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THE WEYL PROBLEM WITH NONNEGATIVE GAUSS CURVATURE

PENGFEI GUAN & YANYAN LI

1. Introduction

Weyl posed the following problem in 1916 [21]: consider the 2-sphere S^2 and suppose g^0 is a Riemannian metric on S^2 whose Gauss curvature is everywhere positive. Does there exist a global C^2 isometric embedding $X: (S^2, g^0) \to (R^3, \delta)$ where δ is the standard flat metric in a Euclidean 3-space R^3? The first attempt to solve the problem was made by Weyl himself. He suggested the continuity method and obtained a priori estimates up to the second derivatives. Later Lewy [13] solved the problem in the case of g^0 being analytic. The complete solution was given in 1953 by Nirenberg in a beautiful paper [16] under very mild hypothesis that the metric g^0 has continuous fourth derivatives. His result depends on the strong a priori estimates he had derived for uniformly elliptic equations in dimension two [17]. The result was extended to the case of continuous third derivatives of the metric by Heinz [9] in 1962. In a completely different approach to the problem, Alexandroff [1] obtained a generalized solution of Weyl's problem as a limit of polyhedra. The regularity of this generalized solution was proved by Pogorelov [18], [19].

The uniqueness question was considered by Weyl in [21]. The first proof of the uniqueness of a solution of the problem (within rigid motion and a possible reflection), i.e., a proof of the theorem that two closed isometric convex surfaces are congruent (within a reflection) was given by Cohn-Vossen [6] in 1927, under the assumption that the surfaces are analytic. It was later shortened considerably by Zhitomirsky [22]. In 1943 Herglotz [10] gave a very short proof of the uniqueness, assuming that the surfaces are three times continuously differentiable. Finally in 1962 it was extended to surfaces having merely two times continuously differentiable metrics by Sacksteder [20]. Notice that the rigidity results in [20] hold under

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more general hypotheses than the Gauss curvature of g^0 being positive everywhere.

A natural question to ask is: if the Gauss curvature of the metric is nonnegative instead of positive everywhere, and we assume the smoothness of the metric, do we still have a smooth isometric embedding? Examples given in [11] and [2] show that for some analytic metrics with Gauss curvature positive on S^2 except at one point, there exist no C^3 global isometric embedding into (R^3, δ).

It is proved by Iaia in [11] that if g^0 is a C^4 Riemannian metric on S^2 with Gauss curvature K satisfying, for some $P \in S^2$, that

(i) $K(P) = 0$, $K(Q) > 0$ for $Q \neq P$,
(ii) $\Delta_{g^0} K(Q) \geq 0$ for all Q near P,

then there exists a $C^{1,1}$ isometric embedding $X: (S^2, g^0) \to (R^3, \delta)$, where δ is the standard flat metric on R^3.

The loss of strict positivity of K leads to degenerate Monge-Ampère equations. Certain types of degenerate Monge-Ampère equations have been studied by Caffarelli, Kohn, Nirenberg, and Spruck [3], [4].

In this note we prove the following result.

Theorem. *Suppose that g^0 is a C^4 Riemannian metric on S^2 with Gauss curvature $K^0 \geq 0$. Then there exists a $C^{1,1}$ isometric embedding $X: (S^2, g^0) \to (R^3, \delta)$ where δ is the standard flat metric in R^3.*

The proof of this theorem contains the following proposition which is of independent interest.

Proposition. *Let M be any closed convex surface in R^3, normalized so that the smallest ball containing M is centered at the origin. Set*

$$a_1 = \min_{x \in M} |x|, \qquad a_2 = \max_{x \in M} |x|.$$

Then

$$\max_{x \in M} |H(x)| \leq a_2 e^{((a_2)^2 - (a_1)^2)/6(a_2)^2} \sqrt{\max_{x \in M}(K(x)^2 - \tfrac{3}{2}\Delta_g K(x))},$$

where g denotes the metric on M induced from (R^3, δ), $K(x)$ denotes the Gauss curvature at $x \in M$ with respect to g, and $H(x)$ denotes the mean curvature of M at $x \in M$.

There are many interesting questions which still need to be studied. We mention a few in the following.

