5, implies u is uniformly bounded. In view of Corollary 5, we prove the result.

q.e.d.

We remark that in this paper we only use (1) of Theorem 6. (2) of Theorem 6 will be used in a forthcoming paper.

5. Proof of Theorems 1 and 2

Now for $\nu > 0$ we define

$$a_\nu := \begin{cases} \inf_{g \in \mathcal{C}_1([g_0])} \dfrac{\mathcal{E}_\nu(g)}{\left(\int_M \sigma_1(g) dvol(g)\right)^{\frac{n-4}{n-2}}} & \text{if } n \neq 4; \\ \displaystyle\int_M \sigma_2(g) dvol(g) - \nu & \text{if } n = 4; \end{cases} \tag{81}$$

If $n \neq 4$ and a_ν is achieved by a metric $g = e^{-2u} g_0$, the g satisfies

$$\frac{\sigma_2(g) - \nu e^{4u}}{\sigma_1(g)} = \kappa, \tag{82}$$

for some constant κ. Equivalently, we will consider the energy functional $\mathcal{E}_\nu$ on the normalized cone $\tilde{\mathcal{C}}_1([g_0])$

$$\tilde{\mathcal{C}}_1([g_0]) := \left\{ g \in \mathcal{C}_1([g_0]) \mid \int_M \sigma_1(g) dvol(g) = 1 \right\}. \tag{83}$$

The first step is to solve the perturbed equation (82).

Proposition 1. *Let $\nu > 0$. In the following three cases*

1) *when $n \geq 5$, $a_\nu < 0$,*
2) *when $n = 4$, $a_\nu < 0$, which is equivalent to $\int_M \sigma_2(g) dvol(g) - \nu < 0$,*
3) *when $n = 3$, $a_\nu > 0$,*

flow (31) globally converges to a solution of (82). As a direct application, a_ν is achieved by a function u_ν satisfying (82) for $\kappa = -1$. Moreover, such a solution is unique.

Proof. The proof follows closely the proof given in [**18**]. When $n \geq 5$, without loss of generality we choose $g_0 \in \Gamma_1^+$ such that $\mathcal{E}_\nu(g_0) < 0$ and $\int_M \sigma_1(g_0) dvol(g_0) = 1$. Using Lemma 4, we have $r_\nu(g(t)) \leq r_\nu(g(0)) < 0$, $\forall t \in [0, T^*)$. If $n = 3$, or $n = 4$, by the hypotheses, we have $r_\nu(g(t)) < -c < 0$, $\forall t \in [0, T^*)$. Applying Corollary 6, the solution u of flow (31) has a uniform C^2 bound, which is independent of t. We divide the proof into 3 steps.

Step 1. The flow preserves the positivity of the scalar curvature. This is another crucial point of this paper.

Proposition 2. *There is a constant $C_0 > 0$, independent of $T \in [0, T^*)$ and ν such that $\sigma_1(g(t)) > C_0\nu$ for any $t \in [0, T]$.*

Proof. The proof follows closely the proof given in [**27**] and [**18**]. Recall

$$W = (w_{ij}) = \left(\nabla^2_{ij}u + u_iu_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}\right),$$

$$F_\nu(W) = \frac{\sigma_2(W) - \nu}{\sigma_1(W)}.$$

Hence, $F_\nu = u_t + r_\nu(g(t))e^{-2u} - s_\nu(g(t))$. Without loss of generality, we assume that the minimum of F_ν is achieved at $(x_0, t_0) \in M \times (0, T]$. Near (x_0, t_0), we have

(84)
$$\frac{d}{dt}F_\nu = \sum_{ij} A^{ij}(\nabla^2_g(u_t))_{ij} = \sum_{ij} A^{ij}\left[(\nabla^2_g(F_\nu))_{ij} - r_\nu(g)(\nabla^2_g(e^{-2u}))_{ij}\right],$$

where

$$A^{ij} := \frac{\partial F_\nu}{\partial w_{ij}} = \frac{(\sigma_1^2(W) - \sigma_2(W) + \nu)\delta^{ij} - \sigma_1(W)W^{ij}}{\sigma_1^2(W)}$$

is positive definite. To simplify the notation, we drop the index ν as before. Since (x_0, t_0) is the minimum of F in $M \times [0, T]$, at this point, we have $\dfrac{dF}{dt} \le 0$, $F_l = 0 \quad \forall l$ and (F_{ij}) is non-negative definite. Note that

$$(\nabla^2_g)_{ij}F = F_{ij} + u_iF_j + u_jF_i - \sum_l u_lF_l\delta_{ij} = F_{ij},$$

at (x_0, t_0), where F_j and F_{ij} are the first and second derivatives with respect to the back-ground metric g_0. From the positivity of A and (84), we have

(85)
$$\begin{aligned} 0 &\ge F_t - \sum_{i,j} A^{ij}F_{ij} \\ &\ge -r_\nu(g)\sum_{i,j} A^{ij}\{(e^{-2u})_{ij} + u_i(e^{-2u})_j + u_j(e^{-2u})_i - \sum_l u_l(e^{-2u})_l\delta_{ij}\} \\ &= -r_\nu(g)e^{-2u}\sum_{i,j} A^{ij}\{-2w_{ij} + 2u_iu_j + 2S(g_0)_{ij} + |\nabla u|^2\delta_{ij}\} \\ &\ge -r_\nu(g)e^{-2u}\left(\frac{-2\sigma_2(W) - 2\nu}{\sigma_1(W)}\right) \\ &\quad -r_\nu(g)e^{-2u}\textstyle\sum_{i,j} A^{ij}(2u_iu_j + 2S(g_0)_{ij} + |\nabla u|^2\delta_{ij}). \end{aligned}$$

Here we have used $\sum_{i,j} A^{ij} w_{ij} = \frac{\sigma_2(W)+\nu}{\sigma_1(W)}$. On the other hand, we have

