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HOMOGENEOUS SPACES DEFINED BY LIE GROUP AUTOMORPHISMS. I

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1. Introduction

This paper is a study of the structure and geometry of coset spaces $X = G/K$ where G is a reductive Lie group and K is an open subgroup of the fixed point set G^θ of a semisimple automorphism θ of G. The symmetric spaces are the case $\theta^2 = 1$. There the structure and classification theory for θ is well known, and the geometry of X basically comes down to a knowledge of the linear isotropy representation of K and the "Cartan decomposition" of the Lie algebra of G into eigenspaces of θ. We follow this general outline, starting with a structure theory for θ, obtaining full classifications (including the linear isotropy representations) in the cases which we know to have significant geometric interest, and then turning to geometric applications utilizing the θ-eigenspace decomposition of the Lie algebra. The geometric applications which we pursue are concerned with G-invariant almost complex structures and almost hermitian metrics on X. The almost complex structures themselves are used as a technical tool in passing from compact G to reductive G in the structure theory for θ.

§ 2 gives the structure theory for an inner automorphism θ of a compact Lie algebra $\mathfrak{G}$. Choosing a Cartan subalgebra and simple root system for $\mathfrak{G}$, we obtain a normal form for θ (Proposition 2.6). This gives us a simple root system for the fixed point set $\mathfrak{G}^\theta$ (Proposition 2.8), a criterion for whether $\mathfrak{G}^\theta$ is the centralizer of a torus (Proposition 2.11), and a method for enumerating all θ of any given fixed finite order (Proposition 2.11). § 3 applies these results to a classification of inner automorphisms of order 3 (Theorem 3.3) and a classification of subalgebras $\mathfrak{G}^\theta \subset \mathfrak{G}$ which are not centralizers of tori (Theorem 3.5).

§ 4 is a complete classification and structure theory for invariant almost complex structures on coset spaces G/K where K is a connected[1] subgroup of

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[1] We show that the existence of an invariant almost complex structure on G/K implies connectedness of K.

maximal rank in a compact connected Lie group G. A result of Kostant [11] gives information on the linear isotropy representation of K (Theorem 4.3) which gives a new short proof of Passiencier's criterion [9] on (G, K) for the existence of an almost complex structure (Theorem 4.4), a unified treatment of integrability of almost complex structures on G/K (Theorem 4.5), a shorter sharper version of Hermann's classification [8] of the G/K which admit almost complex structures and for which K is not the centralizer of a torus (Theorem 4.10), and a complete enumeration (Theorems 4.7 and 4.11) of the almost complex structures.

§ 5 gives the structure theory of an outer (not inner) automorphism θ of a compact Lie algebra $\mathfrak{G}$. This comes down to a matter of some permutations of simple factors and the case where $\mathfrak{G}$ is simple with θ of order 3 (Lemma 5.3). In that case $\mathfrak{G}$ is of type D_4 and we show that either θ is the usual triality with $\mathfrak{G}^\theta$ of type G_2, or θ is an (apparently new) modification of the usual triality with $\mathfrak{G}^\theta$ of type A_2 (Theorem 5.5). The section ends with a synthesis which classifies the pairs $(\mathfrak{G}, \theta)$ for which $\theta \neq 1$, θ^m is outer if $\theta^m \neq 1$, and $\mathfrak{G}$ contains no proper θ-invariant ideal (Theorem 5.10).

§ 6 summarizes and reformulates globally, results of earlier sections for coset spaces $X = G/K$ where G is a compact connected Lie group acting effectively and $\mathfrak{K} = \mathfrak{G}^\theta$ for an automorphism θ of order 3. § 7 transforms the results of § 6 by Cartan involutions, changing the compactness hypothesis on G to the hypothesis that the Lie group G be reductive. Theorems 6.1 and 7.10 cover the cases where X is simply connected and $\mathfrak{G}$ has no proper θ-invariant ideal. Theorems 6.4 and 7.17 cover the (much more general) cases where X carries a G-invariant almost complex structure. § 7 ends with some minor results, useful in § 8, extending results of § 4 to the "noncompact case".

§§ 8 and 9 are a study of invariant almost hermitian metrics ds^2 on a reductive homogeneous space $M = G/K$, mostly concerned with the properties "hermitian", "semi-kaehlerian", "almost kaehlerian", "kaehlerian", "quasi-kaehlerian" and "nearly kaehlerian" for those metrics. The results of § 8 are general in that ds^2 is not required to be positive definite and at most we assume that G is reductive. Under various assumptions, such as the requirement that the almost complex structure of ds^2 be induced from an automorphism θ of $\mathfrak{G}$ such that $\mathfrak{K} = \mathfrak{G}^\theta$ or that the riemannian metric of ds^2 be induced by a bi-invariant bilinear form on $\mathfrak{G}$, Theorems 8.9, 8.11, 8.12 and 8.13 explore relations between, and existence questions for, these possible properties of (M, ds^2); some of those results are based on a characterization (Theorem 8.2) for certain types of almost complex structures based on automorphisms of order 3. Theorems 8.14, 8.15 and 8.16 are specific to the case where K is irreducible on the tangent space or $\mathfrak{K} = \mathfrak{G}^\theta$ with θ of order 3. The results of § 9 concern only positive definite ds^2 where G is compact and K is of maximal rank; they are much deeper than those of § 8 in that they describe the various possible properties of ds^2 in terms of the root patterns of the compact Lie groups G and K. The semi-kaehlerian property is automatic, the kaehler and almost kaehler properties turn out to be equivalent and easily characterized, and the quasi-kaehlerian and nearly kaehlerian properties are characterized after a delicate computation based on the interplay between root patterns and covariant derivatives; this comprizes Theorems 9.4, 9.15 and 9.17. We work out the root pattern criteria for the quasi-kaehler and nearly kaehler conditions, ending up with a complete analysis of the quasi-kaehler metrics in Theorem 9.24. The problem is more delicate for nearly kaehler metrics and we are only able to work it out in some special cases (Propositions 9.20, 9.21, 9.22, 9.23, 9.26 and 9.27, and the cases where K is semisimple). We end with the conjecture: *Let $M = G/K$ where G is a compact connected Lie group acting effectively, K is a subgroup of maximal rank, M carries a G-invariant almost complex structure, and G/K is not a hermitian symmetric coset space. Then M carries a G-invariant almost hermitian metric ds^2 which is nearly kaehler but not kaehler, if and only if $\mathfrak{K} = \mathfrak{G}^\theta$ for some automorphism θ of order* 3.

2. Canonical forms of inner automorphisms

In this section we find canonical forms for inner automorphisms of compact Lie algebras, and also describe the fixed point sets of these automorphisms. Any such algebra decomposes as $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1 \oplus \cdots \oplus \mathfrak{G}_t$ where $\mathfrak{G}_0$ is an abelian ideal and the other $\mathfrak{G}_i$ are simple ideals; any inner automorphism of $\mathfrak{G}$ decomposes as $\theta = 1 \times \theta_1 \times \cdots \times \theta_t$ where θ_i is an inner automorphism of $\mathfrak{G}_i$. Thus we need only consider the case where $\mathfrak{G}$ is simple.

Let $\mathfrak{G}$ be a compact simple Lie algebra, G the corresponding connected centerless Lie group, and T a maximal torus of G. Then $\mathfrak{T}$ is a Cartan subalgebra of $\mathfrak{G}$. Let Λ denote the root system and choose a system $\Psi = \{\psi_1, \cdots, \psi_l\}$ of simple roots, so

$$(2.1) \qquad \mathfrak{G}^C = \mathfrak{T}^C + \sum_{\lambda \in \Lambda} \mathfrak{G}_\lambda \quad \text{and} \quad \mathfrak{G} = \mathfrak{T} + \sum_{\substack{\lambda \in \Lambda \\ \lambda > 0}} \mathfrak{G} \cap (\mathfrak{G}_\lambda + \mathfrak{G}_{-\lambda})$$

is the Cartan decomposition. We have the "dual root lattice" in $\mathfrak{T}^* = \sqrt{-1}\,\mathfrak{T}$ given by

$$\Gamma = \frac{1}{2\pi\sqrt{-1}}(\text{kernel } \exp : \mathfrak{T} \to T) = \{x \in \mathfrak{T}^* : \psi_i(x) \in Z \text{ for } 1 \leq i \leq l\} .$$

The maximal root $\mu = \sum m_i \psi_i$ defines the simplex

$$\mathfrak{D}_0 = \{x \in \mathfrak{T}^* : \psi_i(x) \geq 0,\ \mu(x) \leq 1\}$$

in $\mathfrak{T}^{\#}$ with vertices $\{v_0, v_1, \cdots, v_l\}$ given by

$$(2.2) \qquad v_0 = 0 \,, \quad \psi_i(v_j) = 0 \quad \text{for} \quad i \neq j \,, \quad \psi_i(v_i) = 1/m_i \,.$$

Then $\mathfrak{D}_0$ has the property that every element of G is conjugate to element of $\exp(2\pi\sqrt{-1}\,\mathfrak{D}_0)$.

Let $x \in \mathfrak{T}^{\#}$, $g = \exp(2\pi\sqrt{-1}\,x)$ and $\theta = ad(g)$. Then we write $\mathfrak{G}^{\theta}$ for the fixed point set of θ in $\mathfrak{G}$; it is immediate that

$$(2.3) \qquad \mathfrak{G}^{\theta} = \mathfrak{T} + \sum_{\substack{\lambda \in \Lambda_x \\ \lambda > 0}} \mathfrak{G} \cap (\mathfrak{G}_{\lambda} + \mathfrak{G}_{-\lambda}) \,,$$

where $\Lambda_x = \{\lambda \in \Lambda : \lambda(x) \in \boldsymbol{Z}\}$.

Let W be the Weyl group of $\mathfrak{G}$ viewed as the transformation group on $\mathfrak{T}^{\#}$ generated by the reflections in the root hyperplanes $\lambda = 0$, $\lambda \in \Lambda$. Now the semidirect product $W \cdot \Gamma$ (called the extended Weyl group) acts on $\mathfrak{T}^{\#}$ by $(w, \gamma) : x \to w(x) + \gamma$. We write $x \approx y$ if $x, y \in \mathfrak{T}^{\#}$ are in the same $W \cdot \Gamma$-orbit.

We denote by $\overline{W}$ the Cartan group of $\mathfrak{G}$; while W is the group of transformations of $\mathfrak{T}^{\#}$ defined by inner automorphisms of $\mathfrak{G}$ which preserve $\mathfrak{T}$, $\overline{W}$ is defined to be the corresponding group defined by all automorphisms of $\mathfrak{G}$ which preserve $\mathfrak{T}$. Thus W is a normal subgroup of $\overline{W}$ and the quotient of the two is the "group of outer automorphisms". $\overline{W}$ preserves Γ and the semidirect product $\overline{W} \cdot \Gamma$ (called the extended Cartan group) acts on $\mathfrak{T}^{\#}$ as above. We write $x \sim y$ if $x, y \in \mathfrak{T}^{\#}$ are in the same $\overline{W} \cdot \Gamma$-orbit. Obviously $x \approx y$ implies $x \sim y$, and the converse is true if $\mathfrak{G}$ has no outer automorphisms.

2.4. Lemma. *Let* $x_i \in \mathfrak{T}^{\#}$ $(i = 1, 2)$ *and define* $\theta_i = ad(\exp 2\pi\sqrt{-1}\,x_i)$. *Then*

(i) $x_1 \approx x_2$ *if and only if* θ_1 *is conjugate to* θ_2 *in the group of inner automorphisms of* $\mathfrak{G}$;

(ii) $x_1 \sim x_2$ *if and only if* θ_1 *is conjugate to* θ_2 *in the group of all automorphisms of* $\mathfrak{G}$.

Proof. (i) is just the application to $ad(G)$ of the fact that two elements θ_i in a maximal torus $ad(T)$ are conjugate if and only if they are equivalent under the Weyl group. (ii) follows from the definition of the Cartan group.

q.e.d.

The simple root system $\Psi = \{\psi_1, \cdots, \psi_l\}$ of $\mathfrak{G}$ gives the Dynkin diagram of $\mathfrak{G}$; we denote by $\mathrm{Aut}(\Psi)$ the automorphism group of the Dynkin diagram. Adjoining $\psi_0 = -\mu$ to Ψ we have an "extended" system $\overline{\Psi}$; starting with $\overline{\Psi}$ and following the rules for the Dynkin diagram we construct the extended Dynkin diagram of $\mathfrak{G}$. Then $\mathrm{Aut}(\overline{\Psi})$ denotes the automorphism group of the extended Dynkin diagram.

In the following proposition we set $m_0 = 1$; then $\sum_{i=0}^{l} m_i \psi_i = 0$.

2.5. Proposition. (i) *We have*

$$\mathrm{Aut}(\overline{\Psi}) \cong \{\alpha \in \overline{W}\cdot\Gamma : \alpha(\mathfrak{D}_0) = \mathfrak{D}_0\}$$

via a canonical isomorphism. The isomorphism is given by $w^* \to \alpha$ *where* $w^* \in \mathrm{Aut}(\overline{\Psi})$ *is given by* $w^*(\psi_i) = \psi_i \cdot w^{-1}$ $(i = 0, \cdots, l)$ *and* $\alpha \in \overline{W}\cdot\Gamma$ *is defined by* $\alpha(x) = w^{-1}(x) + v_k$ *for* $x \in \mathfrak{T}^\#$ *where* v_k *is the vertex of* $\mathfrak{D}_0$ *with* $m_k = 1$ *such that* $w^*(\psi_k) = \psi_0$. *If* $w^*(\psi_i) = \psi_{a_i}$ *and* $\alpha \in \overline{W}\cdot\Gamma$ *corresponds to* w^* *under the isomorphism, then* $\alpha(v_i) = v_{a_i}$ $(0 \leq i \leq l)$.

(ii) *The canonical isomorphism, when restricted to the subgroup* $\mathrm{Aut}(\Psi)$ *of* $\mathrm{Aut}(\overline{\Psi})$, *yields*

$$\mathrm{Aut}(\Psi) \cong \{\alpha \in \overline{W} : \alpha(\mathfrak{D}_0) = \mathfrak{D}_0\}\ .$$

Proof. Let $w^* \in \mathrm{Aut}(\overline{\Psi})$ and define $w^{-1}: \mathfrak{T}^\# \to \mathfrak{T}^\#$ by $\psi_j \cdot w^{-1} = w^*(\psi_j)$ for $j = 1, \cdots, l$ and linearity. A calculation shows that $\psi_0 \cdot w^{-1} = w^*(\psi_0)$, also. Define $\alpha: \mathfrak{T}^\# \to \mathfrak{T}^\#$ by $\alpha(x) = w^{-1}(x) + v_k$ for $x \in \mathfrak{T}^\#$, where $w^*(\psi_k) = \psi_0$. Then $\alpha(0) = v_k$, $\psi_k(\alpha(v_j)) = 0$ for $j \neq 0$; and for $j \neq 0$, $p \neq k$ we have $\psi_p(\alpha(v_j)) = w^*(\psi_p)(v_j) = m_q^{-1}\delta_{jq}$ where q is such that $w^*(\psi_p) = \psi_q$. Thus α is a rigid motion of $\mathfrak{T}^\#$ which permutes the vertices of $\mathfrak{D}_0$ and so $\alpha(\mathfrak{D}_0) = \mathfrak{D}_0$. This automatically implies that $\alpha \in \overline{W}\cdot\Gamma$ [11, Theorem 8.11.2].

Conversely, if $\alpha \in \overline{W}\cdot\Gamma$ preserves $\mathfrak{D}_0$ it permutes the faces of $\mathfrak{D}_0$, inducing a permutation of the vertices of the extended Dynkin diagram. As α is an isometry on $\mathfrak{T}^\#$, the permutation of the vertices of the diagram preserves inner products of roots, so it is an automorphism of the extended Dynkin diagram. q.e.d.

We reformulate Proposition 2.5 as a conjugacy criterion.

2.6. Proposition. *If* $x_1 = \sum_{1\leq i\leq l} b_i v_i \in \mathfrak{D}_0$ *and* $x_2 = \sum_{1\leq i\leq l} c_i v_i \in \mathfrak{D}_0$, *then the following conditions are equivalent:*

(i) $x_1 \sim x_2$.

(ii) *There is an isometry of* $\mathfrak{D}_0$ *which sends* x_1 *to* x_2.

(iii) $\overline{W}\cdot\Gamma$ *has an element* α *such that* $\alpha(\mathfrak{D}_0) = \mathfrak{D}_0$ *and* $\alpha(x_1) = x_2$.

(iv) *Define* $\psi_0 = -\mu$, $b_0 = 1 - \mu(x_1)$ *and* $c_0 = 1 - \mu(x_2)$. *Then* $\mathrm{Aut}(\overline{\Psi})$ *has an element* γ, $\gamma(\psi_i) = \psi_{a_i}$, *such that* $b_i = c_{a_i}$ *for* $0 \leq i \leq l$.

Equivalence of (ii) and (iii) is the fact that the root system determines the semisimple compact Lie algebra up to isomorphism. Lemma 2.4 is the equivalence of (i) and (iii). Now we need only prove the equivalence of (iii) and (iv).

Let $\alpha \in \overline{W}\cdot\Gamma$ such that $\alpha(\mathfrak{D}_0) = \mathfrak{D}_0$ and let w^* be the corresponding element of $\mathrm{Aut}(\overline{\Psi})$. $w^*(\psi_i) = \psi_{a_i}$ and $\alpha(v_i) = v_{a_i}$. Decompose α into a linear transformation $w \in \overline{W}$ and a translation $t \in \Gamma$, $\alpha(x) = w(x) + t$. Using $\sum_{i=0}^{l} b_i = 1$, a consequence of the definition of b_0, we compute

$$\alpha(x_1) = \alpha(\sum_{i=1}^{l} b_i v_i) = \alpha(\sum_{i=0}^{l} b_i v_i) = w(\sum_{i=0}^{l} b_i v_i) + t = \sum_{i=0}^{l} b_i w(v_i) + t$$
$$= \sum_{i=0}^{l} b_i(w(v_i) + t) = \sum_{i=0}^{l} b_i \alpha(v_i) = \sum_{a_i=1}^{l} b_i v_{a_i} .$$

Thus $\alpha(x_1) = x_2$ if and only if $b_i = c_{a_i}$. This proves equivalence of (iii) and (iv).

In §3 we will need a special case of Proposition 2.6. Let $k > 1$ be an integer. Let $x, y \in \mathfrak{D}_0$ such that the automorphisms

$$ad(\exp 2\pi\sqrt{-1}\, x) \quad \text{and} \quad ad(\exp 2\pi\sqrt{-1}\, y)$$

of $\mathfrak{G}$ have order k. Suppose that x and y lie on edges of $\mathfrak{D}_0$ which meet the origin v_0, i.e. that $x = (m_i r/k)v_i$ and $y = (m_j s/k)v_j$ where r and s are positive integers prime to k. Then $x \sim y$ if and only if

(i) $m_i = m_j = 1$, $r = s$ or $r + s = k$, and $\gamma(\psi_i) = \psi_j$ for some $\gamma \in \mathrm{Aut}(\Psi)$; or

(ii) $m_i = m_j > 1$, $r = s < k/m_i$, and $\gamma(\psi_i) = \psi_j$ for some $\gamma \in \mathrm{Aut}(\Psi)$; or

(iii) $m_i = m_j > 1$, $r = s = k/m_i$, and $\gamma(\psi_i) = \psi_j$ for some $\gamma \in \mathrm{Aut}(\bar{\Psi})$.