Question 1. Does there exist some smooth Riemannian metric on S^2 with nonnegative Gauss curvature which can never be C^2 isometrically embedded into (R^3, δ)? Or conversely, is it possible to improve our theorem to a C^2 embedding (even $C^{2,\alpha}$ $(0<\alpha<1)$ or $C^{2,1}$?) instead of a $C^{1,1}$ embedding?

Question 2. What are the sufficient conditions (even necessary and sufficient conditions) on the metric with nonnegative Gauss curvature which give rise to a smooth isometric embedding into (R^3, δ)?

Notice that the problems above are *global* since Lin proved (see [14]) that for any smooth 2-dimensional Riemannian metric with nonnegative Gauss curvature, there always exists some smooth *local* isometric embedding into (R^3, δ).

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2. Proof of the Theorem

We first approximate g^0 in C^4 by a sequence of C^∞ metrics g^ε, with corresponding Gauss curvature $\{K^\varepsilon\}$ being positive everywhere. This can be achieved very easily, for instance, by first setting

$$\tilde{g}^\varepsilon = e^{2\varepsilon w} g^0$$

with $\varepsilon > 0$ and $w \in C^{4,\alpha}(S^2)$ $(0<\alpha<1)$. It is well known that $\widetilde{K}^\varepsilon$ satisfies

$$-\varepsilon\Delta_{g^0} w + K = \widetilde{K}^\varepsilon e^{2\varepsilon w}.$$

Clearly if we choose w such that $-\Delta_{g^0} w = 1$ in $\{x \in S^2 | K^0(x) = 0\}$ (notice that K^0 has to be positive somewhere due to the Gauss-Bonnet theorem, so it is easy to find such w), and then choose ε to be very small, $\tilde{g}^\varepsilon$ will be close to g^0 in C^4 and $\widetilde{K}^\varepsilon$ will be positive everywhere. Next we fix w, ε, and hence $\tilde{g}^\varepsilon$. Then we choose a C^∞ metric g^ε, which is C^4 as close to $\tilde{g}^\varepsilon$ as we want (hence $K^\varepsilon > 0$ everywhere).

Now we can apply Nirenberg's theorem to g^ε (since $K^\varepsilon > 0$ everywhere and g^ε is smooth) and therefore obtain a C^∞ isometric embedding $X^\varepsilon\colon (S^2, g^0) \to (R^3, \delta)$.

It is not difficult to see that there exist constants $\alpha_1, \beta_1 > 0$ (independent of ε), such that, for al $\varepsilon > 0$,

$$0 < K^\varepsilon(\omega) \le \beta_1 \qquad \forall \omega \in S^2, \tag{1}$$

$$R(X^\varepsilon) = \min_{Y\in R^3} \max_{\omega\in S^2} |X^\varepsilon(\omega) - Y| \le \alpha_1. \tag{2}$$

From now on we can simply assume that the origin is the center of the smallest ball containing the surface. It is elementary to see that $R(X^\varepsilon) = \|X^\varepsilon\|_{C^0(S^2)}$. With this normalization and (1) we immediately have

$$\|X^\varepsilon\|_{C^0} = R(X^\varepsilon) \le C, \tag{3}$$

where C is some constant independent of ε.

Write g^0 and g^ε in local coordinates,

$$g^0 = E^0\, du^2 + 2F^0\, du\, dv + G^0\, dv^2,$$
$$g^\varepsilon = E^\varepsilon\, du^2 + 2F^\varepsilon\, du\, dv + G^\varepsilon\, dv^2.$$

We already known that $X^\varepsilon\colon (S^2, g^\varepsilon) \to (R^3, \delta)$ is an isometric embedding, so we have $dX^\varepsilon \cdot dX^\varepsilon = g^\varepsilon$, namely,

$$X_u^\varepsilon \cdot X_u^\varepsilon = E^\varepsilon, \quad X_u^\varepsilon \cdot X_v^\varepsilon = F^\varepsilon, \quad X_v^\varepsilon \cdot X_v^\varepsilon = G^\varepsilon.$$

It follows easily from the above that

$$\|\nabla_{g^0} X^\varepsilon\|_{C^0} \le C, \tag{4}$$

where C is some constant independent of ε.