(86)
$$\begin{aligned} &\sum_{i,j} A^{ij} S(g_0)_{ij} \\ &= \frac{(\sigma_1^2(W)-\sigma_2(W))\sigma_1(g_0)}{\sigma_1^2(W)} - \frac{1}{\sigma_1(W)}\sum_{i,j} W^{ij} S(g_0)_{ij} + \frac{\nu\sigma_1(g_0)}{\sigma_1^2(W)}. \end{aligned}$$

Going back to (85), we have

(87)
$$\begin{aligned} 0 &\geq F_t - \sum_{i,j} A^{ij} F_{ij} \\ &\geq -r_\nu(g) e^{-2u}\left[\frac{-2\sigma_2(W)-2\nu}{\sigma_1(W)} + \frac{2(\sigma_1^2(W)-\sigma_2(W))\sigma_1(g_0)}{\sigma_1^2(W)}\right. \\ &\qquad \left. - \frac{2}{\sigma_1(W)}\sum_{i,j} W^{ij} S(g_0)_{ij} + \frac{2\nu\sigma_1(g_0)}{\sigma_1^2(W)}\right], \end{aligned}$$

since (A^{ij}) is positive definite and $r_\nu(g) < -c$ is negative. Let us use $O(1)$ denote terms with a uniform bound. One can check $\sigma_2(W) = O(1)$ for $\|u\|_{C^2}$ is uniformly bounded and $\sum_{i,j} W^{ij} S(g_0)_{ij} = O(1)$. Also the term $\sigma_1^2(W) - \sigma_2(W)$ is always non-negative. From (87), we conclude that there is a positive constant $C_2 > 0$ (independent of T and ν) such that

$$\frac{|\sigma_2(W)(x_0,t_0)| + \nu}{\sigma_1(W)(x_0,t_0)} < C_2. \tag{88}$$

Since (x,t) is the minimum of $F_\nu(W)$ in $M \times [0,T]$, for any $(x,t) \in M \times [0,T]$ we have

$$\frac{\sigma_2(W)(x,t) - \nu}{\sigma_1(W)(x,t)} \geq -C_2.$$

Hence, there is a positive constant $C > 0$, independent of T and ν, such that

$$\sigma_1(W)(x,t) \geq C\nu,$$

for $\sigma_2(W) \leq \frac{1}{2}\sigma_1^2(W)$ provided $\sigma_2(W) \geq 0$. This finishes the proof of the Proposition and this step. q.e.d.

Step 2. Now we can prove equation (82) admits a solution. From Step 1, we know that the flow is uniformly parabolic. And, Krylov's theory implies that $u(t)$ has a uniform $C^{2,\alpha}$ bound. Hence, $T^* = \infty$.

One can also show that $u(t)$ globally converges to $u(\infty)$, which clearly is a solution of (82) for $k = r_\nu(g(\infty))$. (Note that $r_\nu(g(t))$ is monotone and bounded so that $r_\nu(g(\infty))$ exists) (see [**27**]). So $u_\nu = u(\infty) - \frac{1}{2}\log|r_\nu(g(\infty))|$ solves (82) for $\kappa = -1$.

Step 3. The solution to equation (82) for $k = -1$ is unique.
Assume u_1 and u_2 are two solutions. We consider the function $v = u_1 - u_2$, which solves the following second order elliptic equation

$$\sum_{i,j} A^{ij}(x)v_{ij} + \sum_i B^i v_i + m(x)v = 0, \tag{89}$$

where

$$A^{ij}(x) = \int_0^1 F^{ij}(tW_2 + (1-t)W_1)dt,$$

$$W_l(x) = \left((u_l)_{ij} + (u_l)_i(u_l)_j - \frac{1}{2}|\nabla u_l|^2(g_0)_{ij} + (S(g_0))_{ij}\right), \quad \text{for } l = 1, 2$$

and

$$m(x) = -2\int_0^1 e^{-2((1-t)u_1(x)+tu_2(x))}dt < 0.$$

This is a uniformly elliptic equation. Now applying the classical strong maximum principle, we deduce that

$$v \equiv 0.$$

Hence we have the uniqueness. It is clear that the unique solution achieves a_ν. Thus, we finish the proof. q.e.d.

In the following, we study a nonlinear eigenvalue problem for the operator

$$\sigma_2\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) \tag{90}$$

in Γ_1^+. The nonlinear eigenvalue problem for the operator (90) in Γ_2^+ was considered in the first version of [**27**]. In that paper, the nonlinear eigenvalue problem can only be considered in Γ_2^+. With our analysis established for σ_2/σ_1, we can consider the nonlinear eigenvalue problem for the operator (90) in a larger class Γ_1^+.

Proposition 3. *Let (M^n, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_1^+$ and $n \geq 3$. Assume that the first eigenvalue $\lambda(g_0, \sigma_2) > 0$. Then $\mathcal{C}_2([g_0])$ is not empty. Namely there exists a metric in $[g_0]$ with*

$$\sigma_1(g) > 0 \text{ and } \sigma_2(g) > 0.$$

Moreover, there is a regular metric $g = e^{-2u}g_0 \in \mathcal{C}_2([g_0])$ satisfying

$$\sigma_2(g) = \lambda e^{4u}, \tag{91}$$

for $\lambda = \lambda(g_0, \sigma_2) > 0$.