A basic step in our normalization is:

2.7. Proposition. *Let $x \in \mathfrak{T}^{\#}$, and $k > 0$ be an integer such that $x \notin \Gamma$ but $kx \in \Gamma$. Replace x by a transform $w(x) + \gamma \in \mathfrak{D}_0$, $w \in W$ and $\gamma \in \Gamma$, and decompose $x = \frac{1}{k}\sum_{i=1}^{l} n_i m_i v_i$ where $n_i = \psi_i(kx) \in Z$. Make this transformation in such a way as to minimize $\sum_{i=1}^{l} n_i m_i$. Then*

(i) $0 \leq n_i < k$ and $0 < \sum n_i m_i \leq k$, and $\sum n_i m_i = \mathrm{k}$ implies that $m_j > 1$ whenever $n_j \neq 0$;

(ii) $n_j \leq k/2$ if $m_j = 1$; and

(iii) the sets $I_t = \{i : n_i \geq t\}$ have cardinality $|I_1| \geq 1$ and $|I_t| < k/t$, and I_t is empty for $t > k/2$.

Proof. By Proposition 2.6 and the conjugacy property of $\mathfrak{D}_0$ let us transform x into $\mathfrak{D}_0$. That done, $\psi_i(x) \geq 0$ and $\mu(x) \leq 1$. $x = \frac{1}{k}\sum n_i m_i v_i$ gives $n_i = \psi_i(kx) \in Z$, so $n_i \geq 0$ and $1 \geq \frac{1}{k}\sum n_i m_i$. Now $k \geq \sum n_i m_i$. Note $n_i < k$; for $n_i = k$ would imply $m_i = 1$ and $x = v_i \in \Gamma$.

Re-order Ψ so that $n_i \neq 0$ for $1 \leq i \leq r$ and $n_i = 0$ for $r < i \leq l$. If $\sum n_i m_i = k$, i.e. if $\mu(x) = 1$, then x is interior to an $(r-1)$-face of the $(l-1)$-face $\mu = 1$ of $\mathfrak{D}_0$. If some $m_j = 1$ where $1 \leq j \leq r$, we apply the translation $z \to z - v_j$ from Γ and then the Weyl group element which carries $\mathfrak{D}_0 - v_j$ back to $\mathfrak{D}_0$; this carries the $(r-1)$-face with vertices $\{v_1, \cdots, v_r\}$ which has x in its interior, to an $(r-1)$-face $\{v_0, v_{i_1}, \cdots, v_{i_{r-1}}\}$. In other words, if $\sum n_i m_i = k$ then we may assume that $m_i = 1$ implies $n_i = 0$. Now (i) is proved.

Let $m_j = 1$ with $n_j > k/2$. We now have $\sum m_i n_i < k$, so x is interior to the r-face of $\mathfrak{D}_0$ with vertices $\{v_0, v_1, \cdots, v_r\}$ which include v_j. We again apply a $W\cdot\Gamma$-equivalence

$$q: z \to w(z) + \gamma\ , \quad q(v_j) = v_0 = 0\ , \quad q(\mathfrak{D}_0) = \mathfrak{D}_0\ .$$

Now, as in the last step of the proof of Proposition 2.6,

$$\begin{aligned} q(x) &= q\left(\sum_{i=1}^{r} \frac{n_i m_i}{k} v_i + \left\{1 - \sum_{i=1}^{r} \frac{n_i m_i}{k}\right\} v_0\right) \\ &= \frac{1}{k} \sum_{\substack{i=1 \\ i\neq j}}^{r} n_i m_i q(v_i) + \frac{1}{k}\Big(k - \sum_{i=1}^{r} n_i m_i\Big) q(v_0)\ . \end{aligned}$$

Write $q(v_i) = v_{q_i}$; for $0 \neq i \neq j$, $m_i = m_{q_i}$; and $m_{q_0} = 1$ because $q(v_0) = v_{q_0} \in \Gamma$; now

$$q(x) = \frac{1}{k} \sum_{\substack{i=0 \\ i\neq j}}^{r} n_i m_{q_i} v_{q_i}\ , \qquad n_0 = k - \sum_{i=1}^{r} n_i m_i < \frac{k}{2}\ .$$

If b were another index with $m_b = 1$ and $n_b > k/2$ then $\sum_{i=1}^{r} n_i m_i \geq n_j + n_b > k$, which is absurd; thus j was the only index with $m_j = 1$ and $n_j > k/2$. Now (ii) is proved.

$x \neq 0$ shows I_1 nonempty so $|I_1| \geq 1$. And $k \geq \sum_{i=1}^{l} n_i m_i \geq \sum_{n_i \geq t} n_i m_i \geq t \sum_{n_i \geq t} m_i \geq t|I_t|$ shows $|I_t| \leq k/t$. If we had equality then

$$k = \sum n_i m_i\ , \text{ so } n_i \neq 0 \text{ implies } m_i > 1;$$

$$\sum_{i=1}^{l} n_i m_i = \sum_{n_i \geq t} n_i m_i\ , \text{ so } n_i \neq 0 \text{ says } n_i \geq t;$$

$$\sum_{n_i \geq t} m_i = |I_t|, \text{ so } n_i \geq t \text{ gives } m_i = 1;$$

which are inconsistent. Thus $|I_t| < k/t$. If $t > k/2$ and $n_i \in I_t$, then $n_i > k/2$ so $m_i = 1$; but that says $n_i \leq k/2$. Thus I_t is empty for $t > k/2$. q.e.d.

Next we determine the root system of the fixed point set of an inner automorphism of a compact simple Lie algebra.

2.8. Proposition. *Let $\mathfrak{G}$ be a compact simple Lie algebra with simple root system $\Psi = \{\psi_1, \cdots, \psi_l\}$, and θ an inner automorphism of $\mathfrak{G}$. Normalize θ, so $\theta = ad(\exp 2\pi\sqrt{-1}\,x)$ where $x = \sum_{i=1}^{l} c_i v_i \in \mathfrak{D}_0$. If $\mu = \sum m_i \psi_i$ is the maximal root of $\mathfrak{G}$ so that $\mu(x) = \sum c_i$, then Ψ_x, defined as follows, is a simple root system for $\mathfrak{G}^\theta$:*

(2.9) $$\Psi_x = \{\psi_i \in \Psi : c_i = 0\}\ , \quad if \quad \mu(x) < 1\ ,$$

(2.10) $$\Psi_x = \{\psi_i \in \Psi : c_i = 0\} \cup \{-\mu\}\ , \quad if \quad \mu(x) = 1\ .$$

Proof. Let $\lambda = \sum a_i\psi_i$ be a positive root of $\mathfrak{G}$. Then $0 \leq a_i \leq m_i$ and $\lambda(x) = \sum \frac{a_i}{m_i} c_i$, so $0 \leq \lambda(x) \leq 1$. Now, using (2.3), λ is a root of $\mathfrak{G}^\theta$ if and only if $\lambda(x)$ is 0 or 1; $\lambda(x) = 0$ if and only if $c_i \neq 0$ implies $a_i = 0$; and $\lambda(x) = 1$ if and only if $c_i \neq 0$ implies $a_i = m_i$. If $\mu(x) < 1$, then $\lambda(x) < 1$ and it follows that λ is generated by the root system of (2.9). If $\lambda(x) = 1$ then $\mu(x) = 1$ and there is a chain $\{\lambda = \lambda_0, \lambda_1, \cdots, \lambda_q = \mu\}$ of positive roots such that $\lambda_{i+1} = \lambda_i + \psi_{j_i}$ with $c_{j_i} = 0$, so $-\lambda = 1(-\mu) + \sum_{c_i=0} (m_i - a_i)\psi_i$ is generated by the root system of (2.10). Now we have proved Ψ_x contains a simple root system of $\mathfrak{G}^\theta$. As $\psi(x) \in Z$ for every $\psi \in \Psi_x$, the converse follows and the proposition is proved.

Using the normalization given by Proposition 2.7 we determine necessary and sufficient conditions that an inner automorphism have a given order or that its fixed point set is the centralizer of a torus.

2.11. Proposition. *Let $\theta = ad(\exp 2\pi\sqrt{-1}\, x)$ be an inner automorphism of a compact simple Lie algebra $\mathfrak{G}$ with maximal root $\mu = \sum m_i\psi_i$, in a simple root system $\Psi = \{\psi_1, \cdots, \psi_l\}$. Normalize $x = \sum_{i=1}^{l} c_i v_i$ in accordance with Proposition* 2.7.

1. *θ has finite order $k > 0$ if and only if* (1a) *the numbers $n_i = kc_i/m_i = \psi_i(kx)$ have the property that $\{n_i : c_i > 0\}$ is a nonempty set of relatively prime integers or* (1b) $x = 0$, $k = 1$.

2. *$\mathfrak{G}^\theta$ is the centralizer of a torus if and only if* (2a) $\mu(x) < 1$; *or* (2b) $\mu(x) = 1$, *$c_i > 0$ implies that $m_i > 1$, and $\{m_i : c_i > 0\}$ is a set of $r \geq 2$ relatively prime integers.*

3. *$\mathfrak{G}^\theta$ is not the centralizer of a torus if and only if and only if* (3a) $\mu(x) = 1$, (3b) *$c_i > 0$ implies $m_i > 1$, and* (3c) *$\{m_i : c_i > 0\}$ either has just one element or is a set of $r \geq 2$ integers with greatest common divisor $p = 2, 3$, or* 4. *In the former case of* (3c), *$x = v_j$ and θ has order $m_j > 1$; in the latter case of* (3c) *if θ has order k then p divides k.*

Proof. For (1) we have $\theta^k = ad(\exp 2\pi\sqrt{-1}\, kx)$, so $\theta^k = 1$ if and only if the $n_i = kc_i/m_i = \psi_i(kx)$ are integers. Factor $k = tu$, $u \geq 1$; again, $\theta^t = 1$ if an only if the n_i/u are integers. Let $\theta^k = 1$; now θ has order k if and only if the set $\{n_i : c_i \neq 0\}$ is a set of relatively prime integers. This proves (1).

Let $I = \{i : c_i > 0\}$ be enumerated as $\{i_1, \cdots, i_r\}$. If $x = 0$ then $\mathfrak{G}^\theta = \mathfrak{G}$, centralizer of the toral subalgebra $\{0\} \subset \mathfrak{T}$, and $0 = \mu(x) < 1$. In the proof of (2) and (3) we may now assume $r \geq 1$.

If $\mu(x) < 1$, so Ψ_x is given by (2.9), then $\mathfrak{G}^\theta$ is the centralizer of the toral algebra $\{\sqrt{-1}\sum_{s=1}^{r} t_{i_s}v_{i_s} : t_{i_s} \text{ real}\}$. In the proof of (2) and (3) we may now assume $\mu(x) = 1$.

If $m_{i_s} = 1$ then the second paragraph of the proof of Proposition 2.7 gives us a $W\cdot\Gamma$-conjugate $x' \in \mathfrak{D}_0$ of x such that $\mu(x') < 1$. As $\mathfrak{G}^\theta$ and $\mathfrak{G}^{\theta'}$ are conjugate by Lemma 2.4, where $\theta' = ad(\exp 2\pi\sqrt{-1}\, x')$, $\mathfrak{G}^\theta$ is the centralizer of a torus. In the proof of (2) and (3) we may now assume each $m_{i_s} > 1$.

If $r = 1$, then $x = v_{i_1}$ and θ has order m_{i_1}, and $\mathfrak{G}^\theta$ cannot be the centralizer of a torus because it is a proper (by $m_{i_1} > 1$) semisimple (by (2.10)) subalgebra of $\mathfrak{G}$. In the proof of (2) and (3) we may now assume $r > 1$.

Re-order Ψ so that $I = \{1, \cdots, r\}$; then $\Psi_x = \{\psi_{r+1}, \cdots, \psi_l, -\mu\}$. Define $z = v_1 \in \mathfrak{D}_0$ so that $\Psi_z = \{\psi_2, \cdots, \psi_l, -\mu\}$ is the simple root system for $\mathfrak{G}^\varphi$, $\varphi = ad(\exp 2\pi\sqrt{-1}\, z)$.

Let $\mathfrak{D}_0'$ be the fundamental simplex of the semisimple algebra $\mathfrak{G}^\varphi$. Then $\mathfrak{D}_0'$ has vertices $\{v_0', v_1', \cdots, v_l'\}$, where $m_1'(-\mu) + \sum_{i=2}^{l} m_i'\psi_i$ is the maximal root of $\mathfrak{G}^\varphi$, where $v_0' = 0$, where $v_1' = -\frac{1}{m_1'}v_1$, and where $v_i' = \frac{m_i}{m_i'}(v_i - v_1)$ for $2 \le i \le l$. Let $\mathfrak{S}$ be the toral subalgebra $\{\sqrt{-1}\sum_{i=2}^{r} t_i v_i' : t_i \in \boldsymbol{R}\}$ of $\mathfrak{G}^\theta$; then $\mathfrak{G}^\theta$ is the centralizer of $\mathfrak{S}$ in $\mathfrak{G}^\varphi$. Further $\mathfrak{S}$ is the center of $\mathfrak{G}^\theta$ because $\mathfrak{S} \subset \mathfrak{G}^\theta$ and both $\mathfrak{S}$ and that center have the same dimension $r - 1$. Let $\mathfrak{L}$ be the centralizer of $\mathfrak{S}$ in $\mathfrak{G}$. Now $\mathfrak{G}^\theta \subset \mathfrak{L}$, and $\mathfrak{G}^\theta$ is the centralizer of a torus if and only if $\mathfrak{G}^\theta = \mathfrak{L}$.

Let $\lambda = \sum a_i\psi_i$ be a root. Then λ is a root of $\mathfrak{L}$ if and only if $\lambda(v_i') = 0$ for $2 \le i \le r$. In that range, $\lambda(v_i') = \frac{m_i}{m_i'}\left(\frac{a_i}{m_i} - \frac{a_1}{m_1}\right)$. Thus λ is a root of $\mathfrak{L}$ if and only if $a_1/m_1 = a_2/m_2 = \cdots = a_r/m_r$. In other words, $\mathfrak{G}^\theta = \mathfrak{L}$ if and only if $\mathfrak{G}^\theta$ has root system given by

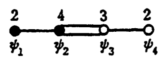
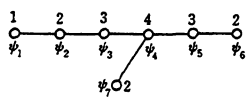
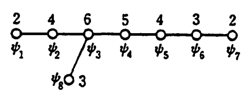
$$(2.11) \qquad \Lambda_x = \{\lambda = \sum a_i\psi_i \in \Lambda : \text{all the } a_{i_s}/m_{i_s} \text{ are equal}\}\,,$$

where we revert to the notation $I = \{i_1, \cdots, i_r\}$.

Suppose $\{m_{i_1}, \cdots, m_{i_r}\}$ relatively prime. Then there are integers u_{i_s} with $\sum_{s=1}^{r} u_{i_s}m_{i_s} = 1$. Let $\lambda = \sum a_i\psi_i$ be a root of $\mathfrak{L}$, so there is a real number t such that $a_{i_s} = tm_{i_s}$ for $1 \le s \le r$. It $t = 0$ then λ is a linear combination of the roots in $\Psi \cap \Psi_x$. It $t \ne 0$ then $\frac{1}{t}\sum_{s=1}^{r} u_{i_s}a_{i_s} = 1$. As the sum is an integer and $|t| \le 1$ this says $t = \pm 1$, so $\lambda(x) = t \in \boldsymbol{Z}$. In either case, now, λ is a root of $\mathfrak{G}^\theta$. Thus $\mathfrak{G}^\theta$ is the centralizer of the toral algebra $\mathfrak{S}$.

Conversely suppose $\{m_{i_1}, \cdots, m_{i_r}\}$ not relatively prime, let $p > 1$ denote the greatest common divisor, and let q be a prime divisor of p. Now q

divides each m_{i_s} and we write $m_{i_s} = qa_{i_s}$. By classification the only possibilities for q are 2 and 3; here 5 is excluded because $r > 1$. If each of the $m_{i_s} = 2$ we define $\lambda = \sum_{i=1}^{l} \psi_i$. If some $m_{i_s} \neq 2$, then $\mathfrak{G}$ is of type F_4, E_7 or E_8, and we define $\lambda = \lambda_q$ where

F_4		$\lambda_2 = \psi_1 + 2\psi_2 + 2\psi_3 + \psi_4$
E_7		$\lambda_2 = \lambda_3 = \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7$
E_8		$\lambda_2 = \psi_1 + 2\psi_2 + 3\psi_3 + 3\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7 + 2\psi_8$ $\lambda_3 = \psi_2 + 2\psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_8$

Now we have a root $\lambda = \sum a_i \psi_i$ of $\mathfrak{G}$ with $m_{i_s} = qa_{i_s}$. λ satisfies the criterion of (2.11), but λ is not a root of $\mathfrak{G}^\theta$ because $\lambda(x) = 1/q$. Thus $\mathfrak{G}^\theta$ is not the centralizer of a torus. If $p \neq 2$ then a glance at F_4, E_7 and E_8 above shows $p = 3$ or $p = 4$. In any case, if θ has order k, then $k = \sum_{i=1}^{l} n_i m_i = \sum_{s=1}^{r} n_{i_s} m_{i_s}$ $= p \sum_{s=1}^{r} n_{i_s}(m_{i_s}/p)$ so p divides k.

This completes the proof of (2) and (3).

3. Classifications involving inner automorphisms

We apply the results of § 2 on an inner automorphism θ of a compact Lie algebra $\mathfrak{G}$, obtaining classifications both for the case where θ is of order 3 and for the case where $\mathfrak{G}^\theta$ is not the centralizer of a torus. We start with two lemmas which allow us to avoid duplication for the case where θ is of order 3.

We specialize the remark just after Proposition 2.6:

3.1. Lemma. *Let $x, y \in \mathfrak{D}_0$ such that $ad(\exp 2\pi\sqrt{-1}\,x)$ and $ad(\exp 2\pi\sqrt{-1}\,y)$ are automorphisms of order 3 on $\mathfrak{G}$. Suppose that x and y are on edges of $\mathfrak{D}_0$ containing v_0,* i.e. *$x = (m_i r/3)v_i$ and $y = (m_j s/3)v_j$. Then $x \sim y$ if and only if*

(i) $m_i = m_j = 1$, $1 \leq r,s \leq 2$, *and* $\gamma(\psi_i) = \psi_j$ *for some* $\gamma \in \mathrm{Aut}\,(\Psi)$; *or*

(ii) $m_i = m_j = 2$, $r = s = 1$, *and* $\gamma(\psi_i) = \psi_j$ *for some* $\gamma \in \mathrm{Aut}\,(\Psi)$; *or*

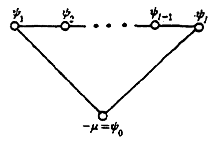
(iii) $m_i = m_j = 3$, $r = s = 1$, *and* $\gamma(\psi_i) = \psi_j$ *for some* $\gamma \in \mathrm{Aut}\,(\Psi)$.