The following will be devoted to establishing a bound on $\|\nabla^2_{g^0} X^\varepsilon\|_{C^0}$. Once we obtain such a bound, the limit of X^ε (along a subsequence) as $\varepsilon \to 0$ will be a $C^{1,1}$ isometric embedding of g^0. For convenience, we drop the dependence on ε in our notation in the following.

Let us recall some elementary differential geometry formulas. Let $X\colon (S^2, g) \to (R^3, \delta)$ be an isometric embedding where $\delta = dx^2 + dy^2 + dz^2$ is the standard flat metric in R^3. Let (u, v) denote local coordinates on part of the sphere and write

$$g = g_{11}\, du^2 + 2g_{12}\, du\, dv + g_{22}\, dv^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2.$$

Suppose $X(u, v) = (x(u, v), y(u, v), z(u, v))$. Then the first fundamental form is given by

$$I = dX \cdot dX = g.$$

Namely,

$$X_u \cdot X_u = E = g_{11}, \quad X_u \cdot X_v = F = g_{12}, \quad X_v \cdot X_v = G = g_{22}.$$

Let the orientation be chosen so that the inner unit normal is given by

$$\overline{X} = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{X_u \times X_v}{\sqrt{EG - F^2}},$$

where $|\cdot|$ denotes the Euclidean norm, and $\times$ the standard cross product between vectors in R^3.

The second fundamental form is then given by

$$II = -dX \cdot d\overline{X} = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where

$$\begin{aligned} L &= -X_u \cdot \overline{X}_u = X_{uu} \cdot \overline{X}, \\ M &= -X_v \cdot \overline{X}_u = -X_u \cdot \overline{X}_v = X_{uv} \cdot \overline{X}, \\ N &= -X_v \cdot \overline{X}_v = X_{vv} \cdot \overline{X}. \end{aligned}$$

Therefore the Gauss and mean curvatures are (determinant and $\frac{1}{2}$ trace of the second fundamental form with respect to the first fundamental form):

$$K = (LN - M^2)/(EG - F^2), \tag{5}$$

$$H = \tfrac{1}{2}(GL - 2FM + EN)/(EG - F^2). \tag{6}$$

Furthermore the Gauss theorem egregium asserts that K can be expressed in terms of up to the second derivatives of E, F, G only.

The Gauss equation takes the form

$$\begin{aligned} X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L\overline{X}, \\ X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M\overline{X}, \\ X_{vv} &= \Gamma^2_{22} X_u + \Gamma^2_{22} X_v + N\overline{X}, \end{aligned} \tag{7}$$

where Γ^k_{ij} are called Christoffel symbols associated with the metric tensor g_{ij}, given by

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{8}$$

with $\partial_1 = \partial_u$ and $\partial_2 = \partial_v$.

The Weingarten equations take the form

$$\begin{aligned} -\overline{X}_u &= L^1_1 X_u + L^2_1 X_v, \\ -\overline{X}_v &= L^1_2 X_u + L^2_2 X_v \end{aligned} \tag{9}$$

where $\{L^i_j\}$ are expressions involving L, M, N and E, F, G.

The Mainardi-Codazzi equations take the form

$$\begin{aligned} L_v - M_u &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}, \\ M_v - N_u &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}. \end{aligned} \tag{10}$$

Let Δ_g denote the Laplace-Beltrami operator associated with the metric g, namely,

$$\Delta_g = \frac{1}{\sqrt{\det_g}} \partial_j (\sqrt{\det g}\, g^{ij} \partial_i) = g^{ij} \nabla_i \nabla_j ,$$

where (g^{ij}) is the inverse matrix of (g_{ij}), and ∇_i the covariant differentiation with respect to g in the direction ∂_i. It follows from (7) that

$$\Delta_g X = 2H\overline{X}. \tag{11}$$

We remark that in order to obtain a C^2 bound on X it is enough to obtain an L^∞ bound on H, since the second fundamental form is positive definite, and an L^∞ bound on H ($\frac{1}{2}$ trace of the second fundamental form with respect to the first fundamental form) will give rise to an L^∞ bound of L, M, N. Now the desired L^∞ bound on X_{uu}, X_{uv}, and X_{vv} will follow immediately from the Gauss equation (7) and the gradient estimate (4).