Proof. We define a function h

$$h : (0, +\infty) \to \mathbb{R}$$
$$\nu \mapsto h(\nu) = a_\nu := \begin{cases} \inf\limits_{g \in \mathcal{C}_1([g_0])} \dfrac{\mathcal{E}_\nu(g)}{\left(\int_M \sigma_1(g) dvol(g)\right)^{\frac{n-4}{n-2}}} & \text{if } n \neq 4; \\ \displaystyle\int_M \sigma_2(g) dvol(g) - \nu & \text{if } n = 4; \end{cases}$$

Thus, h is a non-increasing function if $n \geq 4$ and a non-decreasing function if $n = 3$. First, we consider the case $n \geq 4$. Define a set A

$$A := \{\nu \in (0, +\infty) |\ h(\nu) < 0\}. \tag{92}$$

By the assumption that $\lambda(g_0, \sigma_2) > 0$, it is easy to check that

$$(\lambda(g_0, \sigma_2), +\infty) \subset A \text{ and } (0, \lambda(g_0, \sigma_2)) \cap A = \emptyset. \tag{93}$$

Hence for any $\nu > \lambda(g_0, \sigma_2)$ we have $h(\nu) < 0$. By Proposition 1 we have a smooth metric $g_{u_\nu} = e^{-2u_\nu}g_0 \in \mathcal{C}_1([g_0])$, which solves equation (82) for $\kappa = h(\nu)$ and satisfies $\displaystyle\int_M \sigma_1(g_{u_\nu}) dvol(g_{u_\nu}) = 1$. From Corollary 6, the set of solutions

$$\{u_\nu, \forall \nu \in (\lambda(g_0, \sigma_2), \lambda(g_0, \sigma_2) + 1)\}$$

is uniformly bounded in C^2 norm. From equation (82), $\sigma_1(g_{u_\nu})$ is uniformly bounded from below by a positive constant for $\nu \in (\lambda(g_0, \sigma_2), \lambda(g_0, \sigma_2) + 1)$, since $\nu > \lambda(g_0, \sigma_2) > 0$ and also we have $\dfrac{\sigma_2(W)}{\sigma_1(W)} \leq \dfrac{1}{2}\sigma_1(W)$, provided that $\sigma_1(W) > 0$. By the Krylov's result, the set of solutions $\{u_\nu, \forall \nu \in (\lambda(g_0, \sigma_2), \lambda(g_0, \sigma_2) + 1)\}$ is also uniformly bounded in $C^{2,\alpha}$ for $\alpha > 0$. Define

$$\kappa_0 := \lim_{\nu \to \lambda(g_0, \sigma_2)^+} h(\nu). \tag{94}$$

When $\nu \to \lambda(g_0, \sigma_2)$, u_ν (by taking a subsequence) converges in C^2 to $u_0 \in \Gamma_1^+$, which is a solution of (82) for $\nu = \lambda(g_0, \sigma_2)$ and $\kappa = \kappa_0$. Clearly, $\kappa_0 \leq 0$. We claim

$$\kappa_0 = 0. \tag{95}$$

Otherwise, $\mathcal{E}_{\lambda(g_0,\sigma_2)}(u_0) < 0$, which implies that there is some small $\varepsilon > 0$ such that $\forall \nu \in (\lambda(g_0, \sigma_2) - \varepsilon, \lambda(g_0, \sigma_2)]$, we have $\mathcal{E}_\nu(u_0) < 0$. Hence $h(\nu) < 0$ for all $\nu \in (\lambda(g_0, \sigma_2) - \varepsilon, +\infty)$, that is, $(\lambda(g_0, \sigma_2) - \varepsilon, +\infty) \subset A$.

This contradicts to (93). Hence we have (95), which is equivalent to say that

$$\sigma_2(g_{u_0}) - \lambda(g_0, \sigma_2)e^{4u_0} = 0.$$

This means that $u_0 \in \Gamma_2^+$ solves (91) for $\lambda = \lambda(g_0, \sigma_2)$. In particular, $u_0 \in \Gamma_2^+$.

The proof for the case $n = 3$ is similar by consider a set A defined by

$$A := \{\nu \in (0, +\infty) |\ h(\nu) > 0\}.$$

We leave the proof to the interested reader. q.e.d.

Remark 4. In the case $n \geq 4$, the function $h(\nu)$ is Lipschitz continuous. Indeed, for all $\nu' < \nu$ and for all $g \in \mathcal{C}_1([g_0])$, we have

(96)

$$\begin{aligned} 0 &< \mathcal{E}_{\nu'}(u) - \mathcal{E}_\nu(u) = (\nu - \nu') \int e^{4u} dvol(g) \leq c(\nu - \nu')(Vol(g))^{\frac{n-4}{n}} \\ &\leq cY_1([g])^{\frac{4-n}{n-2}}(\nu - \nu') \left(\int \sigma_1(g) dvol(g) \right)^{\frac{n-4}{n-2}} \end{aligned}$$

This proves the claim.

Proof of Theorem 2. First we discuss the case $Y_{2,1}([g_0]) > 0$. Using Lemma 3, we have $\lambda(g_0, \sigma_2) > 0$. By Proposition 3, the cone Γ_2^+ is not empty. Hence, from the results in [**19**] for the cases $n \geq 5$, in [**11**] for $n = 4$ and [**33**] for the case $n = 3$, there is a solution of (8).

Now suppose $Y_{2,1}([g_0]) = 0$. As in the proof of Proposition 3, we consider the function $h(\nu)$ for all $\nu > 0$. With the same arguments, for any small positive $\nu > 0$, equation (82) admits the unique solution u_ν, since $h(\nu) < 0\ \forall \nu > 0$ if $n \geq 4$ and $h(\nu) > 0\ \forall \nu > 0$ if $n = 3$. This fact is clear for $n \geq 4$. And in the case $n = 3$, we have

$$\inf_{g \in \mathcal{C}_1([g_0])} \int \sigma_1(g) dvol(g) \times \int e^{4u} dvol(g) > 0.$$

Moreover, as $\nu \to 0$, the family of solution $\{u_\nu\}$ is bounded in C^2 norm. Thus, as $\nu \to 0$, u_ν converges in $C^{1,\alpha}$ for any $\alpha \in (0, 1)$ to some $C^{1,1}$ function u. Consequently, this $C^{1,1}$ conformal metric $g_u = e^{-2u} g_0 \in [g_0] \cap \bar{\Gamma}_1^+$ solves equation (9).