Now we need something which is less obvious:

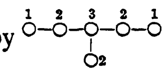
3.2. Lemma. *Let $\mathfrak{G}$ be simple. Suppose that $x = \frac{1}{3}(v_i + v_j)$ with $i \neq j$ and $m_i = m_j = 1$, and $y = \frac{1}{3}(v_r + v_s)$ with $r \neq s$ and $m_r = m_s = 1$. Then*

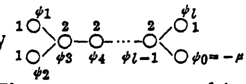
(i) *$\mathfrak{G}$ is of type D_l or E_6 and $x \sim y$; or*

(ii) *$\mathfrak{G}$ is of type A_l, and $x \sim y$ if and only if, assuming $i < j$ and $r < s$, and assuming the ψ_k ordered as at the right, $\{r - 1, s - r - 1, l - s\}$ is a permutation of $\{i - 1, j - i - 1, l - j\}$.*



Proof. As there are two distinct indices i, j with $m_i = m_j = 1$, $\mathfrak{G}$ must be of type $A_l(l > 1)$, $D_l(l > 3)$ or E_6. We run through these cases.

If $\mathfrak{G}$ is of type E_6 the m_k are given by , so $x = y$, so $x \sim y$.

If $\mathfrak{G}$ is of type D_l the m_k are given by , i.e. $m_1 = m_2 = m_l = 1$ and $m_k = 2$ for $2 < k < l$. First suppose $x = \frac{1}{3}(v_1 + v_2)$. If $y = x$ then $x \sim y$; otherwise y is $\frac{1}{3}(v_{(1 \text{ or } 2)} + v_l)$ and we may apply an element of $\mathrm{Aut}(\Psi)$ and assume $y = \frac{1}{3}(v_1 + v_l)$. We then apply an element of $\mathrm{Aut}(\overline{\Psi})$ which interchanges ψ_0 and ψ_1, sending y to x, and see $x \sim y$. Now suppose $x \neq \frac{1}{3}(v_1 + v_2) \neq y$. Then we apply elements of $\mathrm{Aut}(\Psi)$ sending x and y to $\frac{1}{3}(v_1 + v_l)$ and see $x \sim y$. Thus $x \sim y$ in any case.

If $\mathfrak{G}$ is of type A_l the m_k are all 1. Following the numbering in the statement of the lemma, $x \sim y$ if and only if there is either a cyclic permutation of $\{0, 1, \cdots, l\}$ or a cyclic permutation followed by reversal of order, sending $\{0, i, j\}$ to the set $\{0, r, s\}$. The assertion follows.

We now have the classification for inner automorphisms of order 3.

3.3. Theorem. *Let φ be an inner automorphism of order 3 on a compact or complex simple Lie algebra $\mathfrak{G}$. Choose a Cartan subalgebra and let $\Psi = \{\psi_1, \cdots, \psi_l\}$ be a simple root system for $\mathfrak{G}$. Then φ is conjugate in the inner automorphism group of $\mathfrak{G}$ to some $\theta = ad(\exp 2\pi\sqrt{-1}\,x)$ where either $x = \frac{1}{3}m_i v_i$ with $1 \leq m_i \leq 3$ or $x = \frac{1}{3}(v_i + v_j)$ with $m_i = m_j = 1$. A complete list of the possibilities for x, the fixed point set $\mathfrak{G}^\theta$ and a simple root system Ψ_x of $\mathfrak{G}^\theta$, up to conjugacy in the automorphism group of $\mathfrak{G}$, is listed in the table below.*

Remark. To separate the table entries into conjugacy classes under the group of *inner* automorphisms of $\mathfrak{G}$ we operate as follows.

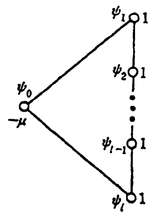
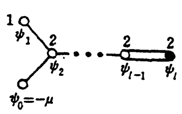
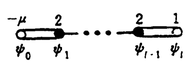
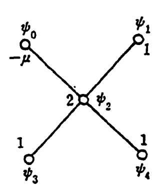
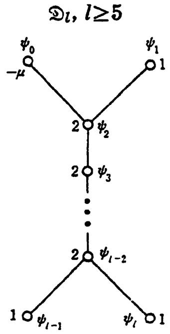
A_l, $l \geq 2$: distinguish between $\frac{1}{3}v_i$ and $\frac{1}{3}v_{l-i+1}$ for $2i \neq l + 1$; between $\frac{1}{3}(v_i + v_j)$ $(i < j)$ and $\frac{1}{3}(v_r + v_s)$ $(r < s)$ if $\{r - 1, s - r - 1, l - s\}$ is an *odd* permutation of $\{i - 1, j - i - 1, l - j\}$.

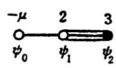
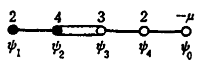
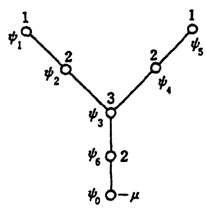
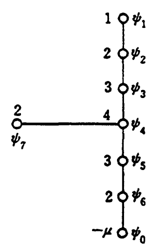
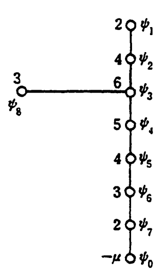
D_4: distinguish between $\frac{1}{3}v_1$, $\frac{1}{3}v_3$ and $\frac{1}{3}v_4$.

D_l, $l \geq 5$: distinguish between $\frac{1}{3}v_l$ and $\frac{1}{3}v_{l-1}$.

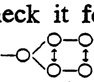
E_6: distinguish between $\frac{1}{3}v_1$ and $\frac{1}{3}v_5$; between $\frac{2}{3}v_2$ and $\frac{2}{3}v_4$.

In any case, conjugacy of θ in the full automorphism group of $\mathfrak{G}$ results in conjugacy of $\mathfrak{G}^\theta$ by the inner automorphism group of $\mathfrak{G}$.

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$
$\mathfrak{A}_1$	$\frac{1}{3}v_1$	empty	$\mathfrak{T}^1$
$\mathfrak{A}_l,\ l\geq 2$ 	$\frac{1}{3}v_i\ [\sim\frac{1}{3}v_{l-i+1}]$	$\{\psi_1, \cdots, \psi_{i-1};\ \psi_{i+1}, \cdots, \psi_l\}$	$\mathfrak{A}_{i-1}\oplus\mathfrak{A}_{l-i}\oplus\mathfrak{T}^1$
	$\frac{1}{3}(v_i+v_j),\ i<j$ $[\sim\frac{1}{3}(v_r+v_s)$ if $\{r-1, s-r-1, l-s\} = \{i-1, j-i-1, l-j\}]$	$\{\psi_1, \cdots, \psi_{i-1};\ \psi_{i+1}, \cdots, \psi_{j-1};\ \psi_{j+1}, \cdots, \psi_l\}$	$\mathfrak{A}_{i-1}\oplus\mathfrak{A}_{j-i-1}\oplus\mathfrak{A}_{l-j}\oplus\mathfrak{T}^2$
$\mathfrak{B}_l,\ l\geq 2$ 	$\frac{1}{3}v_1$	$\{\psi_2, \cdots, \psi_l\}$	$\mathfrak{B}_{l-1}\oplus\mathfrak{T}^1$
	$\frac{2}{3}v_i,\ 2\leq i\leq l$	$\{\psi_1, \cdots, \psi_{i-1};\ \psi_{i+1}, \cdots, \psi_l\}$	$\mathfrak{A}_{i-1}\oplus\mathfrak{B}_{l-i}\oplus\mathfrak{T}^1$
$\mathfrak{C}_l,\ l\geq 2$ 	$\frac{2}{3}v_i,\ 1\leq i\leq l-1$	$\{\psi_1, \cdots, \psi_{i-1};\ \psi_{i+1}, \cdots, \psi_l\}$	$\mathfrak{A}_{i-1}\oplus\mathfrak{C}_{l-i}\oplus\mathfrak{T}^1$
	$\frac{1}{3}v_l$	$\{\psi_1, \cdots, \psi_{l-1}\}$	$\mathfrak{A}_{l-1}\oplus\mathfrak{T}^1$
$\mathfrak{D}_4$ 	$\frac{1}{3}v_1\ [\sim\frac{1}{3}v_3\sim\frac{1}{3}v_4]$	$\{\psi_2, \psi_3, \psi_4\}$	$\mathfrak{A}_3\oplus\mathfrak{T}^1$
	$\frac{2}{3}v_2$	$\{\psi_1; \psi_3; \psi_4\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{T}^1$
	$\frac{1}{3}(v_1+v_3)$ $[\sim\frac{1}{3}(v_1+v_4)\sim\frac{1}{3}(v_3+v_4)]$	$\{\psi_2, \psi_4\}$	$\mathfrak{A}_2\oplus\mathfrak{T}^2$
$\mathfrak{D}_l,\ l\geq 5$ 	$\frac{1}{3}v_1$	$\{\psi_2, \psi_3, \cdots, \psi_l\}$	$\mathfrak{D}_{l-1}\oplus\mathfrak{T}^1$
	$\frac{1}{3}v_l\ [\sim\frac{1}{3}v_{l-1}]$	$\{\psi_1, \psi_2, \cdots, \psi_{l-1}\}$	$\mathfrak{A}_{l-1}\oplus\mathfrak{T}^1$
	$\frac{2}{3}v_i,\ 2\leq i\leq l-3$	$\{\psi_1, \cdots, \psi_{i-1};\ \psi_{i+1}, \cdots, \psi_l\}$	$\mathfrak{A}_{i-1}\oplus\mathfrak{D}_{l-1}\oplus\mathfrak{T}^1$
	$\frac{1}{3}(v_{l-1}+v_l)$ $[\sim\frac{1}{3}(v_1+v_{l-1})$ $\sim\frac{1}{3}(v_1+v_l)]$	$\{\psi_1, \psi_2, \cdots, \psi_{l-2}\}$	$\mathfrak{A}_{l-2}\oplus\mathfrak{T}^2$

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$
$\mathfrak{G}_2$ 	v_1	$\{\psi_2, -\mu\}$	$\mathfrak{A}_2$
	$\frac{2}{3}v_2$	$\{\psi_1\}$	$\mathfrak{A}_1 \oplus \mathfrak{T}^1$
$\mathfrak{F}_4$ 	$\frac{2}{3}v_1$	$\{\psi_2, \psi_3, \psi_4\}$	$\mathfrak{B}_3 \oplus \mathfrak{T}^1$
	v_3	$\{\psi_1, \psi_2; \psi_4, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2$
	$\frac{2}{3}v_4$	$\{\psi_1, \psi_2, \psi_3\}$	$\mathfrak{C}_3 \oplus \mathfrak{T}^1$
$\mathfrak{E}_6$ 	$\frac{1}{3}v_1$ $[\sim\frac{1}{3}v_5]$	$\{\psi_2, \psi_3, \psi_4, \psi_5, \psi_6\}$	$\mathfrak{D}_5 \oplus \mathfrak{T}^1$
	$\frac{2}{3}v_2$ $[\sim\frac{2}{3}v_4]$	$\{\psi_1; \psi_3, \psi_4, \psi_5, \psi_6\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_4 \oplus \mathfrak{T}^1$
	$\frac{2}{3}v_6$	$\{\psi_1, \cdots, \psi_5\}$	$\mathfrak{A}_5 \oplus \mathfrak{T}^1$
	v_3	$\{\psi_1, \psi_2; \psi_4, \psi_5, \psi_6, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2$
	$\frac{1}{3}(v_1+v_5)$	$\{\psi_2, \psi_3, \psi_4, \psi_6\}$	$\mathfrak{D}_4 \oplus \mathfrak{T}^2$
$\mathfrak{E}_7$ 	$\frac{1}{3}v_1$	$\{\psi_2, \cdots, \psi_7\}$	$\mathfrak{E}_6 \oplus \mathfrak{T}^1$
	$\frac{2}{3}v_2$	$\{\psi_1; \psi_3, \cdots, \psi_7\}$	$\mathfrak{A}_1 \oplus \mathfrak{D}_5 \oplus \mathfrak{T}^1$
	$\frac{2}{3}v_6$	$\{\psi_1, \cdots, \psi_5, \psi_7\}$	$\mathfrak{D}_6 \oplus \mathfrak{T}^1$
	$\frac{2}{3}v_7$	$\{\psi_1, \cdots, \psi_6\}$	$\mathfrak{A}_6 \oplus \mathfrak{T}^1$
	v_3 $[\sim v_5]$	$\{\psi_1, \psi_2; \psi_4, \psi_5, \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_5$
$\mathfrak{E}_8$ 	$\frac{2}{3}v_1$	$\{\psi_2, \cdots, \psi_8\}$	$\mathfrak{D}_7 \oplus \mathfrak{T}^1$
	$\frac{2}{3}v_7$	$\{\psi_1, \cdots, \psi_6, \psi_8\}$	$\mathfrak{E}_7 \oplus \mathfrak{T}^1$
	v_6	$\{\psi_7, -\mu; \psi_1, \cdots, \psi_5, \psi_8\}$	$\mathfrak{A}_2 \oplus \mathfrak{E}_6$
	v_8	$\{\psi_1, \cdots, \psi_7, -\mu\}$	$\mathfrak{A}_8$

Proof. Conjugating φ in the inner automorphism group of $\mathfrak{G}$, we take it to $\theta = ad(\exp 2\pi\sqrt{-1}\,x)$, $x = \frac{1}{3}\sum n_i m_i v_i \in \mathfrak{D}_0$, as in Proposition 2.7. Now $1 \leq |I_1| \leq 2$, $0 \leq |I_2| \leq 1$, and I_t is empty for $t > 2$. If some $n_j = 2$ then Proposition 2.7(i) says $m_j = 1$, contradicting Proposition 2.7(ii); thus I_2 is empty. Now either x is of the form $\frac{1}{3}m_i v_i$ with $|I_1| = 1$, or x is of the form $\frac{1}{3}(m_i v_i + m_j v_j)$ with $i \neq j$ and $|I_1| = 2$. In the former case Proposition 2.7(i) says $1 \leq m_i \leq 3$; in the latter case it says $m_i = 1 = m_j$. Now the proof of the theorem is reduced to checking the lists in the table and to verifying the remark.

The list of possibilities for x in the table is clear, the stated equivalences being the content of Lemmas 3.1 and 3.2. Proposition 2.8 gives Ψ_x and thus $\mathfrak{G}^\theta$. This checks the table. To check the remark for A_l and D_l, the reader need only write out some matrices. To check it for E_6 he must take the standard involutive outer automorphism σ :  with fixed point set of type F_4, follow it by an inner automorphism sending $\sigma(\mathfrak{G}^\theta)$ back to $\mathfrak{G}^\theta$, and verify that the composite acts on the center of $\mathfrak{G}^\theta$ by $z \to -z$. q.e.d.

Note that the situation is much simpler for φ of order 2, i.e. for symmetric spaces. There either $x = v_i$ with $m_i = 2$ (non-hermitian symmetric space) or $x = \frac{1}{2}v_i$ with $m_i = 1$ (hermitian symmetric space).

The classification is still feasible for φ of order 5, although it becomes somewhat more complicated:

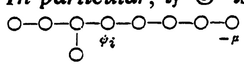
3.4. Proposition. *Let θ be an inner automorphism of order 5 on a compact or complex simple Lie algebra $\mathfrak{G}$, and $\mathfrak{G}^\theta$ the fixed point set of θ. Then θ is conjugate, by an inner automorphism of $\mathfrak{G}$, to $ad(\exp 2\pi\sqrt{-1}\,x)$ where*

(i) $x = \dfrac{m_i}{5}v_i$ *with* $1 \leq m_i \leq 5$, *or* $x = \frac{4}{5}v_i$ *with* $m_i = 2$, *or* $x = \frac{2}{5}v_i$ *with* $m_i = 1$; *or*

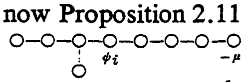
(ii) $x = \frac{1}{5}(v_i + v_j)$ *with* $m_i = m_j = 1$, *or* $x = \frac{1}{5}(2v_i + v_j)$ *with* $1 = m_j \leq m_i \leq 2$, *or* $x = \frac{1}{5}(3v_i + v_j)$ *with* $m_j = 1$, $m_i = 3$, *or* $x = \frac{2}{5}(v_i + v_j)$ *with* $1 \leq m_j \leq m_i \leq 2$, *or* $x = \frac{1}{5}(3v_i + 2v_j)$ *with* $m_i = 3$ *and* $m_j = 2$; *or*

(iii) $x = \frac{1}{5}(v_i + v_j + v_k)$ *with* $m_i = m_j = m_k = 1$, *or* $x = \frac{1}{5}(2v_i + v_j + v_k)$ *with* $1 = m_k = m_j \leq m_i \leq 2$; *or*

(iv) $x = \frac{1}{5}(v_i + v_j + v_k + v_l)$ *with* $m_i = m_j = m_k = m_l = 1$.

In particular, if $\mathfrak{G}^\theta$ is not the centralizer of a torus, then $\mathfrak{G}$ is of type E_8 

with $x = v_i$, $m_i = 5$, and $\mathfrak{G}^\theta$ of type $A_4 \oplus A_4$.

Proof. Proposition 2.7 shows that (i) through (iv) is a complete list of the possibilities for x. If $\mathfrak{G}^\theta$ is not the centralizer of a torus, now Proposition 2.11 shows $x = v_i$ with $m_i = 5$. In that case $\mathfrak{G}$ is of type E_8 

with $\mathfrak{G}^\theta$ of type $A_4 \oplus A_4$ by classification. q.e.d.

Now we go on to a somewhat different type of classification. The following is immediate from Propositions 2.7, 2.8 and 2.11.