Therefore the only thing we need to do is to establish an L^∞ bound on H. Notice that with the choice of inner unit normal, $H \geq 0$.

Define

$$\rho(u, v) = \tfrac{1}{2} X \cdot X .$$

The above function has been used by Darboux (see [7, §707] and also [16]). Differentiating $\rho(u, v)$ and using (7), we have

$$\begin{aligned}
&\rho_u = X \cdot X_u , \qquad \rho_v = X \cdot X_v \\
&\rho_{uu} = X \cdot X_{uu} + E = \Gamma^1_{11}\rho_u + \Gamma^2_{11}\rho_v + LX \cdot \overline{X} + E , \\
&\rho_{uv} = X \cdot X_{uv} + F = \Gamma^1_{12}\rho_u + \Gamma^2_{12}\rho_v + MX \cdot \overline{X} + F , \\
&\rho_{vv} = X \cdot X_{vv} + G = \Gamma^1_{22}\rho_u + \Gamma^2_{22}\rho_v + NX \cdot \overline{X} + G .
\end{aligned} \tag{12}$$

We consider the following function on S^2,

$$f = e^{\alpha\rho} H , \tag{13}$$

with constant $\alpha > 0$ being chosen later.

Consider the maximization

$$f_{\max} = \max_{S^2} f .$$

There exists some point $P \in S^2$, such that $f(P) = f_{\max}$. Let us write the metric $g = g^\varepsilon$ near P in conformal coordinates:

$$g = e^{2h}(du^2 + dv^2) , \tag{14}$$

where $(u, v) = (0, 0)$ corresponds to P, and

$$h = \partial_u h = \partial_v h = 0 \quad \text{at } (0, 0). \tag{15}$$

It is not difficult to calculate in the uv coordinates, using (7), that

$$\Gamma^1_{11} = \Gamma^2_{12} = \partial_1 h, \quad \Gamma^2_{11} = -\partial_2 h, \quad \Gamma^1_{12} = \Gamma^2_{22} = \partial_2 h, \quad \Gamma^1_{22} = -\partial_1 h.$$

Here and in the following, $h_1 = \partial_1 h = \partial_u h$ and $h_2 = \partial_2 h = \partial_v h$.

For the formulas (5) and (6) the Gauss and mean curvatures become

$$K = (LN - M^2)/e^{4h}, \tag{16}$$

$$H = \tfrac{1}{2}(L + N)/e^{2h}, \tag{17}$$

$$K = -\tilde{\Delta} h/e^{2h}, \tag{18}$$

where $\tilde{\Delta} = \partial_{11} + \partial_{22}$. Also, the Mainardi-Codazzi equations take

$$\begin{aligned} L_2 - M_1 &= h_2(L + N) = 2He^{2h} h_2, \\ M_2 - N_1 &= -h_1(N + L) = -2He^{2h} h_1. \end{aligned} \tag{19}$$

Clearly (see (14))

$$\Delta_g K = e^{-2h} \tilde{\Delta} K. \tag{20}$$

Differentiating (16), we have

$$\begin{aligned} K_1 &= \frac{1}{e^{4h}}[L_1 N + L N_1 - 2MM_1 - 4h_1(LN - M^2)] \\ &= \frac{1}{e^{4h}}(L_1 N + L N_1 - 2MM_1) - 4h_1 K, \end{aligned} \tag{21}$$

$$\begin{aligned} K_{11} &= \frac{1}{e^{4h}}(L_{11} N + L N_{11} + 2N_1 L_1 - 2M_1^2 - 2MM_{11}) \\ &\quad - \frac{4h_1}{e^{4h}}(L_1 N + L N_1 - 2MM_1) - 4h_{11} K - 4h_1 K_1 \\ &= \frac{1}{e^{4h}}(L_{11} N + L N_{11} + 2N_1 L_1 - 2M_1^2 - 2MM_{11}) \\ &\quad - 8h_1 K_1 - 4h_{11} K - 16h_1^2 K, \end{aligned} \tag{22}$$

$$\begin{aligned} K_2 &= \frac{1}{e^{4h}}[L_2 N + L N_2 - 2MM_2 - 4h_2(LN - M^2)] \\ &= \frac{1}{e^{4h}}(L_2 N + L N_2 - 2MM_2) - 4h_2 K, \end{aligned} \tag{23}$$

$$K_{22} = \frac{1}{e^{4h}}(L_{22}N + LN_{22} + 2N_2L_2 - 2M_2^2 - 2MM_{22}) \tag{24}$$
$$- 8h_2K_2 - 4h_{22}K - 16h_2^2K,$$

where the last equality of (22) follows from (21).

