

## WHICH AMBIENT SPACES ADMIT ISOPERIMETRIC INEQUALITIES FOR SUBMANIFOLDS?

BRIAN WHITE

### Abstract

We give simple conditions on an ambient manifold that are necessary and sufficient for isoperimetric inequalities to hold.

### 1. Introduction

Let  $N$  be a compact  $(n + 1)$ -dimensional Riemannian manifold with mean convex boundary. Can one bound the  $n$ -dimensional area of a minimal hypersurface in  $N$  in terms of the  $(n - 1)$ -dimensional area of its boundary? The absence of any closed minimal hypersurface in  $N$  is certainly a necessary condition, since such a hypersurface would contradict any such bound. In this paper, we show that this necessary condition is also sufficient. Indeed, we show (Theorem 2.1) that the absence of such a hypersurface implies the existence of a  $c = c_N < \infty$  such that

$$(1) \quad |M| \leq c \left( |\partial M| + \int_M |H| \right)$$

for every  $n$ -dimensional variety  $M$  in  $N$ , where  $H(x)$  is the mean curvature of  $M$  at  $x$ . We also prove an analogous but weaker result (Theorem 2.3) about isoperimetric inequalities for surfaces of any codimension. (For surfaces in compact subsets of Euclidean space, the inequality (1) has a simple, well-known proof: see Theorem 7.2.)

The general codimension theorem involves varifolds in an essential way, but the proof uses only the most elementary facts about them. Indeed, the theorem is a rather trivial consequence of the basic definitions. For readers not familiar with varifolds, we have included all the necessary background (with proofs) in an appendix.

The hypersurface theorem is a consequence of the general theorem together with two facts about mean curvature flow: (1) a hypersurface moving by mean curvature flow cannot bump into a minimal variety, and (2) a mean convex hypersurface moving by mean curvature flow either

---

This research was supported in part by the National Science Foundation grants DMS-0406209 and DMS-0707126.

Received 02/23/08.

vanishes in finite time or converges as  $t \rightarrow \infty$  to a minimal hypersurface with a very small singular set. These facts are not elementary, but otherwise this paper is for the most part self-contained.

After proving the main theorems in Section 2, we give two applications in Section 3. Section 4 discusses the special case of minimal embedded disks in 3-manifolds. Section 5 explains how to extend the hypersurface results to ambient manifolds with piecewise smooth boundaries. Section 6 discusses the nonlinear isoperimetric inequality obtained by replacing  $|M|$  with  $|M|^{(n-1)/n}$  in (1).

## 2. The Main Results

Suppose that  $N$  is a compact Riemannian manifold with smooth boundary. We say that  $N$  is *mean convex* provided that the mean curvature vector at each point of  $\partial N$  is a nonnegative multiple of the inward-pointing unit normal.

We begin with the main result for hypersurfaces. The reader may wish to ignore the “furthermore...” assertion until it is used in Section 3.

**Theorem 2.1.** *Suppose that  $N$  is a compact, connected, mean-convex  $(n+1)$ -dimensional Riemannian manifold with smooth, nonempty boundary, and that  $n < 7$ . The following are equivalent:*

- (a)  $N$  contains no smooth, closed, embedded minimal hypersurface.
- (b) There is an increasing function  $\phi : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $\phi(0) = 0$  such that

$$|M| \leq \phi(|\partial M|)$$

for every smooth  $n$ -dimensional minimal surface  $M$  in  $N$ .

- (c) There is a constant  $c < \infty$  such that if  $M$  is a smooth  $n$ -dimensional manifold in  $N$ , then

$$|M| \leq c \left( |\partial M| + \int_M |H| dA \right).$$

- (d) There is a constant  $c < \infty$  such that if  $V$  is an  $n$ -dimensional varifold in  $N$ , then

$$|V| \leq c |\delta V|.$$

where  $|V|$  and  $|\delta V|$  are the total mass and the total first variation measure, respectively, of  $V$ .

Furthermore, if (a)-(d) fail and if no component of  $\partial N$  is a minimal surface, then the interior of  $N$  contains a smooth, stable, two-sided closed minimal hypersurface that is an embedded submanifold or the double cover of an embedded submanifold.

Two-sidedness of  $M$  means (by definition) that the normal bundle is orientable, i.e. that  $M$  has a continuous unit normal vectorfield. If the

ambient space  $N$  is orientable, then two-sidedness of  $M$  is equivalent to orientability of  $M$ .

**Remark 2.2.** Theorem 2.1 remains true (with the same proof) for  $n \geq 7$  provided “smooth” is replaced by “smooth except for a singular set of Hausdorff dimension at most  $n - 7$ ” in statements (a), (b), (c), and in the “furthermore” assertion.

For general codimensions (and without assuming any mean convexity or boundary regularity of  $N$ ), we have:

**Theorem 2.3.** *Suppose  $N$  is a compact subset of an  $(n + 1)$ -dimensional Riemannian manifold and that  $k \leq n$ . The following are equivalent:*

- (a) *The space  $N$  contains no nonzero, stationary,  $k$ -dimensional varifolds.*
- (b) *There is an increasing function  $\phi : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $\phi(0) = 0$  such that*

$$|V| \leq \phi(|\delta V|)$$

*for every  $k$ -dimensional varifold  $V$  in  $N$  with  $|\delta V| < \infty$ .*

- (c) *There is a constant  $c < \infty$  such that*

$$|V| \leq c|\delta V|$$

*for every  $k$ -dimensional varifold  $V$  in  $N$ .*

*Proof of Theorem 2.3.* Each statement is a special case of the following statement. Hence it suffices to show that statement (a) implies statement (c). We do this by assuming that statement (c) fails and showing that statement (a) must then also fail.