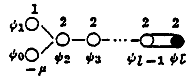
3.5. Theorem. *Let $\mathfrak{G}$ be a compact or complex simple Lie algebra with simple root system $\Psi = \{\psi_1, \cdots, \psi_l\}$, $x \in \mathfrak{D}_0$, $\theta = ad(\exp 2\pi\sqrt{-1}\, x)$, $\mathfrak{G}^\theta$ the fixed point set of θ, and Ψ_x a simple root system of $\mathfrak{G}^\theta$.*

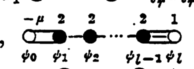
If $\mathfrak{G}$ is of classical type then the following is a complete list of the cases for which $\mathfrak{G}^\theta$ is not the centralizer of a torus:

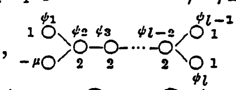
$x = \sum_{s=1}^{r} c_{i_s} v_{i_s}$ *with* $c_{i_s} > 0$ *and* $\sum_{s=1}^{r} c_{i_s} = 1$;

$\Psi = \{-\mu, \psi_1, \cdots, \psi_{i_1-1}; \psi_{i_1+1}, \cdots, \psi_{i_2-1}; \cdots; \psi_{i_r+1}, \cdots, \psi_l\}$; *and*

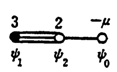
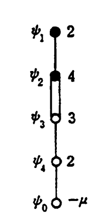
type A_l, $l \geq 1$: does not occur,

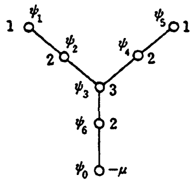
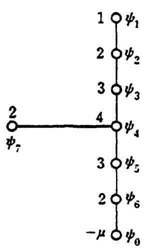
type B_l, $l \geq 2$,  *: here $2 \leq i_1 < \cdots < i_r \leq l$ and $\mathfrak{G}^\theta$ is of type $D_{i_1} \oplus A_{i_2-i_1-1} \oplus \cdots \oplus A_{i_r-i_{r-1}-1} \oplus B_{l-i_r} \oplus T^{r-1}$,*

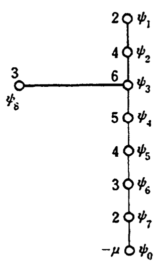
type C_l, $l \geq 2$,  *: here $1 \leq i_1 < \cdots < i_r \leq l-1$ and $\mathfrak{G}^\theta$ is of type $C_{i_1} \oplus A_{i_2-i_1-1} \oplus \cdots \oplus A_{i_r-i_{r-1}-1} \oplus C_{l-i_r} \oplus T^{r-1}$,*

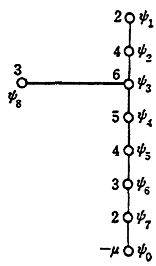
type D_l, $l \geq 4$,  *: here $2 \leq i_1 < \cdots < i_r \leq l-2$ and $\mathfrak{G}^\theta$ is of type $D_{i_1} \oplus A_{i_2-i_1-1} \oplus \cdots \oplus A_{i_r-i_{r-1}-1} \oplus D_{l-i_r} \oplus T^{r-1}$.*

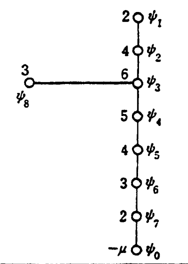
If $\mathfrak{G}$ is of exceptional type then the following table is a complete list up to automorphism of $\mathfrak{G}$ of the cases for which $\mathfrak{G}^\theta$ is not the centralizer of a torus; for each entry, the c_i listed are arbitrary positive numbers with sum 1.

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$
$\mathfrak{G}_2$ 	v_1	$\{\psi_2, -\mu\}$	$\mathfrak{A}_2$
	v_2	$\{\psi_1; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1$
$\mathfrak{F}_4$ 	v_1	$\{\psi_2, \psi_3, \psi_4, -\mu\}$	$\mathfrak{B}_4$
	v_2	$\{\psi_1; \psi_3, \psi_4, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_3$
	v_3	$\{\psi_1, \psi_2; \psi_4, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2$
	v_4	$\{\psi_1, \psi_2, \psi_3; -\mu\}$	$\mathfrak{A}_3 \oplus \mathfrak{A}_1$
	$c_1v_1 + c_2v_2$	$\{\psi_3, \psi_4, -\mu\}$	$\mathfrak{A}_3 \oplus \mathfrak{T}^1$
	$c_1v_1 + c_4v_4$	$\{\psi_2, \psi_3; -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1$
	$c_2v_2 + c_4v_4$	$\{\psi_1; \psi_3; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1$
	$c_1v_1 + c_2v_2 + c_4v_4$	$\{\psi_3; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^2$

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$
$\mathfrak{E}_6$ 	v_2 $[\sim v_4 \sim v_6]$	$\{\psi_1;$ $\psi_3, \psi_4, \psi_5, \psi_6, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_5$
	v_3	$\{\psi_1, \psi_2;$ $\psi_4, \psi_5; \psi_6, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2$
	$c_2v_2+c_4v_4$ $[\sim c_2v_2+c_4v_6 \sim c_4v_4+c_2v_6]$	$\{\psi_1; \psi_5; \psi_3, \psi_6, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_3 \oplus \mathfrak{T}^1$
	$c_2v_2+c_4v_4+c_6v_6$	$\{\psi_1; \psi_3; \psi_5; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1$ $\oplus \mathfrak{T}^2$
$\mathfrak{E}_7$ 	v_2 $[\sim v_6]$	$\{\psi_1; \psi_3, \cdots,$ $\psi_7, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{D}_6$
	v_3 $[\sim v_5]$	$\{\psi_1, \psi_2; \psi_4, \cdots,$ $\psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_5$
	v_4	$\{\psi_1, \psi_2, \psi_3;$ $\psi_5, \psi_6, -\mu; \psi_7\}$	$\mathfrak{A}_3 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_3$
	v_7	$\{\psi_1, \psi_2, \psi_3,$ $\psi_4, \psi_5, \psi_6, -\mu\}$	$\mathfrak{A}_7$
	$c_3v_3+c_5v_5$	$\{\psi_1, \psi_2;$ $\psi_4, \psi_7; \psi_6, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^1$
	$c_2v_2+c_6v_6$	$\{\psi_1; \psi_3, \psi_4, \psi_5, \psi_7;$ $-\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{D}_4 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1$
	$c_2v_2+c_7v_7$ $[\sim c_2v_6+c_7v_7]$	$\{\psi_1;$ $\psi_3, \psi_4, \psi_5, \psi_6, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_5 \oplus \mathfrak{T}^1$
	$c_2v_2+c_4v_4$ $[\sim c_4v_4+c_2v_6]$	$\{\psi_1; \psi_3; \psi_7;$ $\psi_5, \psi_6, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1$ $\oplus \mathfrak{A}_3 \oplus \mathfrak{T}^1$
	$c_4v_4+c_7v_7$	$\{\psi_1, \psi_2, \psi_3;$ $\psi_5, \psi_6, -\mu\}$	$\mathfrak{A}_3 \oplus \mathfrak{A}_3 \oplus \mathfrak{T}^1$
	$c_2v_2+c_6v_6+c_7v_7$	$\{\psi_1; \psi_3, \psi_4, \psi_5; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_3 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^2$
	$c_2v_2+c_4v_4+c_6v_6$	$\{\psi_1; \psi_3; \psi_5; \psi_7; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1$ $\oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^2$
	$c_2v_2+c_4v_4+c_7v_7$ $[\sim c_4v_4+c_2v_6+c_7v_7]$	$\{\psi_1; \psi_3; \psi_5, \psi_6, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_3 \oplus \mathfrak{T}^2$
	$c_2v_2+c_4v_4+c_6v_6+c_7v_7$	$\{\psi_1; \psi_3; \psi_5; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1$ $\oplus \mathfrak{T}^3$

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$
$\mathfrak{E}_8$	v_1	$\{\psi_2, \cdots, \psi_8, -\mu\}$	$\mathfrak{D}_8$
	v_2	$\{\psi_1; \psi_3, \cdots, \psi_8, -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_7$
	v_3	$\{\psi_1, \psi_2; \psi_8; \psi_4, \cdots, \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_5$
	v_4	$\{\psi_1, \psi_2, \psi_3, \psi_8; \psi_5, \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_4 \oplus \mathfrak{A}_4$
	v_5	$\{\psi_1, \cdots, \psi_4, \psi_8; \psi_6, \psi_7, -\mu\}$	$\mathfrak{D}_5 \oplus \mathfrak{A}_3$
	v_6	$\{\psi_1, \cdots, \psi_5, \psi_8; \psi_7, -\mu\}$	$\mathfrak{E}_6 \oplus \mathfrak{A}_2$
	v_7	$\{\psi_1, \cdots, \psi_6, \psi_8; -\mu\}$	$\mathfrak{E}_7 \oplus \mathfrak{A}_1$
	v_8	$\{\psi_1, \cdots, \psi_7, -\mu\}$	$\mathfrak{A}_8$
	$c_6v_6+c_8v_8$	$\{\psi_1, \cdots, \psi_5; \psi_7, -\mu\}$	$\mathfrak{A}_5 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^1$
	$c_3v_3+c_6v_6$	$\{\psi_1, \psi_2; \psi_8; \psi_4, \psi_5; \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^1$
	$c_3v_3+c_8v_8$	$\{\psi_1, \psi_2; \psi_4, \psi_5, \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_5 \oplus \mathfrak{T}^1$
	$c_3v_3+c_6v_6+c_8v_8$	$\{\psi_1, \psi_2; \psi_4, \psi_5; \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^2$
	$c_1v_1+c_7v_7$	$\{\psi_2, \cdots, \psi_6, \psi_8; -\mu\}$	$\mathfrak{D}_6 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1$
	$c_1v_1+c_2v_2$	$\{\psi_3, \cdots, \psi_8, -\mu\}$	$\mathfrak{A}_7 \oplus \mathfrak{T}^1$
	$c_1v_1+c_5v_5$	$\{\psi_2, \psi_3, \psi_4, \psi_8; \psi_6, \psi_7, -\mu\}$	$\mathfrak{D}_4 \oplus \mathfrak{A}_3 \oplus \mathfrak{T}^1$
	$c_2v_2+c_7v_7$	$\{\psi_1; \psi_3, \psi_4, \psi_5, \psi_6, \psi_8; -\mu\}$	$\mathfrak{A}_1 \oplus \mathfrak{A}_5 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1$
	$c_5v_5+c_7v_7$	$\{\psi_1, \psi_2, \psi_3, \psi_4, \psi_8; \psi_6; -\mu\}$	$\mathfrak{D}_5 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1$

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$
$\mathfrak{E}_8$ (cont.)	$c_1v_1+c_3v_3$	$\{\psi_2;\ \psi_8;\ \psi_4, \psi_5, \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_5\oplus\mathfrak{T}^1$
	$c_3v_3+c_7v_7$	$\{\psi_1, \psi_2;\ \psi_4, \psi_5, \psi_6;\ -\mu;\ \psi_8\}$	$\mathfrak{A}_2\oplus\mathfrak{A}_3\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{T}^1$
	$c_2v_2+c_5v_5$	$\{\psi_1;\ \psi_3, \psi_4, \psi_8;\ \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{A}_3\oplus\mathfrak{T}^1$
	$c_2v_2+c_3v_3$	$\{\psi_1;\ \psi_8;\ \psi_4, \psi_5, \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_5\oplus\mathfrak{T}^1$
	$c_3v_3+c_5v_5$	$\{\psi_1, \psi_2;\ \psi_4;\ \psi_8;\ \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_2\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^1$
	$c_1v_1+c_2v_2+c_7v_7$	$\{\psi_3, \psi_4, \psi_5, \psi_6, \psi_8;\ -\mu\}$	$\mathfrak{A}_5\oplus\mathfrak{A}_1\oplus\mathfrak{T}^2$
	$c_1v_1+c_5v_5+c_7v_7$	$\{\psi_2, \psi_3, \psi_4, \psi_8;\ \psi_6;\ -\mu\}$	$\mathfrak{D}_4\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{T}^2$
	$c_1v_1+c_3v_3+c_7v_7$	$\{\psi_2;\ \psi_8;\ -\mu;\ \psi_4, \psi_5, \psi_6\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^2$
	$c_1v_1+c_2v_2+c_5v_5$	$\{\psi_8, \psi_3, \psi_4;\ \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_3\oplus\mathfrak{A}_3\oplus\mathfrak{T}^2$
	$c_2v_2+c_5v_5+c_7v_7$	$\{\psi_1;\ \psi_8, \psi_3, \psi_4;\ \psi_6;\ -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^2$
	$c_1v_1+c_2v_2+c_3v_3$	$\{\psi_4, \psi_5, \psi_6, \psi_7, -\mu;\ \psi_8\}$	$\mathfrak{A}_5\oplus\mathfrak{A}_1\oplus\mathfrak{T}^2$
	$c_2v_2+c_3v_3+c_7v_7$	$\{\psi_1;\ \psi_8;\ -\mu;\ \psi_4, \psi_5, \psi_6\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^2$
	$c_1v_1+c_3v_3+c_5v_5$	$\{\psi_2;\ \psi_8;\ \psi_4;\ \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^2$
	$c_3v_3+c_5v_5+c_7v_7$	$\{\psi_1, \psi_2;\ \psi_8;\ \psi_4;\ \psi_6;\ -\mu\}$	$\mathfrak{A}_2\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{T}^2$
	$c_2v_2+c_3v_3+c_5v_5$	$\{\psi_1;\ \psi_8;\ \psi_4;\ \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^2$
	$c_1v_1+c_2v_2+c_3v_3+c_5v_5$	$\{\psi_8;\ \psi_4;\ \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^3$
	$c_1v_1+c_2v_2+c_3v_3+c_7v_7$	$\{-\mu;\ \psi_8;\ \psi_4, \psi_5, \psi_6\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^3$

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$
$\mathfrak{E}_8$ (cont.) 	$c_1v_1+c_2v_2+c_5v_5+c_7v_7$	$\{-\mu; \psi_6; \psi_8, \psi_3, \psi_4\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_3\oplus\mathfrak{T}^3$
	$c_1v_1+c_3v_3+c_5v_5+c_7v_7$	$\{\psi_2; \psi_8; \psi_4; \psi_6; -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{T}^3$
	$c_2v_2+c_3v_3+c_5v_5+c_7v_7$	$\{\psi_1; \psi_8; \psi_4; \psi_6; -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{T}^3$
	$c_1v_1+c_2v_2+c_3v_3+c_5v_5+c_7v_7$	$\{\psi_8; \psi_4; \psi_6; -\mu\}$	$\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{A}_1\oplus\mathfrak{T}^4$

4. Almost complex coset spaces of positive characteristic

In this section we find all almost complexm anifolds (X, J), $X=G/K$, where G is a compact connected Lie group acting effectively on X, K is a subgroup of maximal rank, and J is a G-invariant almost complex structure on X.

The preliminary step is to show that K is a closed connected subgroup of G:

4.1. Proposition. *Let G be a compact connected Lie group, and K a subgroup of maximal rank. Then K is a closed subgroup, so $X = G/K$ is a manifold. If X carries a G-invariant almost complex structure, then K is connected and X is simply connected.*

Proof. The identity component K_0 is a closed subgroup of G, according to Borel and de Siebenthal [3, p. 220]. Let Z be the center of K_0; so K_0 is the identity component of the centralizer of Z. Z is the product of a finite abelian group and a torus, so K_0 is the identity component of its normalizer. Let N be that normalizer. N/K_0 is finite because N is closed in G; thus K/K_0 is finite because $K \subset N$; this shows that K is closed in G. It follows that $X = G/K$ is a manifold.

It is standard that connectedness of K implies simple connectedness of X, for the maximal rank condition shows that the full inverse image of K_0 in the universal covering group of G is connected.

Let J be a G-invariant almost complex structure on X, and $k \in K$, $\alpha = ad(k)$. Then α is an automorphism of G which preserves K; thus acts (as K) on G/K, and preserves J. Now $k \in K_0$, proving K connected, as a consequence of

4.2. Proposition [12, *Theorem* 13.3(1)]. *Let $X = G/K$ where K is a connected subgroup of maximal rank in a compact connected Lie group G. Let α be an automorphism of G which preserves K, thus acts on X, and preserves some G-invariant almost complex structure on X. Then the following conditions are equivalent, and each implies that α preserves every G-invariant almost complex structure on X:*

(i) *α is an inner automorphism of G.*

(ii) $\alpha|_K$ *is an inner automorphism of* K.

(iii) $\alpha = ad(k)$ *for some element* $k \in K$.

This completes the proof of Proposition 4.1. q.e.d.

We now obtain some information on the linear isotropy representation which will allow us to give a systematic treatment of invariant almost complex structures.

4.3. Theorem. *Let* K *be a connected subgroup of maximal rank in a compact connected Lie group* G, $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$, *orthogonal decomposition under the Killing form of* $\mathfrak{G}$, *and* χ *be the linear (isotropy) representation of* $\mathfrak{K}$ *on* $\mathfrak{M}^C$. *Let* Z *be the center of* K, *and* $\{\alpha_i\}$ *the distinct unitary characters on* Z *such that* $\mathfrak{M}^C = \sum \mathfrak{M}_i$ *where* $\mathfrak{M}_i \neq 0$ *and* $ad_G|_Z$ *is a multiple of* α_i *on* $\mathfrak{M}_i$. *Then*

1. $ad_G|_K$ *preserves* $\mathfrak{M}_i$, *acting there by inequivalent irreducible representations* χ_i, *and* $\chi = \sum \chi_i$;

2. *the following conditions are equivalent:* (2a) G/K *has a* G*-invariant almost complex structure*, (2b) *none of the* α_i *is real valued*, (2c) $\{\alpha_i\}$ *can be enumerated* $\{\alpha_1, \alpha_{-1}; \cdots; \alpha_t, \alpha_{-t}\}$ *with* $\bar{\alpha}_i = \alpha_{-i} \neq \alpha_i$, *and* (2d) $\{\chi_i\}$ *can be enumerated* $\{\chi_1, \chi_{-1}; \cdots; \chi_t, \chi_{-t}\}$ *such that* $\bar{\chi}_i = \chi_{-i} \neq \chi_i$;

3. *in the case and notation of* (2) *the* G*-invariant almost complex structures* J *on* G/K *are precisely the tensor fields constructed as follows:* $\mathfrak{M} = \sum_{i=1}^{t} \tilde{\mathfrak{M}}_i$ *where* $\tilde{\mathfrak{M}}_i$ *is the real form* $\mathfrak{M} \cap (\mathfrak{M}_i + \mathfrak{M}_{-i})$ *of* $\mathfrak{M}_i + \mathfrak{M}_{-i}$, $\tilde{J}$ *is the linear transformation* $\sum_{i=1}^{r} \varepsilon_i \tilde{J}_i$ *where* $\varepsilon_i = \pm 1$ *and* $\tilde{J}_i$ *acts on* $\tilde{\mathfrak{M}}_i$ *as the transformation of square* -1 *with* $\mathfrak{M}_i$ *as* $\sqrt{-1}$*-eigenspace and* $\mathfrak{M}_{-i}$ *as* $-\sqrt{-1}$*-eigenspace*, $\mathfrak{M}$ *is identified with the tangent space to* G/K *at a point, and* $\tilde{J}$ *is extended to an invariant tensor field* J *by the action of* G;

4. *in the case and notation of* (2), *there are precisely* 2^t G*-invariant almost complex structures on* G/K.