Let us consider the last case $Y_{2,1}([g_0]) < 0$. As in the previous case, there holds $h(\nu) \leq h(0) = Y_{2,1}([g_0]) < 0\ \forall \nu > 0$ if $n \geq 4$, and $h(\nu) \geq h(0) = -Y_{2,1}([g_0]) > 0\ \forall \nu > 0$ if $n = 3$ and the family of solution $\{u_\nu\}$ is bounded in C^2 as $\nu \to 0$. We claim this family is also bounded in $C^{2,\alpha}$ for $\alpha > 0$. For this purpose, we will write equation (82) in the new form. Recall $w_{ij} = \nabla^2_{ij} u + u_i u_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}$. We define a new second order tensor $\tilde{W} = (\tilde{w}_{ij})$ with

$$\tilde{w}_{ij} = w_{ij} + \mu(g_0)_{ij},$$

where μ is some small positive constant to be fixed later. It is clear that

$$\sigma_1(\tilde{W}) = \sigma_1(W) + n\mu \tag{97}$$

and

$$2\sigma_2(\tilde{W}) = 2\sigma_2(W) + 2(n-1)\mu\sigma_1(W) + n(n-1)\mu^2. \tag{98}$$

Denote u_ν the unique solution of (82) with $\displaystyle\int \sigma_1(g)dvol(g) = 1$ for $k = h(\nu)$ and for all small $\nu > 0$. Using (97) and (98), u_ν satisfies the following equation

$$\frac{\sigma_2(\tilde{W}) - \frac{n(n-1)\mu^2}{2} - (n-1)\mu(\sigma_1(\tilde{W}) - n\mu) - \nu}{\sigma_1(\tilde{W}) - n\mu} = h(\nu)e^{-2u_\nu}, \tag{99}$$

that is, u_ν solves

$$\frac{\sigma_2(\tilde{W})}{\sigma_1(\tilde{W})} + \frac{-\nu + n\mu h(\nu)e^{-2u_\nu} + \frac{n(n-1)\mu^2}{2}}{\sigma_1(\tilde{W})} = h(\nu)e^{-2u_\nu} + (n-1)\mu. \tag{100}$$

Choose a small $\mu > 0$ such that $\forall \nu \in (0, \nu_0)$ and $\forall x \in M$, we have

$$h(\nu)e^{-2u_\nu(x)} + (n-1)\mu < 0. \tag{101}$$

Now equation (100) is uniformly elliptic and concave for all $\nu \in (0, \nu_0)$. From the classical Krylov's result, the set of solutions $\{u_\nu\}$ is also bounded in $C^{2,\alpha}$ for $\alpha > 0$. Passing to the limit, u_ν converges in $C^{2,\alpha'}$ for any $\alpha' \in (0, \alpha)$ to some $C^{2,\alpha}$ function $u \in \bar{\Gamma}_1^+$, which is a solution of (10). Finally, writing equation in the form (100) for $\nu = 0$, the uniqueness comes from the maximum principle as in the proof of Proposition 1. Therefore, we finish the proof. q.e.d.

Proof of Theorem 1. When $\lambda(g_0, \sigma_2) > 0$, the result follows from Proposition 3. When $\lambda(g_0, \sigma_2) = 0$, the proof is the same as in the case $Y_{2,1}([g_0]) = 0$ in the previous proof. q.e.d.

Proof of Corollary 1. It follows from Proposition 3 directly. q.e.d.

Proof of Corollary 3. From Theorem 2, the cone Γ_2^+ is not empty. Set $g \in \mathcal{C}_2([g_0])$. From a result due to Gursky-Viaclovsky [**30**] the Ricci tensor of g is pointwise positive. Thanks to the result of Hamilton [**35**] by using the Ricci flow, M is diffeomorphic to a compact 3-dimensional Riemannian manifold with constant curvature. Moreover, M is diffeomorphic to $\mathbb{S}^3/\Gamma$, where Γ is a finite isometry subgroup of $\mathbb{S}^3$ in the standard metric. q.e.d.

Remark 5. In Corollary 2, the condition $\int_{M^3} \sigma_2(g_0)dvol(g_0) > 0$ can be easily weakened to $\int_{M^3} \sigma_2(g_0)dvol(g_0) \geq 0$. In this case, we have two cases: either (i) there is a conformal metric $g \in \mathcal{C}^1([g_0])$ with $\int_{M^3} \sigma_2(g)dvol(g) > 0$, or (ii) there is no conformal metric $g \in$

$\mathcal{C}^1([g_0])$ with $\int_{M^3}\sigma_2(g)dvol(g)0$. Case (i) is just Corollary 1. In case (ii), $Y_{2,1}([g_0]) = 0$ and g_0 achieves $Y_{2,1}([g_0])$, hence $\sigma_2(g_0) = 0$. This also implies that $Ric_{g_0} \geq 0$ and M^3 is a quotient of a sphere, since $Y_1([g_0]) > 0$.

6. The Yamabe invariants $Y_{2,1}([g_0])$

In this section, we prove Theorem 3. To this aim, we need a technic result.

Lemma 5. *Let (M^n, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_1^+$ and $n \geq 3$. At most one of the followings holds:*

1) *M admits a regular metric $g \in [g_0] \cap \Gamma_2^+$;*
2) *equation (9) admits a $C^{1,1}$ solution $g \in [g_0] \cap \bar{\Gamma}_1^+$;*
3) *equation (10) admits a regular solution $g \in [g_0] \cap \bar{\Gamma}_1^+$.*

Proof. Let $g \in [g_0] \cap \Gamma_2^+$. Without loss of generality, assume $g = g_0$. From a result of Guan-Wang [**28**], there is no $C^{1,1}$ solution of equation (9) in $\bar{\Gamma}_1^+$. On the other hand, for any $g = e^{-2u}g_0 \in \mathcal{C}_1([g_0])$, let x_0 be a minimum point of u. At this point, we get $\nabla u = 0$ and the Hessian matrix $\nabla^2 u$ is non-negative definite. Thus, the Schouten tensor at point x_0 is in Γ_2^+. As a consequence, equation (10) does not admit a regular solution $g \in [g_0] \cap \bar{\Gamma}_1^+$.