Failure of statement (c) means that there is a sequence  $V_i$  of  $k$ -dimensional varifolds in  $N$  with

$$(2) \quad \frac{|V_i|}{|\delta V_i|} \rightarrow \infty.$$

We may assume that  $|V_i| \equiv 1$ , since otherwise we can replace  $V_i$  with  $V_i/|V_i|$ . Note that  $|\delta V_i| \rightarrow 0$  by (2). By compactness (Theorem 7.5), a subsequence of the  $V_i$  converges to a limit varifold  $V$  with  $|V| = 1$  and with  $|\delta V| = 0$ , which violates (a). q.e.d.

Fix a  $k$  and let  $c_N$  be the supremum (possibly infinite) of  $\frac{|V|}{|\delta V|}$  among nonzero  $k$ -dimensional varifolds  $V$  in  $N$ . Thus  $c_N$ , if finite, is the best constant in the inequality 2.3(c). The reader may enjoy, as an exercise, proving that the supremum is attained, and that  $c_N$  is an upper semicontinuous function of  $N$  with respect to the Hausdorff metric on the space of compact subsets of the Riemannian  $(n + 1)$ -manifold  $N^*$  that contains  $N$ . (The constant  $c_N$  also depends upper semicontinuously on the Riemannian metric on  $N$ .)

**Corollary 2.4.** *Suppose  $N$  is a compact subset of a Riemannian manifold and that  $N$  contains no nonzero, stationary,  $k$ -dimensional varifolds. Then there is a  $c < \infty$  such that*

$$(3) \quad |M| \leq c \left( |\partial M| + \int_M |H| dA \right),$$

*whenever  $M \subset N$  is a compact, smoothly immersed  $k$ -dimensional manifold.*

*Proof.* The corollary follows immediately because if  $V$  is the varifold associated to  $M$ , then the left sides of 2.3(c) and (3) are equal, and the right side of 2.3(c) is less than or equal to the right side of (3). (See Theorem 7.3. If  $M$  is embedded, then the right sides of 2.3(c) and (3) are equal.) q.e.d.

*Proof of Theorem 2.1.* Each of the statements (a)–(d) is a special case of the following statement, so to prove their equivalence, it suffices to assume that statement (d) fails and show that statement (a) must then also fail. By Theorem 2.3,  $N$  must contain a nonzero  $n$ -dimensional stationary varifold  $V$ . We must show that  $N$  also contains a minimal embedded hypersurface with small singular set (in particular, with empty singular set if  $n < 7$ ). That implication, as well as the last assertion of Theorem 2.1, is given by the following theorem. q.e.d.

**Theorem 2.5.** *Suppose that  $N$  is a compact, connected, mean-convex  $(n+1)$ -dimensional Riemannian manifold with smooth, nonempty boundary, and that no connected component of  $\partial N$  is a minimal surface. Suppose also that  $N$  contains a nonzero stationary  $n$ -dimensional varifold. Then the interior of  $N$  contains a closed minimal hypersurface  $M$  such that*

- (i)  *$M$  is a smooth embedded submanifold except for a closed singular set of Hausdorff dimension at most  $n - 7$ ,*
- (ii) *the smooth part of  $M$  is stable, and each one-sided connected component of the smooth part of  $M$  has a stable, two-sided double cover,*
- (iii)  *$M$  is invariant under the isometry group of  $N$ .*

*Proof.* Let  $V$  be the stationary varifold. If  $\text{spt} \|V\|$  (the spatial support of  $V$ ) touched  $\partial N$ , then by the strong maximum principle (Theorem 7.6),  $\text{spt} \|V\|$  would contain an entire connected component of  $\partial N$ , and that component would have to be a smooth minimal surface, contrary to the hypotheses. Thus  $\text{spt} \|V\|$  does not touch  $\partial N$ .

Now we use properties of the mean curvature flow proved in [Whi00, §11]. Let  $M_t$  ( $t \geq 0$ ) be the mean curvature flow with  $M_0 = \partial N$ . Let  $K_t$  be the closed region bounded by  $M_t$ . Since  $K_0 = N$  is mean convex and since no component of  $\partial K_0$  is minimal, the  $K_t$  are nested and mean

convex. Furthermore, the mean curvature does not vanish at any regular point of  $M_t$  for  $t > 0$ .

Since  $V$  is a stationary varifold, and since  $M_t$  and  $\text{spt } \|V\|$  are disjoint at time  $t = 0$ , they must be disjoint at all times by the avoidance principle for mean curvature flow (see Theorem 7.8). That is,  $\text{spt } \|V\| \subset K_t$  for all  $t$ . Consequently,  $K := \bigcap_t K_t$  is nonempty.

For simplicity, assume that  $K$  is connected. (Otherwise argue as below for each of the connected components of  $K$ .) According to [Whi00, §11],  $\partial K$  is an embedded minimal hypersurface with singular set of dimension at most  $n - 7$ . Furthermore, the  $M_t$  converge to a limit  $\tilde{M}$ , where  $\tilde{M} = \partial K$  if  $K$  has nonempty interior, and  $\tilde{M}$  is a double cover of  $\partial K = K$  if  $K$  has empty interior. In either case, the convergence is smooth away from the singular set of  $\partial K$ . The  $M_t$  are two-sided (orient the normal bundle by the mean curvature vector), and therefore  $\tilde{M}$  must also be two-sided.