Proof. (1) is the statement (for the case $A = Z$) of Kostant's theorem [11, Theorem 8.13.3]. Now enumerate $\{\alpha_i\} = \{\alpha_1, \alpha_{-1}; \cdots; \alpha_t, \alpha_{-t}; \alpha_{t+1}, \cdots, \alpha_r\}$ with $\bar{\alpha}_i = \alpha_{-i} \neq \alpha_i$ for $1 \leq i \leq t$ and $\alpha_j = \bar{\alpha}_j$ for $t + 1 \leq j \leq r$. Then $\mathfrak{M} = \sum_{i=1}^{t} \tilde{\mathfrak{M}}_i + \sum_{j=t+1}^{r} \mathfrak{M}'_j$ where $\tilde{\mathfrak{M}}_i = \mathfrak{M} \cap (\mathfrak{M}_i + \mathfrak{M}_{-i})$ and $\mathfrak{M}'_j = \mathfrak{M} \cap \mathfrak{M}_j$. As $\{\chi_i\}$ are mutually inequivalent, the commuting algebra of χ on $\mathfrak{M}$ is $A = \sum_{i=1}^{t} A_i + \sum_{j=t+1}^{r} A'_j$ where $A_i \cong \boldsymbol{C}$ acts on $\tilde{\mathfrak{M}}_i$ and $A'_j \cong \boldsymbol{R}$ acts on $\mathfrak{M}'_j$. In particular, A has an element of square -1 if and only if there no $\alpha_j = \bar{\alpha}_j$, and in that case the elements $\tilde{J}$ of square -1 are the $\sum_{i=1}^{t} \varepsilon_i \tilde{J}_i$, $\varepsilon_i = \pm 1$, $\tilde{J}_i$ acting on $\tilde{\mathfrak{M}}_i$ with $\mathfrak{M}_{\pm i}$ as $\pm\sqrt{-1}$-eigenspace. There are 2^t choices of $\{\varepsilon_i\}$. q.e.d.

Our first consequence of Theorem 4.3 is a short proof of the criterion for the existence of an almost complex structure.

4.4. Theorem (*Passiencier* [9]). *Let K be a connected subgroup of maximal rank in a compact connected Lie group G. Then G/K has a G-invariant almost complex structure, if and only if $\mathfrak{K} = \mathfrak{G}^{\Theta}$, a fixed point set for some finite group Θ of odd order of inner automorphisms of $\mathfrak{G}$.*

Proof. Adopt the notation of Theorem 4.3. If $\mathfrak{K} = \mathfrak{G}^{\Theta}$ where Θ is an odd order finite group of inner automorphisms of $\mathfrak{G}$, then $\Theta = ad(A)$ for a finite subgroup $A \subset Z$, and each $\alpha_i(A)$ has odd order >1. Now no α_i is real and Theorem 4.3(2) gives almost complex structures.

Conversely suppose that G/K carries a G-invariant almost complex structure, so none of the α_i is real. Choose a finite subgroup $A \subset Z$ such that $\mathfrak{K} = \mathfrak{G}^{ad(A)}$; decompose $A = E \times F$ where E is the Sylow 2-subgroup and the order $|F|$ is odd. Let $\Theta = ad(F)$, odd order finite group of inner automorphisms of $\mathfrak{G}$. We must prove $\mathfrak{K} = \mathfrak{G}^{\Theta}$.

If $\mathfrak{K} \neq \mathfrak{G}^{\Theta}$ there is an α_i which annihilates F. Then $u > 0$ where we define 2^u to be the highest of the orders of the $\alpha_i(e)$ such that $e \in E$ and $\alpha_i(F) = 1$. For that α_i and that e, we define $\mathfrak{N}_1 = \mathfrak{M}_i$ and $\mathfrak{N}_{p+1} = [\mathfrak{N}_p, \mathfrak{N}_p]$, and have $ad(e) = -1$ on $\mathfrak{N}_{u-1}$. Then we take $\mathfrak{M}_j \subset \mathfrak{N}_{u-1}$ and have $\alpha_j(F) = 1$, $\alpha_j(E) = \{\pm 1\}$, contradicting the fact that $\alpha_j(A)$ is not real. The contradiction shows $\mathfrak{K} = \mathfrak{G}^{\Theta}$. q.e.d.

A second application of Theorem 4.3, systematizing certain result of A. Frölicher [4], H.-C. Wang [10] and Borel-Hirzebruch [1], is the integrability criterion:

4.5. Theorem. *Let $X = G/K$ where K is a connected subgroup of maximal rank in the compact connected Lie group G. Let J be a G-invariant almost complex structure on X, and $\{\mathfrak{M}_1, \mathfrak{M}_{-1}; \cdots; \mathfrak{M}_t, \mathfrak{M}_{-t}\}$ the irreducible representation spaces of K on $\mathfrak{M}^C$, as in Theorem 4.3, ordered so that $\mathfrak{M}^+ = \sum_{i=1}^{t} \mathfrak{M}_i$ is the $(\sqrt{-1})$-eigenspace of J and $\mathfrak{M}^- = \sum_{i=1}^{t} \mathfrak{M}_{-i}$ is the $(-\sqrt{-1})$-eigenspace. Then the following conditions are equivalent:*

1. *J is integrable,* i.e. *J is induced by a complex structure on X.*
2. *$\mathfrak{K}^C + \mathfrak{M}^+$ is a subalgebra of $\mathfrak{G}^C$.*
3. *$\mathfrak{M}^+, \mathfrak{M}^-, \mathfrak{K}^C + \mathfrak{M}^+$ and $\mathfrak{K}^C + \mathfrak{M}^-$ are subalgebras of $\mathfrak{G}^C$.*
4. *Let Z be the center of K. Then K is the centralizer of the torus Z_0 and there is a maximal torus T of G, $Z \subset T \subset K$, and a system $\Psi = \{\psi_1, \cdots, \psi_l\}$ of simple T-roots of G, such that*

(4a) $\Psi_K = \{\psi_{r+1}, \cdots, \psi_l\}$ *is a system of simple T-roots of K,*

(4b) $\mathfrak{M}^+ = \sum_{\substack{\lambda>0 \\ \lambda \notin \Lambda_K}} \mathfrak{G}_\lambda$ *and* $\mathfrak{M}^- = \sum_{\substack{\lambda<0 \\ \lambda \notin \Lambda_K}} \mathfrak{G}_\lambda$,

(4c) *the $\mathfrak{M}_i$ are the spaces $\sum_{\lambda \in \Lambda_i} \mathfrak{G}_\lambda$ where each Λ_i is a nonempty subset of Λ, $\Lambda_{-i} = -\Lambda_i$, consisting of all roots of the form $\sum a_j\psi_j$ for fixed integers $\{a_1, \cdots, a_r\}$, not all zero and depending on i, and variable integers $\{a_{r+1}, \cdots, a_l\}$.*

Proof. (1) implies (2) because $\mathfrak{M}^+$ is the holomorphic tangent space; for the Poisson bracket of two holomorphic vector fields on a complex manifold is a holomorphic vector field.

Assume (2). Conjugation means conjugation of $\mathfrak{G}^C$ over $\mathfrak{G}$. $\mathfrak{M}^-$ is the conjugate of $\mathfrak{M}^+$ and $\mathfrak{K}^C + \mathfrak{M}^-$ is the conjugate of $\mathfrak{K}^C + \mathfrak{M}^+$. For (3) now we need only check that $\mathfrak{M}^+$ is a subalgebra of $\mathfrak{G}^C$. Let $\mathfrak{L} = \mathfrak{K}^C \cap [\mathfrak{M}^+, \mathfrak{M}^+]$. $\mathfrak{L}$ is an ideal in $\mathfrak{K}^C$; if $\mathfrak{L} \neq 0$ now the representation of $\mathfrak{K}^C$ on $\mathfrak{L}$ has zero-weights. That says that we have $1 \leq i \leq j \leq t$ such that $\alpha_i = \bar{\alpha}_j$, contradicting the hypothesis on indexing the $\{\mathfrak{M}_r\}$ which says $\bar{\alpha}_j = \alpha_{-j}$. Thus $\mathfrak{L} = 0$ and $\mathfrak{M}^+$ is an algebra. We have proved that (2) implies (3).

Assume (3). Let Z be the center of K and suppose $\alpha_i(Z_0) = 1$. Then any subalgebra of $\mathfrak{G}^C$ containing $\mathfrak{M}_i$ has nonzero intersection with $\mathfrak{K}^C$. Now (3) says $\alpha_i(Z_0) \neq 1$. Thus K is the centralizer of the torus Z_0. It follows that $\mathfrak{K}^C + \mathfrak{M}^+$ is a parabolic subalgebra of $\mathfrak{G}^C$ with reductive part $\mathfrak{K}^C$ and unipotent radical $\mathfrak{M}^+$; (4a) and (4b) are immediate, and (4c) follows from Theorem 4.3(1). Thus (3) implies (4).

(4) implies $G/K \cong G^C/P$, where P is the complex parabolic subgroup with Lie algebra $\mathfrak{K}^C + \mathfrak{M}^-$, the isomorphism $gK \mapsto gP$ sending J to the natural complex structure on G^C/P. Thus (4) implies (1).

4.6. Corollary. *Let G be a compact connected Lie group, K the centralizer of a toral subgroup, S the identity component of the center of K, and $N_G(K)$ the normalizer of K in G. Let $T \subset K$ be a maximal torus of G, $W_K \subset W_G$ the Weyl groups of K and G relative to T, and $N_G(W_K)$ the normalizer of W_K in W_G. Then*

(i) *W_K is the centralizer of $\mathfrak{S}$ in W_G, $N_G(W_K)$ is the normalizer of $\mathfrak{S}$ in W_G, and so $N_G(W_K)/W_K$ is the group of linear transformations of $\mathfrak{S}$ by Weyl group elements of G;*

(ii) *$N_G(K) = N_G(W_K)\cdot K$, normalizer of S in G, and $N_G(K) \to N_G(K)/K$ restricts to the projection $N_G(W_K) \to N_G(W_K)/W_K$; in particular $N_G(K)/K \cong N_G(W_K)/W_K$;*

(iii) *the common order of $N_G(K)/K$ and $N_G(W_K)/W_K$ is the number of distinct G-invariant complex structures on G/K.*

Proof. Statements (i) and (ii) are clear, and they show that the order $|N_G(K)/K|$ is the number of decompositions $\mathfrak{M}^C = \mathfrak{M}^+ + \mathfrak{M}^-$ such that some system of simple T-roots of G has properties (4a) and (4b) of Theorem 4.5.

q.e.d.

The third application of Theorem 4.3 is a counting of almost complex structures comparable to Corollary 4.6; it refines a result [1, Prop. 13.4] of Borel-Hirzebruch.

4.7. Theorem. *Let K be the centralizer of a torus in a compact connected Lie group G and let S be the identity component of the center of K. Decompose $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$.*

1. *There is a maximal torus $T \subset G$ and a simple root system $\Psi =$*

$\{\phi_1, \cdots, \phi_l\}$ *such that* K *has simple root system* $\Psi_K \subset \Psi$ *and* $\mathfrak{S} = \{x \in \mathfrak{T} : \phi(x) = 0$ *for every* $\phi \in \Psi_K\}$.

2. *Enumerate the complement* $\Psi - \Psi_K = \{\phi_{i_1}, \cdots, \phi_{i_r}\}$. *Let* $\{a_{i_s}\}$ *be integers such that there is a root of the form* $\sum_{s=1}^{r} a_{i_s}\phi_{i_s} + \sum_{\phi_j \in \Psi_K} b_j\phi_j$ *and let* $\mathfrak{M}_{a_{i_1}, a_{i_2}, \ldots, a_{i_r}}$ *denote the sum of all root spaces* $\mathfrak{G}_\lambda$ *with* λ *of that form. Then* $\mathfrak{T}^C + \mathfrak{M}_{0, \ldots, 0} = \mathfrak{K}^C$, $\mathfrak{K}$ *acts by an irreducible representation* $\chi_{a_{i_1}, \ldots, a_{i_r}}$ *on any other* $\mathfrak{M}_{a_{i_1}, \ldots, a_{i_r}}$, *and* $\bar{\chi}_{a_{i_1}, \ldots, a_{i_r}} = \chi_{-a_{i_1}, \ldots, -a_{i_r}} \neq \chi_{a_{i_1}, \ldots, a_{i_r}}$ *if some* $a_{i_s} \neq 0$.

3. *Let* t *be the number of nonzero linear functionals on the center* $\mathfrak{S}$ *of* $\mathfrak{K}$ *obtained by restriction of positive roots*[2]. *Then* G/K *has precisely* 2^t G-*invariant almost complex structures, as defined in Theorem* 4.3.

Proof. (1) is the standard result on the existence of consistent orderings for the roots of G and K. For (2), observe that we may replace the center of K by S in applications of [11, Theorem 8.13.3], hence in Theorem 4.3, because K is the connected centralizer of S in G. And for (3) observe that t is the number of $\mathfrak{M}_{a_{i_1}, \ldots, a_{i_r}}, a_{i_s} \geq 0$, distinct from $\mathfrak{M}_{0, \ldots, 0}$. q.e.d.

In order to proceed we need a few trivial remarks. Let $X = G/K$ where K is a connected subgroup of maximal rank in the compact connected Lie group G. Assume that G acts effectively on X. As K contains the center of G, this means that G is semisimple and centerless. Now

$$G = G_1 \times \cdots \times G_r\,, \quad K = K_1 \times \cdots \times K_r\,, \quad X = X_1 \times \cdots \times X_r\,, \tag{4.8a}$$

where

$$G_i \text{ is simple}, \quad K_i = K \cap G_i\,, \quad X_i = G_i/K_i\,. \tag{4.8b}$$

For lack of a better term, we refer to (4.8) as the "decomposition of $X = G/K$ into simple factors". The point is that, in Theorem 4.3, each $\mathfrak{M}_j$ must be contained in one of the $\mathfrak{G}_i^C$. Now Theorem 4.3 gives us

4.9. Proposition. *Let* $\mathcal{T}$ *be the class of tensor fields of one of the types: complex structure, almost complex structure, riemannian metric, almost hermitian metric or hermitian metric. Then the G-invariant tensor fields of type* $\mathcal{T}$ *on* X *are just the tensor fields* $\xi = \xi_1 \oplus \cdots \oplus \xi_r$ *where, in the notation* (4.8), ξ_i *is an arbitrary* G_i-*invariant tensor field of type* $\mathcal{T}$ *on* X_i.

Now we can reduce questions of classification to the case where G is simple. Our next application does this for the case where K is not the centralizer of a torus, shortening and sharpening a classification of R. Hermann [8]:

4.10. Theorem. *Let* G *be a compact connected simple Lie group and let* K *be a connected subgroup of maximal rank which is not the centralizer of a torus. Suppose that* G/K *has an invariant almost complex structure*, i.e.

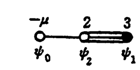
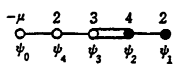
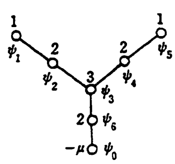
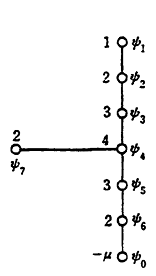
[2] These linear functionals are, of course, just the positive (restricted) $\mathfrak{S}$-roots of $\mathfrak{G}$.

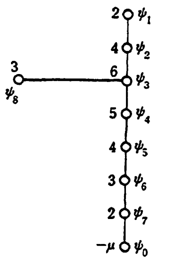
$\mathfrak{K} = \mathfrak{G}^{\Theta}$ where Θ is an odd order finite group of inner automorphisms of $\mathfrak{G}$.

1. *Suppose that $\mathfrak{G}$ has an automorphism with fixed point set $\mathfrak{K}$. Then $\mathfrak{K}$ is conjugate to $\mathfrak{G}^{\theta}$ where $\theta = ad(\exp 2\pi\sqrt{-1}x)$, $x \in \mathfrak{D}_0$, has odd order $k > 1$; all possibilities are listed in the following table; there the n_i are positive integers, arbitrary up to the specified relatively prime conditions and the condition under k.*

2. *Suppose that K is not conjugate to one of the $\mathfrak{G}^{\theta}$ listed above. Then $\mathfrak{K} = \mathfrak{G}^{\Theta}$ where*

(i) *Θ is the product of cyclic subgroups $\{\alpha\}$ and $\{\beta\}$ of orders 3, $\alpha = ad(a)$ and $\beta = ad(b)$, K having center $\{a, b : a^3 = b^3 = 1, ab = ba\}$;*

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^{\theta}$	p	k
$\mathfrak{G}_2$ 	v_1	$\{\psi_2, -\mu\}$	$\mathfrak{A}_2$	3	3
$\mathfrak{F}_4$ 	v_3	$\{\psi_1, \psi_2; \psi_4, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2$	3	3
$\mathfrak{E}_6$ 	v_3	$\{\psi_1, \psi_2; \psi_4, \psi_5; \psi_6, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2$	3	3
$\mathfrak{E}_7$ 	v_3 $[\sim v_5]$	$\{\psi_1, \psi_2; \psi_4, \cdots, \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_5$	3	3
	$\frac{3}{k}(n_3v_3 + n_5v_5)$ $(n_3, n_5) = 1$	$\{\psi_1, \psi_2; \psi_4, \psi_7; \psi_6, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^1$	3	$3n_3 + 3n_5$

$\mathfrak{G}$	x	Ψ_x	$\mathfrak{G}^\theta$	p	k
$\mathfrak{E}_8$	v_6	$\{-\mu, \psi_7; \psi_1, \cdots, \psi_5, \psi_8\}$	$\mathfrak{A}_2 \oplus \mathfrak{E}_6$	3	3
	v_8	$\{\psi_1, \cdots, \psi_7, -\mu\}$	$\mathfrak{A}_8$	3	3
	v_4	$\{\psi_1, \psi_2, \psi_3, \psi_8; \psi_5, \psi_6, \psi_7, -\mu\}$	$\mathfrak{A}_4 \oplus \mathfrak{A}_4$	5	5
	$\frac{3}{k}(n_6 v_6 + n_8 v_8)$ $(n_6, n_8) = 1$	$\{-\mu, \psi_7; \psi_1, \cdots, \psi_5\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_5 \oplus \mathfrak{T}^1$	3	$3n_6 + 3n_8$
	$\frac{3}{k}(2n_3 v_3 + n_8 v_8)$ $(n_3, n_8) = 1$	$\{\psi_1, \psi_2; \psi_4, \cdots, \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_5 \oplus \mathfrak{T}^1$	3	$6n_3 + 3n_8$
	$\frac{3}{k}(2n_3 v_3 + n_6 v_6)$ $(n_3, n_6) = 1$	$\{-\mu, \psi_7; \psi_1, \psi_2; \psi_4, \psi_5; \psi_8\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1$	3	$6n_3 + 3n_6$
	$\frac{3}{k}(2n_3 v_3 + n_6 v_6 + n_8 v_8)$ $(n_3, n_6, n_8) = 1$	$\{\psi_1, \psi_2; \psi_4, \psi_5; \psi_7, -\mu\}$	$\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^2$	3	$6n_3 + 3n_6 + 3n_8$

(ii) *$K \subset L \subset G$ with $\mathfrak{G}$ of type E_8, with $\mathfrak{L} = \mathfrak{G}^\alpha$ of type $A_2 \oplus E_6$ and $\alpha = ad(\exp 2\pi\sqrt{-1}v_6)$ under $\mathfrak{E}_8$ in the table above, and with $\mathfrak{K} = \mathfrak{L}^\beta$ of type $A_2 \oplus (A_2 \oplus A_2 \oplus A_2)$ and $\beta = ad(\exp 2\pi\sqrt{-1}v_3)$ under $\mathfrak{E}_6$ in the table.*

Proof. (1) is immediate from Propositions 2.8 and 2.11. We go on to prove (2).