Now suppose $g_1 = e^{-2u_1}g_0$ is a regular solution of equation (10) in $[g_0] \cap \bar{\Gamma}_1^+$. As above, $[g_0] \cap \Gamma_2^+ = \emptyset$. As in the proof of Theorem 2, u_1 solves

$$\frac{\sigma_2(\tilde{W})}{\sigma_1(\tilde{W})} + \frac{-n\mu e^{-2u_1} + \frac{n(n-1)\mu^2}{2}}{\sigma_1(\tilde{W})} = -e^{-2u_1} + (n-1)\mu, \tag{102}$$

where $\mu > 0$ is a small positive number such that $f(x) := -e^{-2u_1} + \frac{(n-1)\mu}{2} < 0$ and $\tilde{w}_{ij} = w_{ij} + \mu(g_0)_{ij}$. Let $g_2 = e^{-2u_2}g_0 \in [g_0] \cap \bar{\Gamma}_1^+$ be a $C^{1,1}$ solution of equation (9). Thus, u_2 is a subsolution to (102), that is,

$$\frac{\sigma_2(\tilde{W})}{\sigma_1(\tilde{W})} + \frac{n\mu f(x)}{\sigma_1(\tilde{W})} \geq -e^{-2u_1} + (n-1)\mu, \tag{103}$$

since u_2 solves

$$\frac{\sigma_2(\tilde{W})}{\sigma_1(\tilde{W})} + \frac{n(n-1)\mu^2}{2\sigma_1(\tilde{W})} = (n-1)\mu \tag{104}$$

and

$$\sigma_1(\tilde{W}) \geq n\mu. \tag{105}$$

Set $v = u_2 - u_1$ and $H(W, x) = \dfrac{\sigma_2(W) + n\mu f(x)}{\sigma_1(W)}$. Denote

$$H^{ij} := \frac{\partial H(W, x)}{\partial w_{ij}}.$$

On the other hand, for any $c \in \mathbb{R}$, $u_2 + c$ is also a $C^{1,1}$ solution of equation (9). Without loss of generality, we could suppose $v \leq 0$ and $\max v = 0$. From (102) and (103), v is a subsolution of some uniformly elliptic second order operator, that is,

$$Lv := \sum_{i,j} A^{ij} \nabla^2_{ij} v + \sum_i B^i \nabla_i v \geq 0,$$

where

$$A^{ij} = \int_0^1 H^{ij}(s\tilde{W}_2 + (1-s)\tilde{W}_1) ds.$$

From the strong maximum principle, $v \equiv 0$. This contradiction yields the desired result and we finish the proof of Lemma. q.e.d.

Proof of Theorem 3. It follows from Theorem 2 and Lemma 5. q.e.d.

By Theorem 2 and Lemma 5, we have

Proposition 4. *Let (M^n, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_1^+$ and $n \geq 3$. The following holds*
1) If there is a regular conformal metric $g \in [g_0] \cap \Gamma_2^+$, then $Y_{2,1}([g_0]) > 0$;
2) If there is a $C^{1,1}$ conformal metric $g \in [g_0] \cap \bar{\Gamma}_1^+$ satisfying equation (9), then $Y_{2,1}([g_0]) = 0$;
3) If there is a regular conformal metric $g \in [g_0] \cap \bar{\Gamma}_1^+$ solving equation (10), then $Y_{2,1}([g_0]) < 0$.

Remark 6. With the same arguments, we know : let (M^n, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_1^+$ and $n \geq 3$. At most one of the followings holds:

1) M admits a regular metric $g \in [g_0] \cap \bar{\Gamma}_2^+$;
2) equation (10) admits a regular solution $g \in [g_0] \cap \bar{\Gamma}_1^+$.

To this aim, let $g_2 = e^{-2u_2} g_0 \in [g_0] \cap \bar{\Gamma}_2^+$. Thus, u_2 is also a subsolution to (102) and the desired result yields. A direct consequence is that $Y_{2,1}([g_0]) \geq 0$ provided $[g_0] \cap \bar{\Gamma}_2^+ \neq \emptyset$.

Finally, we state the following result.

Corollary 7. *Let (M, g_0) be a compact Riemannian manifold with positive scalar curvature and of dimension $n \geq 4$. The sign of $\lambda(g_0, \sigma_2)$, equivalently the sign of $Y_{2,1}([g_0])$, is a conformal invariant.*

7. Geometric Applications

First we show that any manifold in class (1_+) admits a psc metric with negative Yamabe constant $Y_{2,1} < 0$. Therefore by Theorem 2 we can deform this metric in its conformal class to a metric with non-negative scalar curvature and negative σ_2-scalar curvature. Then we show that $Y_{2,1}$ is continuous in a suitable sense.

Proof of Theorem 4. With the analysis established above, in order to show the Theorem, We need only to show that for any manifold in (1_+) there is a psc metric g with $\int \sigma_2(g) dvol(g) < 0$.

Here we use the well-known construction of Gromov-Lawson [**20**] for positive scalar curvature metrics. See also [**51**]. Let S^p be an embedded sphere in M with trivial normal bundle of codimension $q = n - p \geq 3$. Let $\mathbb{S}^p \times D^q(\bar{r})$ be an embedding into M for some small constant $\bar{r} > 0$. Let r_0 be a constant fixed later as small as we want.