Furthermore, since the  $M_t$  have nonzero mean curvature vector pointing toward  $M$  and since they foliate one side of  $\tilde{M}$  (away from the singular set of  $M$ ),  $\tilde{M}$  must be stable. (Indeed, as explained in [Whi00, 3.5],  $\tilde{M}$  has a one-sided minimizing property that is stronger than stability.)

Invariance of  $K$  and therefore of  $M = \partial K$  under the isometry group of  $N$  is clear from the construction. q.e.d.

**Remark 2.6.** One can also prove Theorem 2.5 without using mean curvature flow. Roughly speaking, one obtains  $M$  by minimizing area among hypersurfaces that enclose  $\text{spt } \|V\|$ . More precisely, one first shows that for any sufficiently small  $\delta > 0$ , there is a open set  $U = U_\delta$  containing  $\text{spt } \|V\|$  that minimizes  $|\partial U| - \delta|U|$  among all such open sets. (Some work is required to show that in a minimizing sequence of open sets, the boundaries can be kept away from  $\text{spt } \|V\|$ .) Such a surface is smooth except for a singular set of dimension at most  $n - 7$ . One then gets  $M$  as a subsequential limit as  $\delta \rightarrow 0$  of the varifolds associated to the  $\partial U_\delta$ .

To get a  $G$ -invariant  $M$  (where  $G$  is the isometry group of  $N$ ), one considers only those sets  $U$  that are  $G$ -invariant.

**Remark 2.7.** In Theorem 2.1, we assumed that  $\partial N$  was nonempty. In the case  $\partial N = \emptyset$ , Schoen and Simon [SS81], using the work of Pitts [Pit81], proved that  $N$  must contain a closed, embedded hypersurface with singular set of dimension at most  $n - 7$ . Thus the statements (a)–(d) in Theorem 2.1 (as modified in Remark 2.2) all fail for such  $N$ .

**Remark 2.8.** It would be interesting either to prove an analog of Theorem 2.5 for surfaces of codimension  $> 1$ , or to construct counterexamples. In particular, let  $N$  be a compact,  $k$ -convex Riemannian manifold  $N$ . (We say that  $N$  is  $k$ -convex if  $\kappa_1 + \cdots + \kappa_k \geq 0$  at each

boundary point, where  $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n$  are the principal curvatures of  $\partial N$  with respect to the inward unit normal.) Suppose that  $N$  contains a nonzero stationary  $k$ -varifold. Must  $N$  then also contain a stationary  $k$ -varifold with some additional properties? For example, must it contain an *integral* stationary  $k$ -dimensional varifold? If so, must it contain a closed  $k$ -dimensional minimal submanifold with a small singular set? A positive answer would imply an extension of the main theorem, Theorem 2.1, from hypersurfaces to  $k$ -dimensional surfaces.

### 3. Examples

**Theorem 3.1.** *Suppose that  $N$  is a compact, connected, mean-convex Riemannian manifold with smooth, nonempty boundary, and that no connected component of  $\partial N$  is a minimal surface. Suppose also that the dimension of  $N$  is at most 7 and that the Ricci curvature of  $N$  is everywhere positive. Then the isoperimetric inequalities listed in Theorem 2.1 hold.*

*More generally, if  $N$  has nonnegative Ricci curvature, then those isoperimetric inequalities hold unless  $N$  contains a closed, embedded, totally geodesic hypersurface  $M$  such that  $\text{Ric}(\nu, \nu) = 0$  for every unit normal  $\nu$  to  $M$ .*

*Proof.* Suppose  $N$  has positive Ricci curvature and that those isoperimetric inequalities fail. Then by Theorem 2.1,  $N$  contains a smooth, stable, two-sided minimal hypersurface  $M$ . As is well-known [Sch05, §5], such a surface is incompatible with positive Ricci curvature for the following reason. The stability of  $M$  means that

$$(4) \quad \int_M (\text{Ric}(\nu, \nu) + |A|^2) |f|^2 \leq \int_M |\nabla f|^2$$

for all smooth functions  $f : M \rightarrow \mathbf{R}$ , where  $\nu$  is a unit normal vectorfield to  $M$  and  $A$  is the second fundamental form of  $M$ . But if we set  $f \equiv 1$ , the left side of (4) is positive and the right side is 0, a contradiction.

If  $N$  has nonnegative curvature and if the isoperimetric inequalities fail, then (by letting  $f \equiv 1$  in (4)) we see that  $|A| \equiv 0$  (i.e., that  $M$  is totally geodesic) and that  $\text{Ric}(\nu, \nu) \equiv 0$ . q.e.d.

In recent years there have been a number of investigations of minimal surfaces in ambient spaces of the form  $M \times \mathbf{R}$ . (See for example [HdLR06], [Ros02], [NR02], [MR04], and [MR05].) Note that  $M \times \mathbf{R}$  is foliated by the minimal surfaces  $M \times \{z\}$ . Using Theorem 2.3, we can prove isoperimetric inequalities in very general compact subsets of ambient spaces admitting such foliations:

**Theorem 3.2.** *Let  $N^*$  be an  $(n+1)$ -dimensional Riemannian manifold. Let  $f : N^* \rightarrow \mathbf{R}$  be a smooth function with nowhere vanishing gradient such that the level sets of  $f$  are minimal hypersurfaces or, more*

generally, such that the sublevel sets  $\{x : f(x) \leq z\}$  are mean convex. Let  $N$  be a compact subset of  $N^*$  such that for each  $z \in \mathbf{R}$ , no connected component of  $f^{-1}(z)$  is a minimal hypersurface lying entirely in  $N$ . Then the isoperimetric inequalities in Theorem 2.3 and in Corollary 2.4 hold for  $k = n$ .