Let Z be the center of K. Then the identity component of Z is a torus S, and $\mathfrak{K} = \mathfrak{K}' \oplus \mathfrak{S}$ where $'$ denotes derived group or algebra. Let H denote the centralizer of S in G. Now $\mathfrak{K} \subsetneqq \mathfrak{H}$ by hypothesis. As $\mathfrak{K}' \subset \mathfrak{H}'$, now $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{S}$.

Let $F \subset Z$ be an odd order finite group with $\mathfrak{K} = \mathfrak{G}^{ad(F)}$ and define $\Phi = ad(F)|_{H'}$. Let $\{\varphi_1, \cdots, \varphi_r\} \subset \Phi$ and define Φ_i to be the subgroup generated by $\{\varphi_1, \cdots, \varphi_i\}$, such that

$$\mathfrak{K}' = \mathfrak{L}_0 \subsetneqq \mathfrak{L}_1 \subsetneqq \cdots \subsetneqq \mathfrak{L}_r = \mathfrak{H}', \quad \mathfrak{L}_i = \mathfrak{H}'^{\Phi_i}.$$

As each $\mathfrak{L}_i$ is semisimple, each inclusion $\mathfrak{L}_i \subset \mathfrak{L}_{i+1}$ is a "direct sum" of inclusions $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{A}$ semisimple and $\mathfrak{B}$ simple, listed under the chart of (1).

Those $\mathfrak{A} \subset \mathfrak{B}$ are $\mathfrak{A}_2 \subset \mathfrak{G}_2$, $\mathfrak{A}_2 \oplus \mathfrak{A}_2 \subset \mathfrak{F}_4$, $\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \subset \mathfrak{E}_6$, $\mathfrak{A}_2 \oplus \mathfrak{A}_5 \subset \mathfrak{E}_7$, $\mathfrak{A}_8 \subset \mathfrak{E}_8$, $\mathfrak{A}_2 \oplus \mathfrak{E}_6 \subset \mathfrak{E}_8$ and $\mathfrak{A}_4 \oplus \mathfrak{A}_4 \subset \mathfrak{E}_8$.

First suppose $S = \{1\}$ so $\mathfrak{H}' = \mathfrak{G}$ and $\mathfrak{K}' = \mathfrak{K}$. We have $r > 1$ because we are in case (2) of the theorem. Now $\mathfrak{H}'$ is simple and we have a composition of the inclusions $\mathfrak{A} \subset \mathfrak{B}$ just listed. The only possibility is

$$\mathfrak{K} = \mathfrak{A}_2 \oplus (\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2) \subset \mathfrak{A}_2 \oplus \mathfrak{E}_6 \subset \mathfrak{E}_8 = \mathfrak{G},$$

which is the conclusion of (2).

If $S \neq \{1\}$ then $\mathfrak{H}' \neq \mathfrak{H}$ but Dynkin diagrams satisfy $\Delta_{H'} \subset \Delta_G$. The listing of $\mathfrak{A} \subset \mathfrak{B}$ now shows that $\mathfrak{H}'$ has a simple ideal $\mathfrak{J}$ such that $\mathfrak{J} \subset \mathfrak{G}$ is given by

$$\text{(a)} \quad \mathfrak{E}_6 \subset \mathfrak{E}_7\,, \quad \text{(b)} \quad \mathfrak{E}_6 \subset \mathfrak{E}_8\,, \quad \text{or} \quad \text{(c)} \quad \mathfrak{E}_7 \subset \mathfrak{E}_8\,.$$

In case (a), $\mathfrak{H}' = \mathfrak{J}$ and $\mathfrak{K}$ is of type $\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2$; then $\mathfrak{K} \subset \mathfrak{G}$ is $\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^1 \subset \mathfrak{E}_7$ which comes under (1). In case (b) there are two subcases,

$$\mathfrak{H}' = \mathfrak{J},\ \mathfrak{K} \subset \mathfrak{G} \ \text{ is } \ \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{T}^2 \subset \mathfrak{E}_8\,,$$
$$\mathfrak{H}' \neq \mathfrak{J},\ \mathfrak{K} \subset \mathfrak{G} \ \text{ is } \ \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_1 \oplus \mathfrak{T}^1 \subset \mathfrak{E}_8\,,$$

which both come under (1). In case (c), $\mathfrak{H}' = \mathfrak{J}$ and $\mathfrak{K} \subset \mathfrak{G}$ is $\mathfrak{A}_2 \oplus \mathfrak{A}_5 \oplus \mathfrak{T}^1 \subset \mathfrak{E}_8$, which again comes under (1). In case (2) of the theorem, now, $S = \{1\}$ and our statement is proved. q.e.d.

Finally we work out the linear isotropy representation and number of invariant almost complex structures of the spaces of Theorem 4.10.

4.11. Theorem. *Let G be a compact connected simple Lie group, and K a closed connected subgroup of maximal rank which is not the centralizer of a torus. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ and* let *χ be the linear isotropy representation of $\mathfrak{K}$ on $\mathfrak{M}^C$. If G/K admits an invariant almost complex structure, then χ and the number 2^r of such structures are given in the following table:*

{Here $\chi = \sum_{i=-r}^{i=r} \chi_i$ as in Theorem 4.3. χ_i has some greatest weight λ_i. We describe χ_i by writing the integer $\dfrac{2\langle \lambda_i, \psi_\nu \rangle}{\langle \psi_\nu, \psi_\nu \rangle}$ at the vertex ψ_ν of the Dynkin diagram of K, except that we do not write the zeroes; we extend that diagram by an x for every circle factor of K, writing x^s if χ_i gives the s tensor power of the usual representation of that circle.}

Proof. If G/K is of type G_2/A_2, F_4/A_2A_2, $E_6/A_2A_2A_2$, E_7/A_2A_5, E_8/A_2E_6 or E_8/A_8, then the assertion on χ is contained in [12, Theorem 11.1]. We now run through the list of the other spaces in Theorem 4.10.

$E_7/A_2A_2A_2T^1$. We label local factors so that $K = A_2A_2'A_2''T^1 \subset L = A_2A_5 \subset G$ with $A_2'A_2''T^1 \subset A_5$. L acts on $(\mathfrak{G}/\mathfrak{L})^C$ by $\pi \oplus \bar{\pi}$ where $\pi = \alpha_2 \otimes \Lambda^2(\alpha_5)$ $= \overset{1}{\circ}\!-\!\circ \otimes \circ\!-\!\overset{1}{\circ}\!-\!\circ\!-\!\circ\!-\!\circ$; here $\alpha_r : A_r \to SU(r+1)$ denotes the usual vector representation $\overset{1}{\circ}\!-\!\circ\!-\!\cdots\!-\!\circ$ and Λ^k denotes k-th alternation. Now $\alpha_5|_{A_2'A_2''T^1}$ is

$\mathfrak{G}$	$\mathfrak{K}$	$\chi=\sum_{-r}^{r}\chi_i,\ \bar{\chi}_i=\chi_{-i}\neq\chi_i$	2^r
$\mathfrak{G}_2$	$\mathfrak{A}_2$	$\chi_1\colon\ \overset{1}{\circ}-\circ$	2
$\mathfrak{F}_4$	$\mathfrak{A}_2\oplus\mathfrak{A}_2$	$\chi_1\colon\ \overset{1}{\circ}-\circ\otimes\overset{2}{\circ}-\circ$	2
$\mathfrak{E}_6$	$\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_2$	$\chi_1\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ$	2
$\mathfrak{E}_7$	$\mathfrak{A}_2\oplus\mathfrak{A}_5$	$\chi_1\colon\ \overset{1}{\circ}-\circ\otimes\circ-\overset{1}{\circ}-\circ-\circ-\circ$	2
	$\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{T}^1$	$\chi_1\colon\ \overset{1}{\circ}-\circ\otimes\circ-\overset{1}{\circ}\otimes\circ-\circ\otimes x^2$ $\chi_2\colon\ \overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\circ-\overset{1}{\circ}\otimes x^{-2}$ $\chi_3\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x$ $\chi_4\colon\ \circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x^1$	16
$\mathfrak{E}_8$	$\mathfrak{A}_8$	$\chi_1\colon\ \circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ-\circ-\circ$	2
	$\mathfrak{A}_2\oplus\mathfrak{E}_6$	$\chi_1\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ-\underset{\substack{\vert\\ \circ}}{\circ}-\circ-\circ$	2
	$\mathfrak{A}_4\oplus\mathfrak{A}_4$	$\chi_1\colon\ \overset{1}{\circ}-\circ-\circ-\circ\otimes\overset{1}{\circ}-\circ-\circ-\circ$ $\chi_2\colon\ \overset{1}{\circ}-\circ-\circ-\circ\otimes\overset{2}{\circ}-\circ-\circ-\circ$	4
	$\mathfrak{A}_2\oplus\mathfrak{A}_5\oplus\mathfrak{T}^1$	$\chi_1\colon\ \circ-\circ\otimes\circ-\circ-\circ-\circ-\circ\otimes x^6$ $\chi_2\colon\ \circ-\overset{1}{\circ}\otimes\overset{1}{\circ}-\circ-\circ-\circ-\circ\otimes x^3$ $\chi_3\colon\ \overset{1}{\circ}-\circ\otimes\circ-\overset{1}{\circ}-\circ-\circ-\circ\otimes x$ $\chi_4\colon\ \circ-\circ\otimes\circ-\circ-\overset{1}{\circ}-\circ-\circ\otimes x^{-3}$ $\chi_5\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ-\circ-\circ-\circ\otimes x^1$	32
	$\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_2$	$\chi_1\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ-\circ$ $\chi_2\colon\ \overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ$ $\chi_3\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}-\circ$ $\chi_4\colon\ \circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ$	16
	$\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_1\oplus\mathfrak{T}^1$	$\chi_1\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ\otimes x$ $\chi_2\colon\ \overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}\otimes x^1$ $\chi_3\colon\ \overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ\otimes x^{-2}$ $\chi_4\colon\ \overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\circ\otimes x^1$	256

$\mathfrak{G}$	$\mathfrak{K}$	$\chi=\sum_{-r}^{r}\chi_i,\ \chi_i=\bar{\chi}_{-i}\neq\chi_i$	2^r
	$\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_1\oplus\mathfrak{T}^1$ (cont.)	χ_5: $\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\circ\otimes x^{-2}$	256
		χ_6: $\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}\otimes x^{1}$	
		χ_7: $\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ\otimes x^{-2}$	
		χ_8: $\circ-\circ\otimes\circ-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}\otimes x^{1}$	
$\mathfrak{E}_8$ (cont.)	$\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{A}_2\oplus\mathfrak{T}^2$	χ_1: $\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x\quad\otimes x$	8192
		χ_2: $\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes x^{1}\quad\otimes x$	
		χ_3: $\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes x\quad\otimes x^{1}$	
		χ_4: $\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes x^{-1}\otimes x^{-1}$	
		χ_5: $\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes x^{1}\quad\otimes x$	
		χ_6: $\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes x\quad\otimes x^{1}$	
		χ_7: $\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes x^{-1}\otimes x^{-1}$	
		χ_8: $\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x^{1}\quad\otimes x$	
		χ_9: $\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x\quad\otimes x^{1}$	
		χ_{10}: $\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x^{-1}\otimes x^{-1}$	
		χ_{11}: $\circ-\circ\otimes\circ-\circ\otimes\circ-\circ\otimes x^{1}\quad\otimes x^{-1}$	
		χ_{12}: $\circ-\circ\otimes\circ-\circ\otimes\circ-\circ\otimes x^{1}\quad\otimes x^{2}$	
		χ_{13}: $\circ-\circ\otimes\circ-\circ\otimes\circ-\circ\otimes x^{2}\quad\otimes x^{1}$	

given by $(\overset{1}{\circ}-\circ\otimes\circ-\circ\otimes x^{1})\oplus(\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes x^{-1})$, where x^k is the k-th tensor power of a certain obvious representation of degree 1 of T^1. Alternating, $\Lambda^2(\alpha_5)|_{A_2'A_2''T^1}$ is $(\circ-\overset{1}{\circ}\otimes\circ-\circ\otimes x^2)\oplus(\circ-\circ\otimes\circ-\overset{1}{\circ}\otimes x^{-2})\oplus(\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x)$, which gives us χ_1,χ_2 and χ_3. For χ_4 we note that K acts on $(\mathfrak{L}/\mathfrak{K})^C$ by $(\circ-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes x^1)$ + (dual).

E_8/A_4A_4. Here the center Z of K has order 5. Let ε be a primitive 5-th root of 1, z be a generator of Z, and $\mathfrak{M}_i$ denote the subspace of $(\mathfrak{G}/\mathfrak{K})^C$ on which z acts as scalar multiplication ε^i. Theorem 4.3 says that the action χ_i of K on $\mathfrak{M}_i$ is irreducible, and [11, Lemma 8.12.5, p. 293] says $\mathfrak{M}_i\neq 0$. Thus $\chi=\chi_1\oplus\chi_{-1}\oplus\chi_2\oplus\chi_{-2}$ with $\deg\chi_i$ divisible by 5. The representations of A_4 of lowest degree are $\overset{1}{\circ}-\circ-\circ-\circ$ (degree 5), $\overset{2}{\circ}-\circ-\circ-\circ$ (degree 15), $\circ-\overset{1}{\circ}-\circ-\circ$ (degree 10) and their duals. As $\deg\chi_1+\deg\chi_2=100$, and as $\chi_2(z)=\chi_1(z)^2$ as scalars, it follows that we may take $\chi_1=\overset{1}{\circ}-\circ-\circ-\circ\otimes\overset{1}{\circ}-\circ-\circ-\circ$ (degree 25) and $\chi_2=\overset{1}{\circ}-\circ-\circ-\circ\otimes\overset{2}{\circ}-\circ-\circ-\circ$ (degree 75).

$E_8/A_2A_5T^1$. Here $K\subset L\subset G$ with L of type A_8. L acts on $(\mathfrak{G}/\mathfrak{L})^C$ by

$\pi \oplus \bar{\pi}$ where $\pi = \Lambda^3(\alpha_8) = \circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ-\circ-\circ$. For a certain faithful representation x^1 of $\mathfrak{T}^1$,

$$\alpha_8|_{A_2A_5T^1} = (\overset{1}{\circ}-\circ \otimes \circ-\circ-\circ-\circ-\circ \otimes x^2) \oplus (\circ-\circ \otimes \overset{1}{\circ}-\circ-\circ-\circ-\circ \otimes x^{-1})\ ,$$

which has third alternation $\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$ as listed. And finally the action of K on $(\mathfrak{L}/\mathfrak{K})^C$ is the direct sum of $\overset{1}{\circ}-\circ \otimes \overset{1}{\circ}-\circ-\circ-\circ-\circ \otimes x^1$ and its dual.

Let ω be the representation $\overset{1}{\circ}-\circ-\underset{\circ}{\underset{|}{\circ}}-\circ-\circ$ of degree 27 of E_6. We need the fact that $\omega|_{A_2A_2A_2} = (\overset{1}{\circ}-\circ \otimes \overset{1}{\circ}-\circ \otimes \circ-\circ) \oplus (\circ-\circ \otimes \overset{1}{\circ}-\circ \otimes \overset{1}{\circ}-\circ) \oplus (\overset{1}{\circ}-\circ \otimes \circ-\circ \otimes \overset{1}{\circ}-\circ)$. To see that let γ be the highest weight of ω and number the simple roots of of E_6 as usual. Then $\dfrac{2\langle\gamma, \mu\rangle}{\langle\mu, \mu\rangle} = \dfrac{2\langle\gamma, \psi_i\rangle}{\langle\psi_i, \psi_i\rangle} = 1$ and γ is orthogonal to the other simple roots of $\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2$. Thus the restriction of ω has $\overset{1}{\circ}-\circ \otimes \overset{1}{\circ}-\circ \otimes \circ-\circ$ as the irreducible summand with highest weight γ. But that restriction is stable under the inner automorphism of order 3 on E_6 which induces a cyclic permutation of the three simple summands of $\mathfrak{A}_2 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2$ by rotation of the extended Dynkin diagram. Thus $\circ-\circ \otimes \overset{1}{\circ}-\circ \otimes \overset{1}{\circ}-\circ$ and $\overset{1}{\circ}-\circ \otimes \circ-\circ \otimes \overset{1}{\circ}-\circ$ are also summands of the restriction of ω. These three summands add up to a subrepresentation of degree 27 of the restriction of ω, so they exhaust the restriction. Our assertion is proved.

$E_8/A_2A_2A_2A_2$. Here $K \subset L \subset G$ with L of type A_2E_6. L acts on $(\mathfrak{G}/\mathfrak{L})^C$ by $\pi \oplus \bar{\pi}$ where $\pi = a_2 \otimes \omega$, ω given as just above. Now $\pi|_K = \chi_1 \oplus \chi_2 \oplus \chi_3$ as listed. And finally the action of K on $(\mathfrak{L}/\mathfrak{K})^C$ is the sum of $\circ-\circ \otimes \overset{1}{\circ}-\circ \otimes \overset{1}{\circ}-\circ \otimes \overset{1}{\circ}-\circ$ and its dual.

$E_8/A_2A_2A_2A_1T^1$. Here $K \subset L \subset G$ where L is the group $A_2A_2A_2A_2$ considered just above and where the A_1T^1 factor of K is contained in the last A_2 factor of L. We have just found the action of L on $(\mathfrak{G}/\mathfrak{L})^C$. Let β be an irreducible summand. If $\beta = (\cdots) \otimes \circ-\circ$ then $\beta|_K = (\cdots) \otimes \circ \otimes x$. If $\beta = (\cdots) \otimes \overset{1}{\circ}-\circ$ then $\beta|_K [(\cdots) \otimes \overset{1}{\circ} \otimes x^1] \oplus [(\cdots) \otimes \circ \otimes x^{-2}]$. Now K acts on $(\mathfrak{G}/\mathfrak{L})^C$ by $\chi_1 \oplus \chi_{-1} \oplus \cdots \oplus \chi_7 \oplus \chi_{-7}$ as listed. And finally the representation of K on $(\mathfrak{L}/\mathfrak{G})^C$ is the sum of $\circ-\circ \otimes \circ-\circ \otimes \circ-\circ \otimes \overset{1}{\circ} \otimes x^1$ and its dual.