By the construction of Gromov-Lawson, we have a manifold (N_1, h_1) with an end $\mathbb{S}^p \times \mathbb{S}^{q-1} \times [0, +\infty)$ such that $N_1 - (\mathbb{S}^p \times \mathbb{S}^{q-1} \times [1, \infty], h_1)$ is isometric to $M - \mathbb{S}^p \times D^q(\bar{r})$ and $(\mathbb{S}^p \times \mathbb{S}^{q-1} \times [2, +\infty), h_1)$ is isometric to $\mathbb{S}^p \times \mathbb{S}^{q-1}(r_\infty) \times [2, +\infty)$ with the product metric. Here $\mathbb{S}^p$ is standard sphere with radius 1 and $\mathbb{S}^{q-1}(r_\infty)$ is the standard sphere with a small radius r_∞. The crucial point in the construction of Gromov-Lawson is that the scalar curvature of h_1 is positive. Now we glue on $N_1 - \mathbb{S}^p \times \mathbb{S}^{q-1}(r_\infty) \times [\tau_0, +\infty)$ a product manifold $\mathbb{S}^p \times D^q$, where D^q is not equipped with the flat metric, but a product metric of $\mathbb{S}^{q-1} \times [0, b]$ in a neighborhood of the boundary and of positive scalar curvature on D^q. The result manifold is of positive scalar curvature and is diffeomorphic to M. It contains a product $\mathbb{S}^p \times \mathbb{S}^{q-1}(r_\infty) \times [2, \tau_0]$ for a small r_∞ and a large τ_0.

Now we consider the case $q = 3$. In this case, its Ricci curvature, written as a diagonal matrix, is $\operatorname{diag}\{n-4, \cdots, n-4, r_\infty^{-2}, r_\infty^{-2}, 0\}$. Its Schouten tensor, also written as a diagonal matrix and up to a multiple constant independent of r_∞, is

$$\operatorname{diag}\{n-4-\alpha, \cdots, n-4-\alpha, r_\infty^{-2} - \alpha, r_\infty^{-2} - \alpha, -\alpha\}$$

with

$$\alpha = \frac{1}{2(n-1)} \left((n-3)(n-4) + \frac{2}{r_\infty^2} \right).$$

It can be written as

$$\frac{1}{n-1} r_\infty^{-2} \operatorname{diag}\{-1, \cdots, -1, n-2, n-2, -1\} + C,$$

Where C is a matrix whose entries are independent of r_∞. To decide the sign of σ_2-scalar curvature for this product, we only need to compute $\sigma_2(-1, \cdots, -1, n-2, n-2, -1) = -\frac{1}{2}(n-1)(n-2)$. Hence choosing

r_∞ small, we have negative σ_2-scalar curvature in this product part. By choosing τ_0 large enough, we obtain a metric $\tilde{g}$ on M with

$$\int_M \sigma_2(\tilde{g}) < 0.$$

Therefore for the conformal class $[\tilde{g}]$ we have $Y_{2,1}([\tilde{g}]) < 0$. By Theorem 2, we have a conformal metric $\tilde{h}$ with $\sigma_2(\tilde{h}) \le 0$ and with non-negative scalar curvature. q.e.d.

When $n = 4$, we have another proof. In this case,

$$\int \left(\sigma_2(g) + \frac{1}{16}|W|^2\right) dvol(g)$$

is the Euler characteristic of M up to a constant multiple. Since $\int |W|^2 dvol(g)$ is invariant in a conformal class, so is $\int \sigma_2(g) dvol(g)$. In [**1**] the authors constructed a sequence of metrics g_i satisfying that

$$Y_1([g_i]) \to Y_1([g_0]) \text{ and } \int |W|^2 dvol(g_i) \to \infty,$$

as $i \to \infty$. Since $R_{g_0} > 0$, from this sequence we can find a psc metric g_i with $\int \sigma_2(g_i) dvol(g_i) < 0$.

Lemma 6. *Let $n \ge 4$. If the space of psc metric is equipped with the $C^{4,\alpha}$-topology, then map $g \to Y_{2,1}([g])$ is continuous.*

Proof. A similar Lemma for the first Yamabe constant Y_1 was given in [**4**], see also [**5**]. For $Y_{2,1}$, the proof becomes complicated. We use solutions of equations considered here.

Let g_j be a sequence of psc metric converging to g in $C^{4,\alpha}$ topology. The proof is trivial for the case $n = 4$, since $\int_M \sigma_2(g) dvol(g)$ is constant in a conformal class. Now we consider the cases $n \ge 5$. By taking a subseqeunce we may assume that $\lim_{j\to\infty} Y_{2,1}([g_j]) = Y_0$. It is easy to see that

$$Y_0 = \lim_{j\to\infty} Y_{2,1}([g_j]) \le Y_{2,1}([g]). \tag{106}$$

To see this, for any $\varepsilon > 0$ find $g_u = e^{-2u} g \in \mathcal{C}_1([g])$ such that

$$\int \sigma_2(g_u) dvol(g_u) < Y_{2,1}([g]) + \frac{1}{2}\varepsilon \qquad \text{and} \qquad \int \sigma_1(g_u) dvol(g_u) = 1.$$

For sufficiently large j, we have $e^{-2u} g_i \in \mathcal{C}_1([g_j])$. Since $\int \sigma_2$ and $\int \sigma_1$ are continuous for this fixed function u, we have

$$Y_{2,1}([g_j]) \le Y_{2,1}([g]) + \varepsilon,$$

for sufficiently large j, and hence (106).

Now we show that $Y_0 \ge Y_{2,1}([g])$. Let $Y_{2,1}(\mathbb{S}^n)$ be the second Yamabe constant for the standard sphere $\mathbb{S}^n$. Using a method given in [**54**], one can show that

$$Y_{2,1}(\mathbb{S}^n) \ge Y_{2,1}([g]),$$

for any psc metric g. Without loss of generality, we may assume $Y_0 < Y_{2,1}(\mathbb{S}^n)$. Otherwise, we have $\lim_{j\to\infty} Y_{2,1}([g_j]) = Y_0 = Y_{2,1}(\mathbb{S}^n) \geq Y_{2,1}([g])$ and hence we are done.