Of course if  $N$  does contain a connected component  $M$  of  $f^{-1}(z)$  that is a minimal hypersurface, then that component must be a compact minimal hypersurface without boundary, and thus all of the isoperimetric inequalities fail. (They all fail for  $M$ .)

*Proof.* Although this theorem is about hypersurfaces, we cannot apply Theorem 2.1 because we are not making any assumptions about  $\partial N$ . However, by Theorem 2.3, it suffices to show that  $N$  contains no nonzero stationary  $n$ -varifolds.

Suppose to the contrary that  $V$  is such a varifold. Since  $N$  is compact,  $\text{spt } \|V\|$  (the spatial support of  $V$ ) is compact, and thus the function  $f$  has a maximum value  $z$  on  $\text{spt } \|V\|$ . Hence  $\text{spt } \|V\|$  touches but lies on the  $\{f \leq z\}$  side of the smooth minimal hypersurface  $M = f^{-1}(z)$ . By the strong maximum principle (Theorem 7.6),  $\text{spt } \|V\|$  must contain an entire component of  $M$  and that component must be a minimal hypersurface. But by hypothesis,  $N$  does not contain any such component.

q.e.d.

#### 4. Minimal Disks in a 3-Manifold

Suppose  $N$  is a compact Riemannian 3-manifold with smooth, mean-convex boundary. Often embedded minimal disks in  $N$  with boundary curves in  $\partial N$  are of particular interest. Could  $N$  admit isoperimetric inequalities for such disks even if it did not admit isoperimetric inequalities for other kinds of minimal surfaces? The answer is, in general, no:

**Theorem 4.1.** *Suppose  $N$  is a compact, mean convex Riemannian 3-manifold whose boundary is a sphere with nowhere vanishing mean curvature. Then one and only one of the following holds:*

- (a)  $N$  is diffeomorphic to a ball, and the isoperimetric inequalities listed in Theorem 2.1 hold.
- (b) There is a compact family  $\mathcal{C}$  of smooth embedded curves in  $\Sigma$  such that

$$\sup \text{area}(D) = \infty,$$

the sup being over all smooth, embedded minimal disks  $D$  in  $N$  with  $\partial D \in \mathcal{C}$ .

Note that (b) implies that there is *no* bound of the form

$$(5) \quad |D| \leq \phi \left( |\partial D| + \|\partial D\|_{C^k} + \sup_{x,y \in \partial D} \frac{\text{dist}_{\partial D}(x,y)}{\text{dist}_N(x,y)} \right)$$

for embedded minimal disks  $D$  with  $\partial D \subset \partial N$ . In other words, there is no  $k$  and no increasing function  $\phi : \mathbf{R} \rightarrow \mathbf{R}$  for which (5) holds.

*Proof.* Clearly (b) violates the isoperimetric inequalities, so (b) implies failure of (a).

Now suppose that (a) fails. If the isoperimetric inequalities fail, then  $N$  contains a smooth, closed minimal surface by Theorem 2.1. Thus either  $N$  is not diffeomorphic to a ball, or  $N$  contains a closed minimal surface. According to [Whi89, §3], in either case we have (b). q.e.d.

## 5. Nonsmooth Boundaries

In Theorems 2.1, 2.5, and 3.1,  $\partial N$  was assumed to be smooth. Sometimes it is convenient to work in domains  $N$  with boundaries that are only piecewise smooth. In fact, the theorems are true under very mild boundary regularity and mean convexity assumptions.

Suppose  $N$  is an arbitrary compact, connected subset of an Riemannian  $(n+1)$ -manifold. Examination of the proofs of Theorems 2.1, 2.5, and 3.1 show that they remain valid for  $N$  provided the  $N$  has the following two properties:

- 1) If  $V$  is a stationary  $n$ -varifold in  $N$ , then  $\text{spt} \|V\|$  is contained in the interior of  $N$ .
- 2) If  $X$  is a compact subset of the interior of  $N$ , then there is a strictly mean convex set  $K_0$  with smooth boundary such that  $X$  is contained in the interior of  $K_0$ . (One uses  $\partial K_0$  as the initial surface for the mean curvature flow in the proof of Theorem 2.5, with  $X = \text{spt} \|V\|$ .)

Suppose, for example, that  $\partial N$  is the union of smooth  $n$ -manifolds-with-boundary that meet in pairs along common edges at interior angles that are everywhere strictly between 0 and  $\pi$ . Suppose also that no connected component of  $\partial N$  is a smooth minimal surface. Then  $N$  has properties (1) and (2), and therefore Theorems 2.1, 2.5, and 3.1 hold for  $N$ . Property (1) follows easily from the maximum principle 7.6. The  $K_0$  of property (2) is obtained by rounding off the corners of  $N$ . (The rounding may be accomplished as follows. Let  $K^*$  be the union of all closed geodesic balls in  $N$  of radius  $\epsilon$ . For small enough  $\epsilon$ , the boundary of  $K^*$  will be  $C^{1,1}$ , and under mean curvature flow it will immediately move into the interior of  $N$  and be smooth with everywhere positive mean curvature.)