Apply ∂_2 to the first equation of (19), and ∂_1 to the second, and add together to obtain

$$(-L_2 + 2He^{2h}h_2)_2 = (-N_1 + 2He^{2h}h_1)_1,$$

which yields

$$N_{11} = L_{22} + 2H_1h_1e^{2h} - 2H_2h_2e^{2h} + 2(h_{11} - h_{22})He^{2h} \tag{25}$$
$$+ 4(h_1^2 - h_2^2)He^{2h}.$$

Differentiating (17) gives

$$2H_1 = \frac{L_1 + N_1}{e^{2h}} - 2h_1\frac{L+N}{e^{2h}} = \frac{L_1 + N_1}{e^{2h}} - 4h_1H, \tag{26}$$

$$\begin{aligned} 2H_{11} &= \frac{L_{11} + N_{11}}{e^{2h}} - 2h_1\frac{L_1 + N_1}{e^{2h}} - 4h_1H_1 - 4h_{11}H \\ &= \frac{L_{11} + N_{11}}{e^{2h}} - 2h_1(2H_1 + 4h_1H) - 4h_1H_1 - 4h_{11}H \\ &= \frac{L_{11} + N_{11}}{e^{2h}} - 8h_1H_1 - 8h_1^2H - 4h_{11}H, \end{aligned} \tag{27}$$

$$2H_2 = (L_2 + N_2)/e^{2h} - 4h_2H, \tag{28}$$

$$2H_{12} = \frac{L_{22} + N_{22}}{e^{2h}} - 4h_2H_1 - 4h_{12}H - 4h_1H_2 - 8h_1h_2H, \tag{29}$$

$$2H_{22} = \frac{L_{22} + N_{22}}{e^{2h}} - 8h_2H_2 - 8h_2^2H - 4h_{22}H. \tag{30}$$

Applying ∂_1 to the first equation of (19) and ∂_2 to the second, we obtain

$$\begin{aligned} M_{11} &= (L_2 - 2h_2He^{2h})_1 \\ &= L_{12} - 2H_1e^{2h}h_2 - 2He^{2h}h_{12} - 4Hh_1h_2e^{2h}, \end{aligned} \tag{31}$$

$$\begin{aligned} M_{22} &= (N_1 - 2h_1He^{2h})_2 \\ &= N_{12} - 2H_2e^{2h}h_1 - 2He^{2h}h_{12} - 4Hh_1h_2e^{2h}. \end{aligned} \tag{32}$$

A use of (22) and (24), leads to

$$\begin{aligned}
e^{6h}\Delta_g K &= e^{4h}\tilde{\Delta} K \\
&= L_{11}N + LN_{11} + 2L_1N_1 - 2M_1^2 - 2MM_{11} \\
&\quad + L_{22}N + LN_{22} + 2L_2N_2 - 2M_2^2 - 2MM_{22} \\
&\quad - e^{4h}[8h_1K_1 + 8h_2K_2 + 4(h_{11} + h_{22})K + 16(h_1^2 + h_2^2)K] \\
&= N(L_{11} + L_{22}) + L(N_{11} + N_{22}) - 2M(M_{11} + M_{22}) \\
&\quad + 2(L_1N_1 + L_2N_2 - M_1^2 - M_2^2) \\
&\quad - e^{4h}[8h_1K_1 + 8h_2K_2 + 4(h_{11} + h_{22})K + 16(h_1^2 + h_2^2)K].
\end{aligned}$$