Consider the following perturbed energy functional

$$J_{\varepsilon,\nu}(g) := \frac{\int(\sigma_2(g) - \nu e^{4u})dvol(g))}{\left(\int e^{2\varepsilon u}\sigma_1(g)dvol(g)\right)^{\frac{n-4}{n-2-2\varepsilon}}}$$

for small $\nu \geq 0$ and $\varepsilon \geq 0$ in the cone $\mathcal{C}_1$. The analysis established above implies that the above functional admits a minimizer in $\mathcal{C}_1([g])$ for any psc metric. It is easy to check that for given $j \in \mathbb{N}$, there are small constants $\varepsilon_j \geq 0$ and $\nu_j \geq 0$ such that a minimizer $\tilde{g}_j = e^{-2u_j}g_j \in \mathcal{C}_1([g_j])$ of the perturbed functional J_{ε_j,ν_j} satisfies

$$J_{\varepsilon_j,\nu_j}(\tilde{g}_j) \leq Y_{2,1}([g_j]) + \frac{1}{j}.$$

(When $Y_0 \leq 0$, we can take $\varepsilon_j \equiv 0$. When $Y_0 > 0$, we can take $\nu_j \equiv 0$.) As a minimizer of $J_{\varepsilon,\nu}$, $\tilde{g}_j = e^{-2u_j}g_j$ satisfies an equation similar to (8), for which we have local estimates. Since g_j converges to g in $C^{4,\alpha}$, the local estimates and $Y_0 < Y_{2,1}(\mathbb{S}^n)$ imply that $\tilde{g}_j$ converges to a metric $\tilde{g} = e^{-2\tilde{u}}g \in \overline{\mathcal{C}}_1([g])$ in C^0-topology, and hence in $C^{1,\beta}$-topology for any $\beta \in (0,1)$. We first show that

(107)
$$\lim_{j\to\infty}\int \sigma_2(e^{-2u_j}g_j)dvol(e^{-2u_j}g_j) = \lim_{j\to\infty}\int \sigma_2(e^{-2u_j}g)dvol(e^{-2u_j}g)$$

and

(108)
$$\lim_{j\to\infty}\int e^{2\varepsilon_j u_j}\sigma_1(e^{-2u_j}g_j)dvol(e^{-2u_j}g_j) = \lim_{j\to\infty}\int \sigma_1(e^{-2u_j}g)dvol(e^{-2u_j}g).$$

This is clear, for u_j has a uniform C^2 bound and g_j converges to g in $C^{4,\alpha}$-topology and $\varepsilon_j \to 0$. Then, we claim that there exists metrics $e^{-2\tilde{u}_j}g \in \mathcal{C}_1([g])$ such that

(109)
$$\lim_{j\to\infty}\int \sigma_2(e^{-2\tilde{u}_j}g)dvol(e^{-2\tilde{u}_j}g) = \lim_{j\to\infty}\int \sigma_2(e^{-2u_j}g)dvol(e^{-2u_j}g)$$

and

(110)
$$\lim_{j\to\infty}\int \sigma_1(e^{-2\tilde{u}_j}g)dvol(e^{-2\tilde{u}_j}g) = \lim_{j\to\infty}\int \sigma_1(e^{-2u_j}g)dvol(e^{-2u_j}g).$$

Given $j \in \mathbb{N}$, we choose $t_j \in (0,1)$ with $\lim t_j = 1$ such that $e^{-2t_j u_j} g \in \mathcal{C}_1([g])$. To see this, we compute

(111)

$$\begin{aligned}
& e^{-2t_j u_j}\sigma_1(e^{-2t_j u_j} g) \\
= \quad & t_j \Delta_g u_j - t_j^2 \frac{n-2}{2}|\nabla u_j|_g^2 + \sigma_1(g) \\
= \quad & t_j e^{-2u_j}\sigma_1(e^{-2u_j} g_j) + t_j\left(\Delta_g u_j - \Delta_{g_j} u_j + \sigma_1(g) - \sigma_1(g_j)\right) \\
& + t_j\left(\frac{n-2}{2}|\nabla u_j|_{g_j}^2 - \frac{n-2}{2}|\nabla u_j|_g^2\right) \\
& + (1-t_j)\left(t_j \frac{n-2}{2}|\nabla u_j|_g^2 + \sigma_1(g)\right).
\end{aligned}$$

It is clear that

(112)

$$\begin{aligned}
& \left| t_j(\Delta_g u_j - \Delta_{g_j} u_j + \sigma_1(g) - \sigma_1(g_j)) + t_j(\frac{n-2}{2}|\nabla u_j|_{g_j}^2 - \frac{n-2}{2}|\nabla u_j|_g^2) \right| \\
& \quad \le C\|g - g_j\|_{C^2}(|\nabla u_j|_g^2 + 1).
\end{aligned}$$

Thus, we can choose t_j close to 1 such that $e^{-2t_j u_j} g \in \mathcal{C}_1([g])$. From (20), (23) and the fact that u_j has a uniform C^2 bound, we have (109) and (110), which imply that $Y_0 = \lim_{j\to\infty} Y_{2,1}([g_j]) \ge Y_{2,1}([g])$. Hence we finish the proof of the Lemma. q.e.d.

In fact, the Lemma is true for C^4-topology.

Proof of Theorem 5. If M belongs to class (2_+), then there is a psc metric of positive σ_2-scalar curvature. By Theorem 3, $Y_{2,1}([g]) > 0$. From Theorem 4 we have another psc metric $\tilde{g}$ with $Y_{2,1}([\tilde{g}]) < 0$. Using the previous Lemma we have the third psc metric $\hat{g}$ with $Y_{2,1}([\hat{g}]) = 0$. Now 1. follows from Theorem 2. The proof for 2. and 3. is the same. q.e.d.

In order to prove the Conjecture (Trichotomy Theorem) as in [**41**] and [**42**], we need an implicit function theorem for the linearization operator of $\sigma_2(g)$ in $L^{2,p}$. This seems to be rather difficult for us, at least at the moment.