## 6. Nonlinear Inequalities

We now consider isoperimetric inequalities of the form

$$(6) \quad |M|^{1-1/k} \leq c \left( |\partial M| + \int_M |H| dA \right)$$

for  $k$ -dimensional surfaces  $M$ , or the corresponding inequalities

$$(7) \quad |V|^{1-1/k} \leq c |\delta V|$$

for  $k$ -varifolds  $V$ .

First we observe that the inequality (7) can never be valid in any ambient manifold (even Euclidean space) if we allow arbitrary varifolds. For suppose  $V = V_\epsilon$  is  $\epsilon$  times the varifold associated to a smooth  $k$ -manifold in  $N$ . Then the left and right sides of (7) are proportional to  $\epsilon^{1-1/k}$  and  $\epsilon$ , respectively, so the inequality necessarily fails for small  $\epsilon$ .

On the other hand, Allard proved that (7) is true in Euclidean space if we require that  $V$  be a finite-mass integer-multiplicity rectifiable  $k$ -varifold or, more generally, that  $V$  be a finite-mass rectifiable  $k$ -varifold with density  $\geq 1$  almost everywhere: see [All72, 7.1] or [Sim83, 18.6]. In particular, we have (6) for any  $k$ -manifold  $M$  with  $|M|$  finite.

For general ambient manifolds we have the following theorem

**Theorem 6.1.** *Suppose  $N$  is a compact region in a Riemannian manifold. Then there are constants  $\alpha > 0$  (depending on  $N$ ) and  $c' = c'_k < \infty$  (depending only on  $k$ ) such that if  $M$  is a smooth  $m$ -dimensional surface in  $N$  with  $|M| \leq \alpha$ , then*

$$(8) \quad |M|^{1-1/k} \leq c' \left( |\partial M| + \int_M |H| \right).$$

*More generally, if  $V$  is a rectifiable  $k$ -varifold in  $N$  with density  $\geq 1$  almost everywhere and with  $|V| \leq \alpha$ , then*

$$(9) \quad |V|^{1-1/k} \leq c' |\delta V|.$$

*Proof.* We describe the proof of (8); the proof of (9) is essentially the same. Embed the Riemannian manifold isometrically in a Euclidean space  $E$ . Let  $H_e$  be the mean curvature of  $M$  as a submanifold of  $E$ . Then

$$(10) \quad |H_e| \leq |H| + K$$

holds at each point for some  $K$  depending only on the second fundamental form of  $N$  at that point. In particular, since  $N$  is compact there is a constant  $K$  (independent of  $M$ ) such that (10) holds at all points.

By the isoperimetric inequality in Euclidean space, we have

$$|M|^{1-1/k} \leq c \left( |\partial M| + \int |H_e| dA \right) \leq c \left( |\partial M| + \int |H| dA + K|M| \right).$$

Thus

$$|M|^{1-1/k}(1 - cK|M|^{1/k}) \leq c_m \left( |\partial M| + \int |H| dA \right).$$

Thus for  $|M| \leq \alpha$ ,

$$|M|^{1-1/k}(1 - cK\alpha^{1/k}) \leq c \left( |\partial M| + \int |H| dA \right).$$

Now we let  $\alpha = (2cK)^{-k}$  and  $c' = 2c$ . q.e.d.

**Remark 6.2.** Theorem 6.1 remains true (with the same proof) for noncompact  $N \subset E$  provided the norm of second fundamental form of  $N$  is bounded.

It remains to consider whether the nonlinear isoperimetric inequality (8) holds (perhaps with a worse constant) for  $M$  with  $|M| \geq \alpha$ . Here we consider the case of  $n$ -dimensional surfaces  $M$  in a compact, mean-convex Riemannian  $(n+1)$ -manifold  $N$  with smooth (or piecewise smooth) boundary. (Exactly the same reasoning applies to rectifiable varifolds with density bounded below by 1 almost everywhere.) The answer is then simple: a suitable constant  $c'$  exists if and only if  $N$  contains no closed minimal hypersurface. The “only if” is immediate since a closed minimal hypersurface would be a counterexample to the inequality. Thus suppose  $N$  contains no closed minimal hypersurface. Then by Theorem 2.1, we have the linear inequality

$$|M| \leq c \left( |\partial M| + \int |H| dA \right)$$

for all  $M$ . If  $|M| \geq \alpha$ , then  $|M|^{1-1/n} \leq \alpha^{-1/n}|M|$ , so we get the nonlinear inequality (8) with constant  $c' = c\alpha^{-1/n}$ .

## 7. Appendix: Varifolds

Let  $N$  be a compact subset of a Riemannian  $(n+1)$ -manifold  $N^*$ . Let

$$G_k(N) = \{(p, S) : p \in N, S \text{ is a } k\text{-dimensional subspace of } \text{Tan}_p(N^*)\}.$$

**Definition 7.1.** A  $k$ -dimensional varifold (or  $k$ -varifold, for short) in  $N$  is a finite Borel measure on  $G_k(N)$ .

(This definition and some of the discussion below need to be modified slightly for noncompact  $N$ . However, in this paper  $N$  is always compact.)