In the above equation substituting (25) for L_{22} and also for N_{11}, and substituting (31) for M_{11} and (32) for M_{22}, we get

$$\begin{aligned}
e^{6h}\Delta_g K = {} & N[L_{11} + N_{11} - 2H_1h_1e^{2h} + 2H_2h_2e^{2h} \\
&\qquad - 2(h_{11} - h_{22})He^{2h} - 4(h_1^2 - h_2^2)He^{2h}] \\
& + L[L_{22} + N_{22} + 2H_1h_1e^{2h} - 2H_2h_2e^{2h} \\
&\qquad + 2(h_{11} - h_{22})He^{2h} + 4(h_1^2 - h_2^2)He^{2h}] \\
& - 2M[L_{12} + N_{12} - 2(H_1h_2 + H_2h_1)e^{2h} \\
&\qquad - 4He^{2h}h_{12} - 8Hh_1h_2e^{2h}] \\
& + 2(L_1N_1 + L_2N_2 - M_1^2 - M_2^2) \\
& - e^{4h}[8h_1K_1 + 8h_2K_2 + 4(h_{11} + h_{22})K + 16(h_1^2 + h_2^2)K].
\end{aligned}$$

Using (27), (29), and (30) to replace $L_{11} + N_{11}$, $L_{12} + N_2$, $L_{22} + N_{22}$ by $2H_{11}e^{2h}$, $2H_{12}e^{2h}$, $2H_{22}e^{2h}$ respectively, plus some lower order terms, we obtain

$$\begin{aligned}
e^{6h}\Delta_g K = {} & N[2e^{2h}H_{11} + 6H_1h_1e^{2h} \\
&\qquad + 2H_2h_2e^{2h} + 2(h_{11} + h_{22})He^{2h} + 4(h_1^2 + h_2^2)He^{2h}] \\
& + L[2e^{2h}H_{22} + 6h_2H_2e^{2h} \\
&\qquad + 2H_1h_1e^{2h} + 2(h_{11} + h_{22})He^{2h} + 4(h_1^2 + h_2^2)He^{2h}] \\
& - 2M[2e^{2h}H_{12} + 2(H_1h_2 + H_2h_1)e^{2h}] \\
& + 2(L_1N_1 + L_2N_2 - M_1^2 - M_2^2) \\
& - e^{4h}[8h_1K_1 + 8h_2K_2 + 4(h_{11} + h_{22})K + 16(h_1^2 + h_2^2)K].
\end{aligned}$$

Regrouping the terms and using (18) to replace $h_{11}+h_{22}$ by $-e^{2h}K$ yield

$$
\begin{aligned}
e^{6h}\Delta_g K &= 2e^{2h}(NH_{11}-2MH_{12}+LH_{22}) \\
&\quad + 2(L_1N_1+L_2N_2-M_1^2-M_2^2) \\
&\quad + N[6H_1h_1e^{2h} \\
&\qquad + 2H_2h_2e^{2h}+2(h_{11}+h_{22})He^{2h}+4(h_1^2+h_2^2)He^{2h}] \\
&\quad + L[6h_2H_2e^{eh} \\
&\qquad + 2H_1h_1e^{2h}+2(h_{11}+h_{22})He^{2h}+4(h_1^2+h_2^2)He^{2h}] \\
&= 4M(H_1h_2+H_2h_1)e^{2h} \\
&\quad - e^{4h}[8h_1K_1+8h_2K_2-4K^2e^{2h}+16(h_1^2+h_2^2)K].
\end{aligned} \tag{33}
$$

Notice that we derive from (26) and (28) that

$$
\begin{aligned}
L_1N_1 &\le \tfrac{1}{4}(L_1+N_1)^2 = e^{4h}(H_1+2h_1H)^2, \\
L_2N_2 &\le \tfrac{1}{4}(L_2+N_2)^2 = e^{4h}(H_2+2h_2H)^2.
\end{aligned} \tag{34}
$$