$E_8/A_2A_2A_2T^2$. Here $K \subset L \subset G$ where L is $A_2A_2A_2A_2$ as above, where K and L have their first three A_2 factors in common, and where the T^2 factor of K is in the last A_2 factor of L. We know the representation of L on $(\mathfrak{G}/\mathfrak{L})^C$. Let β be an irreducible summand. If $\beta\,(\cdots) \otimes \circ-\circ$ then $\beta|_K = (\cdots) \otimes x \otimes x$. If $\beta = (\cdots) \otimes \overset{1}{\circ}-\circ$ then $\beta|_K = [(\cdots) \otimes x^1 \otimes x] \oplus [(\cdots) \otimes x \otimes x^1] \oplus [(\cdots) \otimes x^{-1} \otimes x^{-1}]$. Now K acts on $(\mathfrak{G}/\mathfrak{L})^C$ by $\chi_1 \oplus \chi_{-1} \oplus \cdots \oplus \chi_{10} \oplus \chi_{-10}$ as listed. Finally the representation of K on $(\mathfrak{L}/\mathfrak{K})^C$, which is given by the representation of T^2 on $(\mathfrak{A}_2/\mathfrak{T}^2)^C$, is the sum of $(\circ-\circ \otimes \circ-\circ \otimes \circ-\circ \otimes x^1 \otimes x^{-1}) \oplus (\circ-\circ \otimes \circ-\circ \otimes \circ-\circ \otimes x^1 \otimes x^2) \oplus (\circ-\circ \otimes \circ-\circ \otimes \circ-\circ \otimes x^2 \otimes x^1)$ and its dual.

5. Canonical forms of outer automorphisms

Let θ be an automorphism of a compact Lie algebra $\mathfrak{G}$. Decompose $\mathfrak{G} = \mathfrak{G}_0 \oplus {}^1\mathfrak{G} \oplus \cdots \oplus {}^r\mathfrak{G}$ where $\mathfrak{G}_0$ is abelian and the ${}^s\mathfrak{G}$ are simple. Then θ preserves $\mathfrak{G}_0$ and permutes the ${}^s\mathfrak{G}$; let $\{\mathfrak{G}_i^{(1)}, \cdots, \mathfrak{G}_i^{(r_i)}\}$ be an orbit of the permutation group defined by θ and define $\mathfrak{G}_i = \mathfrak{G}_i^{(1)} \oplus \cdots \oplus \mathfrak{G}_i^{(r_i)}$. We may assume the simple summands of $\mathfrak{G}_i$ ordered so that, if $r_i > 1$,

$$\theta(\mathfrak{G}_i^{(s)}) = \mathfrak{G}_i^{(s+1)} \quad \text{for} \quad s < r_i\,, \tag{5.1a}$$

$$\theta(\mathfrak{G}_i^{(r_i)}) = \mathfrak{G}_i^{(1)}\,. \tag{5.1b}$$

The trick is to think of (5.1a) as an identification, so $\mathfrak{G}_i$ is viewed as the sum of r_i copies of the simple algebra $\mathfrak{G}_i^{(1)}$. Then (5.1b) becomes an automorphism φ_i of $\mathfrak{G}_i^{(1)}$; in other words, $\mathfrak{G}_i$ and $\theta|_{\mathfrak{G}_i}$ are given by

$$\mathfrak{G}_i = \mathfrak{G}_i^{(1)} \oplus \cdots \oplus \mathfrak{G}_i^{(1)} \quad (r_i \text{ times})\,, \tag{5.2a}$$

$$\theta(x_1, \cdots, x_{r_i}) = (\varphi_i x_{r_i}, x_1, \cdots, x_{r_i-1}) \quad \text{where} \quad x_s \in \mathfrak{G}_i^{(1)}\,. \tag{5.2b}$$

Finally, $\theta|_{\mathfrak{G}_0}$ is an arbitrary linear automorphism; we are not assuming that θ defines a Lie group automorphism.

In the paragraph above, notice that θ has finite order $k > 0$ if and only if k is the least common multiple of positive integers $\{k_0, \cdots, k_r\}$ where $\mathfrak{G} = \mathfrak{G}_0 \oplus \cdots \oplus \mathfrak{G}_r$ and $\theta|_{\mathfrak{G}_i}$ has order k_i; if $i \geq 1$ then $k_i = r_i s_i$ where φ_i has finite order s_i. We also observe that θ is an outer (not inner) automorphism if and only if either (i) $\theta|_{\mathfrak{G}_0} \neq 1$, or (ii) some $r_i > 1$, or (iii) some φ_i is outer.

Now we need only study the outer automorphisms of simple compact Lie algebras. There we use

5.3. Lemma. *Let $\mathfrak{G}$ be a compact simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{T}$ and a system $\Psi = \{\psi_1, \cdots, \psi_l\}$ of simple roots. Relative to these choices let Δ be the Dynkin diagram, and $\{h_\psi, e_\lambda : \psi \in \Psi, \lambda \in \Lambda\}$ a Weyl basis* (Λ = root system) *of $\mathfrak{G}^C$.*

1. *Let $s : \Psi \to \Psi$ be a symmetry of Δ; so s has some finite order p. Extend s to Λ by linearity and let σ be the linear transformation of $\mathfrak{G}^C$ defined by $\sigma(h_\psi) = h_{s(\psi)}$, $\sigma(e_\lambda) = e_{s(\lambda)}$. Then* (1a) *$\sigma$ is an automorphism of order p on $\mathfrak{G}^C$ which preserves $\mathfrak{G}$,* (1b) *σ is inner if and only if $s = 1$,* i.e. *$p = 1$, and* (1c) *the automorphism group of $\mathfrak{G}$ is the disjoint union of the $\sigma \cdot ad(G)$, where G is a fixed connected group with algebra $\mathfrak{G}$, and s runs through the symmetries of Δ.*

2. *Given s, the fixed point set $\mathfrak{G}^\sigma$ of σ has Cartan subalgebra $\mathfrak{T}^\sigma = \mathfrak{T} \cap \mathfrak{G}^\sigma$.*

3. *Let $\theta \in \sigma \cdot ad(G)$, automorphism differing by σ from an inner automorphism. Then there exist $x \in \mathfrak{T}^\sigma$ and $\alpha \in ad(G)$ such that $\alpha \cdot \theta \cdot \alpha^{-1} = \sigma \cdot ad(\exp x)$, and θ has finite order $k > 0$ if and only if $ad(\exp x)$ has finite order $q > 0$ and k is the least common multiple of p and q.*

{Statements (1) and (2) are standard and straightforward; see [11, Theorem 8.11.2, p, 285] for an exposition. Statement (3) is the de Siebenthal conjugacy theorem [11, Theorem 8.6.9, p. 256] in disguise. This method is used in [11, § 8.11] to classify the riemannian symmetric spaces.}

The case $p = 1$ is the case of inner automorphisms, settled in § 2. The case $p = 2$ is the case of symmetric spaces, which is standard and can, for example, be found in [11, § 8.11]. If $p > 2$ then, by classification, $\mathfrak{G}$ is of type D_4 $\underset{\psi_1}{\circ}-\underset{\psi_2}{\circ}\!<^{\circ\,\psi_3}_{\circ\,\psi_4}$ with s given by $\psi_2 \to \psi_2$, $\psi_1 \to \psi_3 \to \psi_4 \to \psi_1$ or its inverse, so σ or σ^{-1} is triality:

5.4. Definition. Let $\mathfrak{G}$ be a Lie algebra of type D_4 $\underset{\psi_1}{\circ}-\underset{\psi_2}{\circ}\!<^{\circ\,\psi_3}_{\circ\,\psi_4}$. Let t be the symmetry $\psi_2 \to \psi_2$, $\psi_1 \to \psi_3 \to \psi_4 \to \psi_1$ of the Dynkin diagram. Let τ be the automorphism defined in Lemma 4.3(1). Then τ is called the *triality automorphism* $\mathfrak{G}$.

5.5. Theorem. *Let θ be an outer automorphism of order* 3 *on a compact or complex simple Lie algebra G. Then $\mathfrak{G}$ is of type D_4 $\underset{\psi_1}{\circ}-\underset{\psi_2}{\circ}\!<^{\circ\,\psi_3}_{\circ\,\psi_4}$ and triality τ is defined as above; the fixed point set $\mathfrak{G}^\tau$ is of type G_2 and has linear isotropy representation*

$$\mathfrak{G}^\tau = \mathfrak{G}_2 \text{ on } (\mathfrak{D}_4/\mathfrak{G}_2)^C : \overset{1}{\bullet}\!\equiv\!\circ \oplus \overset{1}{\bullet}\!\equiv\!\circ \,.$$

Define $\tau' = \tau \cdot ad(\exp 2\pi\sqrt{-1} \cdot \frac{2}{3} v_2)$; *then τ' is an outer automorphism of order* 3 *with fixed point set $\mathfrak{G}^{\tau'}$ of type A_2 and has linear isotropy representation*

$$\mathfrak{G}^{\tau'} = \mathfrak{A}_2 \text{ on } (\mathfrak{D}_4/\mathfrak{A}_2)^C : \circ-\overset{3}{\circ} \oplus \overset{3}{\circ}-\circ \,.$$

Finally, θ is conjugate to τ or τ' in the full group of automorphisms of G, and θ is conjugate to $\tau^{\pm 1}$ or to $\tau'^{\pm 1}$ by an inner automorphism.

Proof. Let $\varepsilon = e^{2\pi\sqrt{-1}/3}$ and define

$$\text{(5.6a)} \quad \begin{aligned} &a_\pm = e_{\pm\psi_1} + e_{\pm\psi_3} + e_{\pm\psi_4}, \qquad a'_\pm = e_{\pm\psi_1} + \varepsilon e_{\pm\psi_3} + \varepsilon^2 e_{\pm\psi_4}, \\ &a''_\pm = e_{\pm\psi_1} + \varepsilon^2 e_{\pm\psi_3} + \varepsilon e_{\pm\psi_4}; \end{aligned}$$

$$\text{(5.6b)} \quad \begin{aligned} b_\pm &= e_{\pm(\psi_1+\psi_2)} + e_{\pm(\psi_3+\psi_2)} + e_{\pm(\psi_4+\psi_2)}, \\ b'_\pm &= e_{\pm(\psi_1\pm\psi_2)} + \varepsilon e_{\pm(\psi_3+\psi_2)} + \varepsilon^2 e_{\pm(\psi_4+\psi_2)}, \\ b''_\pm &= e_{\pm(\psi_1+\psi_2)} + \varepsilon^2 e_{\pm(\psi_3+\psi_2)} + \varepsilon e_{\pm(\psi_4+\psi_2)}; \end{aligned}$$

$$\text{(5.6c)} \quad \begin{aligned} c_\pm &= e_{\pm(\psi_1+\psi_3+\psi_2)} + e_{\pm(\psi_3+\psi_4+\psi_2)} + e_{\pm(\psi_4+\psi_1+\psi_2)}, \\ c'_\pm &= e_{\pm(\psi_1+\psi_3+\psi_2)} + \varepsilon e_{\pm(\psi_3+\psi_4+\psi_2)} + \varepsilon^2 e_{\pm(\psi_4+\psi_1+\psi_2)}, \\ c''_\pm &= e_{\pm(\psi_1+\psi_3+\psi_2)} + \varepsilon^2 e_{\pm(\psi_3+\psi_4+\psi_2)} + \varepsilon e_{\pm(\psi_4+\psi_1+\psi_2)}. \end{aligned}$$

Then the complex eigenspaces $\mathfrak{E}(\tau, \varepsilon^k)$ have bases

$$\text{(5.7a)}\qquad \mathfrak{E}(\tau, 1): \{h_{\phi_2}, h_{\phi_1} + h_{\phi_3} + h_{\phi_4}, e_{\pm\phi_2}, e_{\pm(\phi_1+\phi_2+\phi_3+\phi_4)}, e_{\pm(\phi_1+2\phi_2+\phi_3+\phi_4)}, a_\pm, b_\pm, c_\pm\},$$

$$\text{(5.7b)}\qquad \mathfrak{E}(\tau, \varepsilon): \{h_{\phi_1} + \varepsilon^2 h_{\phi_3} + \varepsilon h_{\phi_4}, a''_\pm, b''_\pm, c''_\pm\},$$

$$\text{(5.7c)}\qquad \mathfrak{E}(\tau, \varepsilon^2): \{h_{\phi_1} + \varepsilon h_{\phi_3} + \varepsilon^2 h_{\phi_4}, a'_\pm, b'_\pm, c'_\pm\}.$$

As $\mathfrak{E}(\tau, 1)$ is $\mathfrak{G}^\tau$ or its complexification, now $\mathfrak{G}^\tau$ is of rank 2 and dimension 14, hence of type G_2. Furthermore the complex eigenspaces of τ' have bases

$$\text{(5.8a)}\qquad \mathfrak{E}(\tau', 1): \{h_{\phi_2}, h_{\phi_1} + h_{\phi_3} + h_{\phi_4}, a_\pm, b'_+, b''_-, c'_+, c''_-\},$$

$$\text{(5.8b)}\qquad \mathfrak{E}(\tau', \varepsilon): \{h_{\phi_1} + \varepsilon^2 h_{\phi_3} + \varepsilon h_{\phi_4}, e_{\phi_2}, e_{\phi_1+\phi_2+\phi_3+\phi_4}, e_{-(\phi_1+2\phi_2+\phi_3+\phi_4)}, a''_\pm, b_+, b'_-, c_+, c'_-\},$$

$$\text{(5.8c)}\qquad \mathfrak{E}(\tau', \varepsilon^2): \{h_{\phi_1} + \varepsilon h_{\phi_3} + \varepsilon^2 h_{\phi_4}, e_{-\phi_2}, e_{-(\phi_1+\phi_2+\phi_3+\phi_4)}, e_{\phi_1+2\phi_2+\phi_3+\phi_4}, a'_\pm, b_-, b''_+, c_-, c''_+\}.$$

As before, a glance at $\mathfrak{E}(\tau', 1)$ shows that $\mathfrak{G}^{\tau'}$ is of rank 2 and dimension 8, hence of type A_2.

We check the adjoint representations. A glance at (5.7) shows that $\mathfrak{G}_2 = \mathfrak{E}(\tau, 1)$ acts nontrivially on the 7-dimensional spaces $\mathfrak{E}(\tau, \varepsilon^{\pm 1})$, so it must act on each by its representation of lowest degree $\overset{1}{\bullet}\!\Rrightarrow\!\circ$ which is of degree 7. A glance at (5.8) and a short calculation shows that $\mathfrak{G}^{\tau'}$ $\underset{\beta_1}{\circ}\!-\!\underset{\beta_2}{\circ}$ has a Weyl basis $\{h_{\beta_1}, h_{\beta_2}; f_{\pm\beta_1}, f_{\pm\beta_2}, f_{\pm(\beta_1+\beta_2)}\}$ given by

$$\text{(5.9a)}\qquad h_{\beta_1} = h_{\phi_1} + h_{\phi_3} + h_{\phi_4}, \quad h_{\beta_2} = -3h_{\phi_2} - 2(h_{\phi_1} + h_{\phi_3} + h_{\phi_4});$$

$$\text{(5.9b)}\qquad f_{\pm\beta_1} = a_\pm; \quad f_{\beta_2} = c''_-, \quad f_{-\beta_2} = c'_+, \quad f_{\beta_1+\beta_2} = b''_-, \quad f_{-\beta_1-\beta_2} = b'_+.$$

Now (5.8) and another short calculation show that $\mathfrak{G}^{\tau'}$ acts on $\mathfrak{E}(\tau', \varepsilon)$ by $\circ\!-\!\overset{3}{\circ}$ and acts on $\mathfrak{E}(\tau', \varepsilon^2)$ by $\overset{3}{\circ}\!-\!\circ$.

Finally, let θ be an outer automorphism of order 3. As the symmetry group of the Dynkin diagram of D_4 is the dihedral group of order 6, the class represented by θ is that of τ or τ^{-1} and the latter two are conjugate. Now we may assume $\theta \in \tau \cdot ad(\exp \mathfrak{T}^\tau)$, i.e. $\theta = \tau \cdot ad(\exp 2\pi\sqrt{-1}x)$ with $x = av_2 + b(v_1 + v_3 + v_4)$. Then $\theta^3 = 1$ reduces our considerations to the cases (i) $a = b = 0$ so $\theta = \tau$, (ii) $a = \frac{2}{3}$ and $b = 0$ so $\theta = \tau'$, and (iii) $a = 0$ and $b = \frac{1}{3}$. In that third case $\theta = \tau \cdot ad(\exp 2\pi\sqrt{-1} \cdot \frac{1}{3}(v_1 + v_3 + v_4))$ has $\mathfrak{G}^\theta$ of dimension 14 with basis $\{h_{\phi_2}, h_{\phi_1} + h_{\phi_3} + h_{\phi_4}, e_{\pm\phi_2}, e_{\pm(\phi_1+\phi_2+\phi_3+\phi_4)}, e_{\pm(\phi_1+2\phi_2+\phi_3+\phi_4)}, a'_+, a''_-, b'_-, b''_-, c'_-, c''_+\}$; as for $\mathfrak{G}^\tau$ this says that $\mathfrak{G}^\theta$ is of type G_2; then $\mathfrak{G}^\theta$ and $\mathfrak{G}^\tau$ are conjugate; as τ is the only element of $\tau \cdot ad(G)$ with fixed point set $\mathfrak{G}^\tau$ it follows that θ and τ are conjugate. q.e.d.

Now we can classify the outer automorphisms in some sense, using the "irreducibility" conditions that the algebra has no invariant direct sum decomposition and that every power $\neq 1$ is outer:

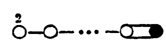
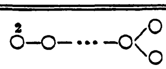
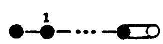
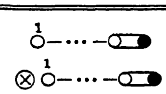
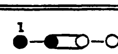
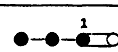
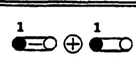
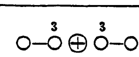
5.10. Theorem. *Let θ be an outer automorphism of a compact Lie algebra $\mathfrak{G}$ such that* (i) *there is no proper θ-invariant ideal in $\mathfrak{G}$ and* (ii) *every power θ^m not equal to the identity is an outer automorphism*[3]. *Let χ be the linear isotropy representation of the fixed point set $\mathfrak{G}^\theta$ on $\mathfrak{G}/\mathfrak{G}^\theta$.*

1. *If $\mathfrak{G}$ is not semisimple then it is abelian of dimension $r = 1$ or 2, $\mathfrak{G}^\theta = 0$, and χ is trivial. If $r = 1$ then θ is scalar multiplication by some real λ, $0 \neq |\lambda| \neq 1$. If $r = 2$ then θ is scalar multiple of a rotation,*

$$\theta = \rho\begin{pmatrix} \cos 2\pi\varphi & \sin 2\pi\varphi \\ -\sin 2\pi\varphi & \cos 2\pi\varphi \end{pmatrix}, \quad \rho > 0, 0 < \varphi < \tfrac{1}{2}.$$

2. *If $\mathfrak{G}$ is semisimple but not simple then $\mathfrak{G} = \mathfrak{K} \oplus \cdots \oplus \mathfrak{K}$ ($r > 1$ summands) with $\mathfrak{K}$ simple and $\theta(x_1, \cdots, x_r) = (\varphi x_r, x_1, \cdots, x_{r-1})$ for an automorphism φ of $\mathfrak{K}$. If $\varphi = 1$ then $\mathfrak{G}^\theta$ is $\mathfrak{K}$ embedded diagonally, θ has order r, and $\chi = ad_{\mathfrak{K}} \oplus \cdots \oplus ad_{\mathfrak{K}}$ ($r - 1$ summands). If $\varphi \neq 1$ then φ has order $p = 2$ or 3 with $\mathfrak{K}$ and φ listed below in* (3); *then θ has order rp, $\mathfrak{G}^\theta$ is $\mathfrak{K}^\varphi$ embedded diagonally, and $\chi = \overbrace{ad_{K^\varphi} \oplus \cdots \oplus ad_{K^\varphi}}^{r-1} \oplus \overbrace{\gamma \oplus \cdots \oplus \gamma}^{r}$ where γ is the linear isotropy of $\mathfrak{K}^\varphi$ on $(\mathfrak{K}/\mathfrak{K}^\varphi)^C$ as listed in* (3).