Every  $C^1$  embedded, compact  $k$ -dimensional submanifold  $M$  (with or without boundary) in  $N$  determines a varifold  $V = \mathbf{v}(M)$  (called the varifold associated to  $M$ ) by setting

$$V(U) = \text{area}\{p \in M : (p, \text{Tan}_p M) \in U\}$$

for every borel set  $U \subset G_k(N)$ . Consequently  $k$ -varifolds can be regarded as generalized  $k$ -dimensional surfaces.

One can also define, in a similar way, a varifold  $V = \mathbf{v}(M)$  associated to a  $C^1$  immersed submanifold with boundary. Indeed, if  $\iota : M \rightarrow N$  is an immersion, we let

$$V(U) = \text{area} \{p \in M : (\iota(p), \iota_{\#}(\text{Tan}_p M)) \in U\}$$

where area is with respect to the induced metric on  $M$ .

If  $V$  is a  $k$ -varifold, we let  $|V|$  denote its mass:

$$|V| = V(G_k(N)).$$

The mass of  $V$  is also written (for reasons that need not concern us here) as  $\|V\|(N)$ . Note that if  $V = \mathbf{v}(M)$ , then the mass  $|V|$  of  $V$  is just the area of  $M$ .

Since  $V$  is a measure on  $G_k(N)$ , its support is a closed subset of  $G_k(N)$ . If we project that closed set to  $N$  by the projection  $(x, S) \mapsto x$ , then result is the *spatial support* of  $V$ , written  $\text{spt } \|V\|$ . It is the smallest closed subset  $K$  of  $N$  such that

$$V\{(x, S) \in G_k(N) : x \notin K\} = 0.$$

Now suppose  $X$  is a smooth vectorfield on  $N^*$ . For a smoothly embedded, compact submanifold  $M$ , one has

$$(11) \quad \int_M \text{div}^M X = \int_{\partial M} X \cdot \nu - \int_M H \cdot X$$

where  $H(p)$  is mean curvature of  $M \subset N$  at  $p$ , where  $\nu(p) \in \text{Tan}_p M$  is the outward pointing unit normal to  $\partial M$  at  $p$ , and where  $\text{div}^M X(p)$  is the divergence of  $X$  over  $M$  at  $p$ . That is,

$$\text{div}^M X(p) = \sum_{i=1}^k (\nabla_{e(i)} X) \cdot e(i)$$

where  $e(1), \dots, e(k)$  is an orthonormal basis for  $\text{Tan}_p M$ .

(To prove (11), one breaks  $X$  into tangential and normal parts: see the proof of [Sim83, 9.6]. Incidentally, the quantity (11) occurs in the first variation formula for area: it is equal to the initial rate of change of area of any one-parameter family of surfaces starting at  $M$  and moving with initial velocity  $X$ . See [Sim83, §9].)

**Theorem 7.2.** *Suppose that  $M$  is a compact  $m$ -dimensional surface in Euclidean space. Then*

$$(12) \quad |M| \leq \frac{r}{m} \left( |\partial M| + \int_M |H| \right)$$

where  $r$  is the radius of the smallest ball containing  $M$ .

*Proof.* We may assume that the smallest ball containing  $M$  is centered at the origin. Let  $X(x) = x$ . Then  $\operatorname{div}^M X(p) \equiv m$ , so (12) follows immediately from (11). q.e.d.

The left side of (11) can be generalized to an arbitrary  $k$ -varifold in  $N$  as follows. We define a linear functional  $\delta V$  on the space of smooth tangent vectorfields  $X$  on  $N^*$  by

$$\delta V(X) = \int \operatorname{div}_S X(p) dV(p, S),$$

the integral being over all  $(p, S) \in G_k(N)$ . Here

$$\operatorname{div}_S X(p) = \sum_{i=1}^k (\nabla_{e(i)} X) \cdot e(i)$$

where  $e(1), \dots, e(k)$  is an orthonormal basis for  $S$ .

The linear functional  $\delta V$  has a norm  $|\delta V| \in [0, \infty]$  given by

$$|\delta V| = \sup_X \delta V(X)$$

where the sup is over all smooth tangent vectorfields  $X$  on  $N^*$  such that  $|X(p)| \leq 1$  for all  $p$ . Equivalently,  $|\delta V|$  is the smallest number  $c \in [0, \infty]$  such that

$$\delta V(X) \leq c \|X\|_0$$

for every smooth vectorfield  $X$ , where  $\|X\|_0$  is the  $C^0$  norm (i.e., the sup norm) of  $X$ .

(The norm  $|\delta V|$  would be written  $\|\delta V\|(N)$  in the notation of [All72] and [Sim83].)

Although  $\delta V(X)$  is finite for every varifold  $V$  and  $C^1$  vectorfield  $X$ , the norm  $|\delta V|$  may be infinite.

**Theorem 7.3.** *Let  $V = \mathbf{v}(M)$  be the varifold associated to a smoothly immersed manifold  $M$  in  $N$ . Then*

$$(13) \quad |\delta V| \leq |\partial M| + \int_M |H|.$$

*If  $M$  is embedded, then equality holds.*

*Proof.* Note by (11), which also holds for immersed surfaces, we have

$$(14) \quad |\delta V(X)| \leq \left( |\partial M| + \int_M |H| \right) \|X\|_0$$

which implies (13). In the embedded case, one can choose  $X$  with  $\|X\|_0 \leq 1$  so that  $X = \nu$  on  $\partial M$ , and so that, except for a small set in  $M$ ,  $X = -H/|H|$  provided  $H \neq 0$ . Then  $\delta V(X)$  will be arbitrarily close to the right side of (13). By definition of  $|\delta V|$ , this implies that the left side of (14) is greater than or equal to the right side, and thus that the two sides must be equal. (In the immersed case, this choice of  $X$  is not always possible because  $X$  must be well-defined on  $N$ .) q.e.d.