Example 1. Let us consider the manifold $M = \mathbb{S}^3 \times \mathbb{S}^1$. It is a locally conformally flat manifold with $\sigma_1 > 0$ and $\sigma_2 = 0$ for the product metric. From

$$2\pi^2\chi(M) = \int_M \sigma_2(g) + \frac{1}{16}|W|^2,$$

we have $\int_{\mathbb{S}^3\times\mathbb{S}^1} \sigma_2(g) = -\frac{1}{16}\int |W|^2 \le 0$ for any metric g on $\mathbb{S}^3 \times \mathbb{S}^1$. Hence, $\mathbb{S}^3 \times \mathbb{S}^1$ belongs to class (2_0).

Example 2. For $n > 4$ let us consider the locally conformally flat manifold $\mathbb{S}^p \times H^{n-p}$, where $p > n/2$ and H^{n-p} is a compact quotient of the hyperbolic space $\mathbb{H}^{n-p}$ with sectional curvature -1. Let g_0 be the product metric. It is clear that $\sigma_1(g_0) > 0$, for $p > n/2$. Its σ_2-scalar curvature, up to a positive constant multiple, is

$$\left(p - \frac{n}{2}\right)^2 - \frac{n}{4}.$$

Hence, $p > \frac{n+\sqrt{n}}{2}$ if and only if $\sigma_2(g_0) > 0$. Now we consider such p with $\sigma_2(g_0) = 0$, namely $p = \frac{n+\sqrt{n}}{2}$. It is clear that we have to consider $n = (2m+1)^2$ for some $m \in \mathbb{N}$. For example $m = 1$, $M = \mathbb{S}^6 \times H^3$. In general $M_m := \mathbb{S}^{(m+1)(2m+1)} \times H^{m(2m+1)}$. We conjecture that on such a manifold there is a psc metric with positive σ_2-scalar curvature. A rough idea to check this can be made as follows. By Theorem 2, if it is not true, we know that any psc metric g on M_m has $Y_{2,1}([g]) \le 0$, which is equivalent to

$$\inf_{\tilde{g} \in \mathcal{C}_1([g])} \int_M \sigma_2(\tilde{g}) dvol(\tilde{g}) \le 0.$$

Hence 0 is be a minimax value of $\int_M \sigma_2(g) dvol(g)$ and the product metric g_0 would be a critical point of $\int \sigma_2$ on the space of all metrics. If this is true, then a result in [**40**] implies that g_0 is a metric of constant sectional curvature. This certainly is false. From this example, we propose a

Conjecture. *When $n > 4$, class (2_0) is empty.*

8. Further applications in fully nonlinear equations

With the method considered in this paper, we can also deal with the following problems.

1. When (M, g_0) is a locally conformally flat manifold with $g_0 \in \mathcal{C}_{k-1}$ ($k > 1$), the method presented here can be used to study the equation

$$\frac{\sigma_k(g)}{\sigma_{k-1}(g)} = f. \tag{113}$$

Similarly we define a Yamabe type invariant in [**22**] as follows

$$Y_{k,k-1}([g_0]) = \inf \frac{\int_M \sigma_k(g) dvol(g)}{\left(\int_M \sigma_{k-1}(g) dvol(g)\right)^{\frac{n-2k}{n-2k+2}}},$$

for $k \le n/2$. We can show that if $Y_{k,k-1} > 0$ ($Y_{k,k-1} = 0$ and $Y_{k,k-1} < 0$ resp.), then there is a conformal metric g with $\sigma_k > 0$ ($\sigma_k = 0$ and $\sigma_k \le 0$ resp.)

2. Let (M^4, g_0) be a compact 4-dimensional manifold with positive scalar curvature. One can consider the equation

$$\frac{\sigma_2 - s|W|^2}{\sigma_1} = f, \tag{114}$$

where W is the Weyl tensor and s is a non-negative number. In this case for a fixed s

$$\int (\sigma_2 - s|W|^2) dvol(g)$$

is a constant in a given conformal class. With the method presented here, we can show that the number $\int (\sigma_2 - s|W|^2) dvol(g)$ is positive, null or negative resp. if and only if there is a conformal metric with

$$\sigma_2 - s|W|^2 > 0, = 0 \text{ or } \le 0 \text{ resp.}$$

The positive and null cases were studied already in [**12**] as mentioned above. This can be seen as a generalization of the following classical result: Let (M^2, g_0) be a closed surface. Its Euler characteristic $\chi(M^2) = \frac{1}{2\pi} \int_M R_g dvol(g)$ is positive, negative or null resp. if and only if there is a conformal metric g with positive, negative or null scalar curvature resp. It is clear that equation (114) can also be considered on a higher dimensional manifold.

3. Our methods can also be applied to study the following fully nonlinear equations

$$\frac{\sigma_k(\nabla^2 u)}{\sigma_{k-1}(\nabla^2 u)} = f$$

and

$$\frac{\sigma_k(\nabla^2 u + ug)}{\sigma_{k-1}(\nabla^2 u + ug)} = f,$$

in the class of $(k-1)$-admissible functions.

4. Another interesting problem is a generalization of the prescribed scalar curvature problem. Let $M^n = \mathbb{S}^n$ and $f : \mathbb{S}^n \to \mathbb{R}^1$ be a smooth function. We would like to ask if there is a conformal metric $g \in \overline{\mathcal{C}}_1$ such that

$$\sigma_2(g) = f \sigma_1(g).$$

A Kazdan-Warner type necessary condition could be obtained as Han [**36**] did for the prescribed σ_k-scalar curvature problem. Here, and also in the previous problem, the function f need not be positive.

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Département de Mathématiques
Université Paris Est Créteil Val de Marne
61 avenue du Général de Gaulle
94010 Créteil Cedex, France
E-mail address: ge@univ-paris12.fr

Department of Mathematics
National Taiwan University and
Taida Institute for Mathematical Science
National Taiwan University
Taiwan
E-mail address: cslin@math.ntu.edu.tw

Albert-Ludwigs-Universität Freiburg
Mathematisches Institut
Eckerstra.1
D-79104 Freiburg, Germany
E-mail address: guofang.wang@math.uni-freiburg.de