Using the relation $h_{11}+h_{22}=-e^{2h}K$ and (34), evaluating (33) at P (so $h=h_1=h_2=0$), and noticing that $K\le H^2$ we have

$$
\Delta_g K \le 2(NH_{11}-2MH_{12}+LH_{22})+2(H_1^2-H_2^2). \tag{35}
$$

It follows from (13) that the following hold at P, in consequence of $f_i(P)=0$,

$$
\begin{aligned}
&H_1=-\alpha\rho_1H, \qquad H_2=-\alpha\rho_2H, \\
&H_{11}=-\alpha H\rho_{11}+\alpha^2\rho_1^2H+f_{11}e^{-\alpha\rho}, \\
&H_{12}=-\alpha H\rho_{12}+\alpha^2\rho_1\rho_2H+f_{12}e^{-\alpha\rho}, \\
&H_{22}=-\alpha H\rho_{22}+\alpha^2\rho_2^2H+f_{22}e^{-\alpha\rho}.
\end{aligned}
$$

Since f has P as the maximum point, we also have

$$
(f_{ij}(P))\le 0, \tag{36}
$$

and therefore

$$
\begin{aligned}
LH_{22}-2MH_{12}+NH_{11} &\le H[L(-\alpha\rho_{22}+\alpha^2\rho_2^2) \\
&\quad -2M(-\alpha\rho_{12}+\alpha^2\rho_1\rho_2)+N(-\alpha\rho_{11}+\alpha^2\rho_2^2)],
\end{aligned} \tag{37}
$$

$$
(H_1^2+H_2^2)=H^2\alpha^2(\rho_1^2+\rho_2^2). \tag{38}
$$

Using (35), (37), and (38) we obtain, at point P, that

$$\Delta_g K \leq -2\alpha H(L\rho_{22} - 2M\rho_{12} + N\rho_{11}) + 2\alpha^2 H(L\rho_2^2 - 2M\rho_1\rho_2 + N\rho_1^2) + 2(H_1^2 + H_2^2), \tag{39}$$

so that at P in consequence of $H_i^2 = \alpha^2\rho_i^2 H^2$ and $L, M, N \leq 2H$,

$$\begin{aligned}\Delta_g K &\leq -2\alpha H(L\rho_{22} - 2M\rho_{12} + N\rho_{11}) \\ &\quad + 2\alpha^2 H(L+N)(\rho_1^2 + \rho_2^2) + 2H^2\alpha^2(\rho_1^2 + \rho_2^2) \\ &= -2\alpha H(L\rho_{22} - 2M\rho_{12} + N\rho_{11}) + 6H^2\alpha^2(\rho_1^2 + \rho_2^2).\end{aligned}$$

Since we are in a geodesic normal coordinate system, it follows from (12) that, at P,

$$|X \cdot \overline{X}|^2 + \rho_1^2 + \rho_2^2 = |X|^2,$$

and

$$\rho_{11} = LX \cdot \overline{X} + E, \quad \rho_{12} = MX \cdot \overline{X} + F, \quad \rho_{22} = NX \cdot \overline{X} + G.$$

Therefore at P

$$\begin{aligned}\Delta_g K &\leq -4\alpha H(LN - M^2)X \cdot \overline{X} - 2\alpha H(LG - 2MF + NE) \\ &\quad + 6H^2\alpha^2(\rho_1^2 + \rho_2^2) \\ &= -4\alpha HKX \cdot \overline{X} - 2\alpha H(L+N) + 6H^2\alpha^2(\rho_1^2 + \rho_2^2) \\ &\leq 4\alpha HK|X \cdot \overline{X}| - 4\alpha H^2 + 6H^2\alpha^2(\rho_1^2 + \rho_2^2) \\ &\leq -4\alpha H^2 + 6\alpha^2 H^2|X|^2 + \tfrac{2}{3}K^2.\end{aligned}$$

Choose $\alpha = \frac{1}{3}R^{-2}$, where $R = R(X)$ is as in (2), we have at P that

$$H^2 \leq R^2(K^2 - \tfrac{2}{3}\Delta_g k).$$

Thus, $H(P) \leq C$, and it follows easily that

$$\max_{S^2} f = f(P) = e^{\alpha\rho(P)}H(P) \leq C,$$

i.e.,

$$\max_{S^2} H = \max_{S^2} e^{-\alpha\rho} f \leq \max_{S^2} Ce^{-\alpha\rho} \leq C.$$

Hence we have obtained the desired estimate on H.

Notice that if we keep track of the above constants in the above computation, we immediately obtain the proof of the proposition mentioned in §1.

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McMaster University
Rutgers University