**Definition 7.4.** A varifold  $V$  is called *stationary* provided  $\delta V = 0$  (or, equivalently, provided  $|\delta V| = 0$ .)

For a smoothly embedded surface  $M$ , the associated varifold is stationary if and only if  $M$  is a minimal surface without boundary (by Theorem 7.3). Thus stationary varifolds are generalizations of minimal surfaces without boundary.

**Theorem 7.5.** *Let  $V_i$  be a sequence of  $k$ -dimensional varifolds in  $N$ . Suppose that  $N$  is compact and that  $\sup |V_i| < \infty$ . Then there is a subsequence  $V'_i$  that converges to a varifold  $V$ . Furthermore,*

$$(15) \quad |V| = \lim |V'_i|$$

and

$$(16) \quad |\delta V| \leq \liminf |\delta V'_i|.$$

Here convergence of varifolds means weak convergence of measures.

*Proof.* The existence of a convergent subsequence follows from the Riesz Representation Theorem and the Banach-Alaoglu Theorem. Continuity of mass (15) is an immediate consequence of weak convergence (since the space  $G_k(N)$  is compact.)

Note that  $\delta V(X)$  is the integral with respect to the measure  $V$  of the continuous function

$$\begin{aligned} G_k(N) &\rightarrow \mathbf{R} \\ (p, S) &\mapsto \operatorname{div}_S X(p). \end{aligned}$$

Consequently, by definition of weak convergence of measures,

$$(17) \quad \delta V(X) = \lim \delta V'_i(X).$$

But

$$\delta V'_i(X) \leq |\delta V'_i| \cdot \|X\|_0,$$

so from (17) we see that

$$\delta V(X) \leq \liminf |\delta V'_i| \cdot \|X\|_0.$$

Taking the supremum over all  $X$  with  $\|X\|_0 \leq 1$  gives (16). q.e.d.

**Theorem 7.6** (Maximum Principle). *Let  $B$  be an open set in a Riemannian  $(n+1)$ -manifold  $N^*$ . Let  $M$  be a smooth, connected hypersurface properly embedded in  $B$  and dividing  $B$  into two components. Let  $\Omega$  be one of the two components of  $B \setminus M$ .*

*Suppose that  $\Omega$  is mean concave along  $M$ , i.e., that at each point of  $M$ , the mean curvature is a nonnegative multiple of the outward unit normal to  $\Omega$ .*

*Suppose  $S$  is the spatial support of a nonzero stationary  $n$ -varifold in  $N^*$  such that  $S$  is disjoint from  $\Omega$ . If  $S$  contains any point of  $M$ , then it must contain all of  $M$ , and  $M$  must be a minimal surface.*

See [SW89] for the proof. (Note the additional remarks at the end of that paper.)

For a full treatment of varifolds, see Allard's paper [All72] or chapter 8 of Simon's book [Sim83]. Simon's book only considers varifolds in Euclidean space. However, as explained in [All72, §4.4], the study of  $k$ -varifolds in a Riemannian manifold  $N$  can be reduced to the Euclidean case by isometrically embedding  $N$  into a Euclidean space  $E$ . If one does that, the terminology in this paper needs to be interpreted accordingly. For example, " $V$  is a  $k$ -varifold in  $N$ " should be interpreted as:  $V$  is a  $k$ -varifold in  $E$  and  $V$  vanishes outside of

$$\{(x, S) \in G_k(E) : x \in N \text{ and } S \subset \text{Tan}_x N\}.$$

Also the mean curvature  $H$  of a submanifold  $M \subset N$  means the component of the mean curvature of  $M \subset E$  that is tangent to  $N$ . And the norm  $|\delta V|$  (which should perhaps be written  $|\delta V|_N$ ) means the supremum of  $\delta V(X)$  over all  $C^1$  vectorfields  $X$  with  $\|X\|_0 \leq 1$  that are tangent to  $N$  at each point of  $N$ .

We conclude this paper by proving that hypersurfaces moving by mean curvature flow cannot collide with stationary varifolds. (This was used in the proof of Theorem 2.5.) We begin with the smooth case:

**Proposition 7.7.** *Let  $S$  be the spatial support of a nonzero stationary  $n$ -varifold in a Riemannian  $(n + 1)$ -manifold  $N$ . Let  $t \in [0, T] \mapsto M_t$  be a mean curvature flow, where the  $M_t$ 's are compact smooth embedded hypersurfaces in  $N$ . If  $M_t$  and  $S$  are disjoint at time 0, then they must remain disjoint for all  $t \in [0, T]$ .*

*Proof.* Fix a small  $\epsilon$  with  $0 < \epsilon < \text{dist}(M_0, S)$  and a small  $\delta > 0$ . (How small will be specified shortly.) Let  $K_0 = K(\epsilon, \delta)$  be the set of points in  $N$  at distance  $\leq \epsilon$  from  $M_0$ . Note that  $K_0$  is disjoint from  $S$ . Let  $K_t$ ,  $t \in [0, T]$  be the result of letting  $K_0$  evolve so that its boundary moves with velocity  $H - \delta\nu$ , where  $\nu$  is the outward unit normal to  $K_t$ . We choose  $\epsilon$  and  $\delta$  sufficiently small that  $\partial K_t$  is smooth for all  $t \in [0, T]$ . By choosing  $\delta$  small enough (after having fixed  $\epsilon$ ), we can also ensure that  $M_t$  lies in the interior of  $K_t$  for all  $t \in [0, T]$ .