3. *If $\mathfrak{G}$ is simple then the following table is a complete list of the possibilities, up to conjugacy of θ in the full automorphism group of $\mathfrak{G}$.*

$\mathfrak{G}$	θ	$\mathfrak{G}^\theta$	order of θ	χ
$\mathfrak{A}_{2n}$	ν=complex conjugation	$\mathfrak{B}_n$	2	
$\mathfrak{A}_{2n-1}$	ν	$\mathfrak{D}_n$	2	
	$\nu\cdot\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$	$\mathfrak{C}_n$	2	
$\mathfrak{D}_n$	$\begin{pmatrix} I_{2r+1} & \\ & -I_{2n-2r-1} \end{pmatrix}$ $0 \leq r \leq n-1$	$\mathfrak{B}_r \oplus \mathfrak{B}_{n-r-1}$	2	
$\mathfrak{E}_6$	γ from symmetry of Dynkin diagram	$\mathfrak{F}_4$	2	
	$\gamma' = \gamma\cdot$(certain θ')	$\mathfrak{C}_4$	2	
$\mathfrak{D}_4$	τ	$\mathfrak{G}_2$	3	
	τ'	$\mathfrak{A}_2$	3	

[3] This is automatic if θ has prime order, especially in the important special case of order 3.

Proof. Assume $\mathfrak{G}$ not semisimple. Then $\mathfrak{G}$ is an abelian algebra $\mathfrak{T}^r$ of some dimension $r > 0$, for θ preserves the center and the derived algebra and satisfies (i). Now θ is a nonsingular linear transformation of $\mathfrak{G}$, arbitrary except for property (i). Property (i) implies that θ is diagonable over $\boldsymbol{C}$. If λ is an eigenvalue of θ now $r = 1$ if and only if λ is real, and $r > 1$ implies $\theta = |\lambda|\begin{pmatrix} \cos 2\pi\varphi & \sin 2\pi\varphi \\ -\sin 2\pi\varphi & \cos 2\pi\varphi \end{pmatrix}$ where $\lambda = |\lambda| e^{2\pi\sqrt{-1}\varphi}$.

Suppose $\mathfrak{G}$ semisimple but not simple. Again by (i), $\mathfrak{G} = \mathfrak{K} \oplus \cdots \oplus \mathfrak{K}$ ($r > 1$ summands) for some simple Lie algebra $\mathfrak{K}$ such that θ acts by $(x_1, \cdots, x_r) \to (\varphi x_r, x_1, \cdots, x_{r-1})$. As $\theta^r = \varphi \times \cdots \times \varphi$ hypothesis (ii) says that the automorphism φ of $\mathfrak{K}$ is trivial or outer. The fixed point sets satisfy $\mathfrak{G}^\theta \subset (\mathfrak{K}^\varphi \oplus \cdots \oplus \mathfrak{K}^\varphi)^\theta \subset \mathfrak{G}^\theta$, so $\mathfrak{G}^\theta = (\mathfrak{K}^\varphi \oplus \cdots \oplus \mathfrak{K}^\varphi)^\theta$ is $\mathfrak{K}^\varphi$ embedded diagonally in $\mathfrak{G}$. The adjoint action of $\mathfrak{G}^\theta$ on the s-th summand $\mathfrak{K}$ of $\mathfrak{G}$ is the same as the adjoint action of the subalgebra $\mathfrak{K}^\varphi$ of that summand. The latter is $ad_{\mathfrak{K}^\varphi} \oplus \gamma$ where γ is the linear isotropy representation of $\mathfrak{K}^\varphi$ on $(\mathfrak{K}/\mathfrak{K}^\varphi)^C$. Now

$$ad_{\mathfrak{G}^\theta} + \chi = \overbrace{ad_{\mathfrak{K}^\varphi} \oplus \cdots \oplus ad_{\mathfrak{K}^\varphi}}^{r} \oplus \overbrace{\gamma \oplus \cdots \oplus \gamma}^{r} ,$$

so

$$\chi = \overbrace{ad_{\mathfrak{K}^\varphi} \oplus \cdots \oplus ad_{\mathfrak{K}^\varphi}}^{r-1} \oplus \overbrace{\gamma \oplus \cdots \oplus \gamma}^{r} .$$

If $\varphi = 1$ then $\mathfrak{K}^\varphi = \mathfrak{K}$ and γ is of degree 0, so our assertions are immediate. If $\varphi \neq 1$ then $\mathfrak{K}$ and φ come into the scope of (3) and our assertions are reduced to that case.

Finally assume $\mathfrak{G}$ simple. Classification and (ii) say that θ has order $p = 2$ or 3. If $p = 2$ our assertions are standard facts on symmetric spaces [11, pp. 285-288], the calculation of χ being an exercise in multilinear algebra. If $p = 3$ our assertions are the content of Theorem 5.5.

6. Summary and global formulation for automorphisms of order 3

Let X be a compact simply connected coset space G/K where G is a compact connected Lie group acting effectively on X. Let θ be an automorphism of order 3 on such that $\mathfrak{K} = \mathfrak{G}^\theta$. As in the first paragraph of § 5 we now have

$$\mathfrak{G} = \mathfrak{G}_1 \oplus \cdots \oplus \mathfrak{G}_r, \quad \mathfrak{K} = \mathfrak{K}_1 \oplus \cdots \oplus \mathfrak{K}_r, \quad \theta = \theta_1 \oplus \cdots \oplus \theta_r ,$$

where $\mathfrak{K}_i = \mathfrak{K} \cap \mathfrak{G}_i = \mathfrak{G}_i^{\theta_i}$ and θ_i is an automorphism of order 3 on $\mathfrak{G}_i$ which does not preserve any proper ideals. Let G_i and K_i be the analytic subgroups of G for $\mathfrak{G}_i$ and $\mathfrak{K}_i$; they are closed subgroups, and

$$X = X_1 \times \cdots \times X_r, \quad X_i = G_i/K_i ,$$

because X is simply connected. Now a glance at the commuting algebra of the linear isotropy representation of K shows that the G-invariant almost complex structures on X are precisely the tensor fields $J = J_1 \times \cdots \times J_r$, J_i invariant almost complex structure on X_i. Thus our study of X is reduced to the "irreducible" case considered in the following theorem.

6.1. Theorem. *Let X be a simply connected coset space G/K where G is a compact connected Lie group acting effectively. Suppose $\mathfrak{K} = \mathfrak{G}^\theta$ where θ is an automorphism of order 3 on $\mathfrak{G}$ which does not preserve any proper ideals. Let N be the number of G-invariant almost complex structures on X. Then the following tables give a complete list of the possibilities, up to automorphism of G.*

{*Note:* In the tables, if A is a Lie group then A/Z_n denotes the quotient of A by a central subgroup which is cyclic of order n. If A is explicitly written as $B \times C$, then the Z_n has the property that no nontrivial element is contained in either factor, so A/Z_n is $B \times C$ "glued" along central cyclic subgroups of order n.}

Table 1. G centerless simple, K centerlizer of a torus

G	K	N
$SU(n)/Z_n$ $n \geq 2$	$S\{U(r_1) \times U(r_2) \times U(r_3)\}/Z_n$ $0 \leq r_1 \leq r_2 \leq r_3,\ \ 0 < r_2,\ \ r_1 + r_2 + r_3 = n$	2 if $r_1 = 0$ 8 if $r_1 > 0$
$SO(2n+1)$ $n \geq 1$	$U(r) \times SO(2n-2r+1), \quad 1 \leq r \leq n$	2 if $r = 1$ 4 if $r > 1$
$Sp(n)/Z_2$	$\{U(r) \times Sp(n-r)\}/Z_2, \quad 1 \leq r \leq n$	2 if $r = n$ 4 if $r < n$
$SO(2n)/Z_2$ $n \geq 3$	$\{U(r) \times SO(2n-2r)\}/Z_2, \quad 1 \leq r \leq n$	2 if $r = 1$ 2 if $r = n$ 4 if $1 < r < n$
G_2	$U(2)$	4
F_4	$\{Spin(7) \times T^1\}/Z_2$	4
	$\{Sp(3) \times T^1\}/Z_2$	4
E_6/Z_3	$\{SO(10) \times SO(2)\}/Z_2$	2
	$\{S(U(5) \times U(1)) \times SU(2)\}/Z_2$	4
	$\{[SU(6)/Z_3] \times T^1\}/Z_2$	4
	$\{[SO(8) \times SO(2)] \times SO(2)\}/Z_2$	8

G	K	N
E_7/Z_2	$\{E_6 \times T^1\}/Z_3$	2
	$\{SU(2)\times[SO(10)\times SO(2)]\}/Z_2$	4
	$\{SO(2)\times SO(12)\}/Z_2$	4
	$S\{U(7)\times U(1)\}/Z_4$	4
E_8	$SO(14)\times SO(2)$	4
	$\{E_7\times T^1\}/Z_2$	4

Table 2. G centerless simple, rank G=rank K, K semisimple with center of order 3.

G	K	N
G_2	$SU(3)$	2
F_4	$\{SU(3)\times SU(3)\}/Z_3$	2
E_6/Z_3	$\{SU(3)\times SU(3)\times SU(3)\}/\{Z_3\times Z_3\}$	2
E_7/Z_2	$\{SU(3)\times[SU(6)/Z_2]\}/Z_3$	2
E_8	$\{SU(3)\times E_6\}/Z_3$	2
	$SU(9)/Z_3$	2

Table 3. rank $G >$ rank K

G	K	N
Spin (8)	$SU(3)/Z_3$	2
	G_2	one to one correspondence with 2×2 real matrices of square $-I$, i.e. of form $\begin{pmatrix} a & b \\ -(1+a^2)/b & -a \end{pmatrix}$, $b\neq0$
$\{L\times L\times L\}/Z$ where L is compact simple and simply connected and Z is its center embedded diagonally.	L/Z where L is embedded diagonally in $L\times L\times L$ and Z is its center.	

Proof. If θ is inner, then G and K are as stated by Theorem 3.1, N is as stated by Theorems 4.2 and 4.3 and because K is connected, and G is the centerless group as listed because K has maximal rank. Now we must check that K is the global Lie group with Lie algebra $\mathfrak{K}$ as listed. That depends on the following application of Schur's Lemma to isotropy representations:

(6.2) *Let A/B be an irreducible effective symmetric coset space of compact connected Lie groups. Then the center Z_B of B is* (i) *trivial if rank $A >$ rank B*, (ii) *Z_2 if rank $A =$ rank B with B semisimple, and* (iii) *a circle group if B is not semisimple.*

It also depends on the observation

(6.3) *Let B_1 be a simple normal analytic subgroup of B in* (6.2). *In cases* (ii) *and* (iii) *of* (6.2), $Z_B \cap B_1 = Z_2$.

Table 1. $\theta = ad(\exp 2\pi\sqrt{-1}x)$ in the notation of Theorem 3.1, with $x = \frac{1}{3}v_i$ $(m_i = 1)$, $x = \frac{2}{3}v_i$ $(m_i = 2)$ or $x = \frac{1}{3}(v_i + v_j)$ $(m_i = m_j = 1)$. Let L be the analytic subgroup of G with Lie algebra $\mathfrak{L} = \mathfrak{G}^{\varphi}$, $\varphi = ad(\exp 2\pi\sqrt{-1}y)$, y to be specified. If $x = \frac{1}{3}v_i$ then G/K satisfies (6.2) in case (iii); then (6.2) and (6.3) prove our assertions. If $x = \frac{2}{3}v_i$ we let $y = v_i$; then G/L satisfies case (ii) of (6.2), and $(L/L'Z_L)/(K/L'Z_L)$ satisfies case (iii) of (6.2) where L' is the largest analytic subgroup of K normal in L; then the assertions follow from (6.2) and (6.3). If $x = \frac{1}{3}(v_i + v_j)$ we let $y = \frac{1}{2}v_i$, so G/L and $(L/L'Z_L)/(K/L'Z_L)$ satisfy case (iii) of (6.2) and we are finished by (6.2) and (6.3).

Table 2. K has center of order 3 which meets every simple normal analytic subgroup. The assertions follow.

Table 3. $\mathfrak{G}, \mathfrak{K}$ and N are as stated by Theorems 5.5 and 5.10. Global formulation is obvious for $\mathfrak{G} = \mathfrak{L} \oplus \mathfrak{L} \oplus \mathfrak{L}$, and follows for $\mathfrak{G} = \mathfrak{D}_4$ because there the lift of $K = SU(3)/Z_3 = ad(SU(3))$ of $K = G_2$ from $SO(8)$ to $Spin(8)$ remains centerless. q.e.d.

Now we can complete Theorem 6.1 to a global structure-classification theorem.

6.4. Theorem. *The coset spaces $X = G/K$ with properties* (i) *G is a compact connected Lie group acting effectively,* (ii) *$\mathfrak{K} = \mathfrak{G}^{\theta}$ where θ is an automorphism of order* 3 *on $\mathfrak{G}$, and* (iii) *X carries a G-invariant almost complex structure, are precisely the spaces $(X_0 \times X_1 \times \cdots \times X_r)/\Gamma = \{(G_0 \times G_1 \times \cdots \times G_r)/\Gamma\}/K$ constructed as follows.*

X_0 is a complex euclidean space, G_0 is its translation group and $K_0 = \{1\} \subset G_0$.

$r \geq 0$ is an integer. If $1 \leq i \leq r$ then $X_i = G_i/K_i$ is one of the spaces listed in Theorem 6.1. *Z_i is ths center of G_i, trivial if rank $G_i =$ rank K_i (tables* 1 *and* 2), *$Z_2 \times Z_2$ if $G_i = Spin(8)$, and $Z \times Z$ if $G_i = \{L \times L \times L\}/Z$.*

Γ is any discrete subgroup of $G_0 \times Z_1 \times \cdots \times Z_r$ such that $\Gamma \cap G_0$ is a lattice in the vector group G_0.

K is the image of $(K_0 \times K_1 \times \cdots \times K_r)$ in $(G_0 \times G_1 \times \cdots \times G_r)/\Gamma$.

Proof. The universal covering $\tilde{X} \to X$ can be constructed from the universal covering group $\beta : G^* \to G$ as follows. Z^* is the kernel of the action of G^* on $G^*/\beta^{-1}(K)_0$, $\tilde{G} = G^*/Z^*$, $\tilde{K} = \{Z^* \cdot \beta^{-1}(K)_0\}/Z^*$ and $\tilde{X} = \tilde{G}/\tilde{K}$. If $\nu : \tilde{G} \to G$ is the natural covering, then $\tilde{K} = \nu^{-1}(K)_0$ and $\tilde{X} \to X$ is given by $\tilde{g}\tilde{K} \to \nu(\tilde{g})K$. As G is compact and effective and $\tilde{X}$ is simply connected,

$\tilde{G} = G_0 \times G_1 \times \cdots \times G_r$ and $\tilde{K} = K_0 \times K_1 \times \cdots \times K_r$ where G_0 is a vector group, $K_0 = \{1\} \subset G_0$, $\tilde{X} = X_0 \times X_1 \times \cdots \times X_r$, $X_0 = G_0/K_0$ complex euclidean space, and $(1 \leq i \leq r)$ $X_i = G_i/K_i$ is listed in Theorem 6.1.

We can view $X = \tilde{G}/\nu^{-1}(K)$ with $\tilde{X} \to X$ given by $\tilde{g}\tilde{K} \to \tilde{g}\cdot\nu^{-1}(K)$; for $\tilde{K}$ is the identity component of $\nu^{-1}(K)$ and $\tilde{g}\cdot\nu^{-1}(K) \to \nu(\tilde{g})\cdot K$ defines a diffeomorphism of $\tilde{G}/\nu^{-1}(K)$ onto G/K. Now we look to see the possibilities for $\nu^{-1}(K)$.

Let A be the normalizer of $\tilde{K}$ in $\tilde{G}$. Then $\nu^{-1}(K)$ is a closed subgroup of A with identity component $\tilde{K}$. Now $A = G_0 \times A_1 \times \cdots \times A_r$ where $(1 \leq i \leq r)$ A_i is the normalizer of K_i in G_i. Let $a \in \nu^{-1}(K)$, so $a \in A$, and decompose $a_0 \times a_1 \times \cdots \times a_r$, $a_i \in A_i$. Then a_i preserves the almost complex structure on G_i/K_i. If rank $K_i =$ rank G_i now Proposition 4.2 says $a_i \in K_i = Z_iK_i$. If rank K_i is less than rank G_i we go by cases. First let $G_i = \boldsymbol{Spin}(8)$. If $K_i = G_2$ then it has no outer automorphism so $ad(a_i)|_{K_i}$ is inner. If $K_i = \boldsymbol{SU}(3)/\boldsymbol{Z}_3$ then $ad(a_i)$ cannot interchange the summands of the isotropy representation $\circ\!-\!\overset{3}{\circ} \oplus \overset{3}{\circ}\!-\!\circ$, so $ad(a_i)|_{K_i}$ is inner. In either case we have $a' \in a_iK_i$ centralizing K_i. No inner automorphism of $\mathfrak{G}_i$ is trivial on $\mathfrak{K}_i$; thus $a' \in Z_i$; now $a' \in Z_iK_i$. The same argument is valid for the case $G_i = (L \times L \times L)/Z$.

We have now proved that $\nu^{-1}(K)$ is a closed subgroup of $G_0 \times Z_1K_1 \times \cdots \times Z_rK_r$ which has identity component $\tilde{K} = \{1\} \times K_1 \times \cdots \times K_r$. But $G_0 \times Z_1K_1 \times \cdots \times Z_rK_r \cong (G_0 \times Z_1 \times \cdots \times Z_r) \times \tilde{K}$ as topological groups. So $\nu^{-1}(K) = \Gamma \times \tilde{K}$ where Γ is a discrete subgroup of $G_0 \times Z_1 \times \cdots \times Z_r$. Thus $X = (X_0 \times X_1 \times \cdots \times X_r)/\Gamma = G/K$; $G = (G_0 \times G_1 \times \cdots \times G_r)/\Gamma$ and K is the image of $\tilde{K}$ in G. As G is compact $\Gamma \cap G_0$ is a lattice in G_0.

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