We claim that  $K_t$  is disjoint from  $S$  for all  $t \in [0, T]$ . For suppose not. Thus  $K_t$  will collide with  $S$  at some first time  $\tau \leq T$ . At any point  $p$  of contact of  $K_\tau$  and  $S$ , we have  $H - \delta\nu \geq 0$ , so in particular  $H \cdot \nu > 0$  at  $p$ , and therefore also on  $B \cap \partial K_\tau$  for some small ball  $B$  about  $p$ . But that violates the maximum principle (Theorem 7.6). (Let the  $M$  and  $\Omega$  in that theorem be  $B \cap \partial K_\tau$  and  $B \cap \text{interior}(K_\tau)$ .) Thus  $K_t$  remains disjoint from  $S$  for all  $t \in [0, T]$ . Since  $M_t \subset K_t$ , the surfaces  $M_t$  also remain disjoint from  $S$ . q.e.d.

**Theorem 7.8.** *Let  $S$  be the spatial support of a nonzero stationary  $n$ -varifold in a Riemannian  $(n + 1)$ -manifold  $N$ . Let  $M_t$ ,  $t \in [0, \infty)$ ,*

be the family of sets generated from  $M_0$  by the level-set mean curvature flow in  $N$ . If  $M_0$  and  $S$  are disjoint at time 0, then they remain disjoint for all  $t$ .

In particular, the evolution of an initially mean convex hypersurface by mean curvature flow is such a flow.

*Proof.* By Proposition 7.7,  $t \in [0, \infty) \mapsto S$  is a set-theoretic subsolution of mean curvature flow (in the terminology of [Ilm93] or [Ilm94, §10]) or a weak set-flow (in the terminology of [Whi95].) Theorem 7.8 is thus a special case of the avoidance principle for set-theoretic subsolutions [Whi95, 7.1]. q.e.d.

### References

- [All72] W.K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972) 417–491, MR 0307015, Zbl 0252.49028.
- [HdLR06] D. Hoffman, J.H.S. de Lira & H. Rosenberg, *Constant mean curvature surfaces in  $M^2 \times \mathbb{R}$* , Trans. Amer. Math. Soc. **358** (2006), no. 2, 491–507, MR 2177028, Zbl 1079.53088.
- [HS75] D. Hoffman & J. Spruck, *A Sobolev inequality for Riemannian submanifolds*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 1, Amer. Math. Soc., Providence, R.I., 1975, pp. 139–141, MR 0377771. *A correction to: “Sobolev and isoperimetric inequalities for Riemannian submanifolds”*, Comm. Pure Appl. Math. **28** (1975), no. 6, 765–766, MR 0397625,
- [Ilm93] T. Ilmanen, *The level-set flow on a manifold*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 193–204, MR 1216585, Zbl 0827.53014.
- [Ilm94] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no 520, x+90, MR 1196160, Zbl 0798.35066.
- [MR04] W.H. Meeks & H. Rosenberg, *Stable minimal surfaces in  $M \times \mathbb{R}$* , J. Differential Geom. **68** (2004), no. 3, 515–534, MR 2144539, Zbl 1079.53089.
- [MR05] W.H. Meeks & H. Rosenberg, *The theory of minimal surfaces in  $M \times \mathbb{R}$* , Comment. Math. Helv. **80** (2005), no. 4, 811–858, MR 2182702, Zbl 1085.53049.
- [NR02] B. Nelli, & H. Rosenberg, *Minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$* , Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 2, 263–292, MR 1940353, Zbl 1038.53011.
- [Pit81] J.T. Pitts, *Existence and regularity of minimal surfaces on Riemannian manifolds*, Mathematical Notes, vol. 27, Princeton University Press, Princeton, N.J., 1981, MR 626027, Zbl 0462.58003.
- [Ros02] H. Rosenberg, *Minimal surfaces in  $M^2 \times \mathbb{R}$* , Illinois J. Math. **46** (2002) 1177–1195, MR 1988257, Zbl 1036.53008.
- [Sch05] R. Schoen, *Mean curvature in Riemannian geometry and general relativity*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 113–136, MR 2167257, Zbl 1101.53038.
- [SS81] R. Schoen & L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), no 6, 741–797, MR 634285, Zbl 0497.49034.

- [Sim83] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University Centre for Mathematical Analysis, Canberra, 1983, MR 756417, Zbl 0546.49019.
- [SW89] B. Solomon & B. White, *A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals*, Indiana Univ. Math. J. **38** (1989), no. 3, 683–691, MR 1017330, Zbl 0711.49059.
- [Whi89] B. White, *New applications of mapping degrees to minimal surface theory*, J. Differential Geom. **29** (1989), no. 1, 143–162, MR 978083, Zbl 0638.58005.
- [Whi95] B. White, *The topology of hypersurfaces moving by mean curvature*, Comm. Anal. Geom. **3** (1995), no. 1-2, 317–333, MR 1362655, Zbl 0858.58047.
- [Whi00] B. White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **13** (2000), no. 3, 665–695, MR 1758759, Zbl 0961.53039.

MATHEMATICS DEPARTMENT  
STANFORD UNIVERSITY  
STANFORD, CA 94305

*E-mail address:* white@math.stanford.edu