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LINKING AND HOLOMORPHIC HULLS

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1. Introduction

If X and Y are disjoint compact oriented smooth submanifolds of a smooth oriented manifold M and are homologous to zero in M, then the linking number of X and Y, denoted $\mathrm{link}(X, Y)$ (or by $\mathrm{link}(X, Y; M)$ for clarity) is equal to the intersection number of V and Y, where (V, X) is a compact oriented submanifold with boundary in M. This can be taken as one of the several equivalent definitions of linking number; here the dimensions a, k, m of X, Y, and M respectively, satisfy $a+k = m-1$. We say that X and Y are linked if $\mathrm{link}(X, Y)$ is not zero. Our object is to apply this linking notion of Gauss to the geometry of holomorphic hulls. For example, in the case that the underlying manifold M is $\mathbf{C}^n$, our results say that the polynomially convex hull of one of the sets X or Y has a nonempty intersection with the other set, provided that X and Y are linked.

Now take M to be a Stein manifold and let X be a compact subset of M. Then the holomorphic hull of X is

$$\widehat{X} = \{p \in M: |f(p)| \le \max\{|f(q)|: q \in X\} \text{ for all } f \in A(M)\}$$

where $A(M)$ is the space of all holomorphic functions on M. $\widehat{X}$ is a compact subset of M. In special cases arising from the maximum principle, $(\widehat{X}, X)$ is a smooth manifold with boundary which is foliated by complex manifolds with boundaries in X. In general however, $\widehat{X}$ is not so nice and may not contain any complex manifolds, or even continuous ones. Nevertheless the perception persists that the pair $(\widehat{X}, X)$ behaves like a manifold with boundary. This is the motivation for what follows. To adapt the above data on linking to this context we replace (V, X) with $(\widehat{X}, X)$ where now X is an arbitrary compact subset of M. As before Y is an oriented manifold disjoint from X and homologous to zero in M. Then, when X and Y are linked in an appropriate sense, the previous consequence that V and Y have a nonzero intersection number will be

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replaced by the cruder statement that $\widehat{X}$ and Y have a nonempty intersection. To adapt the hypothesis of the manifolds X and Y being linked to the setting in which X is an arbitrary compact set it suffices to require that Y not be homologous to zero in $M\backslash X$; when X is a manifold as above this is equivalent to $\operatorname{link}(X, Y)$ being nonzero.

Theorem 1. *Let M be a Stein manifold of (complex) dimension n and X a compact subset. Let Y be a compact oriented submanifold of M of (real) dimension k, disjoint from X, and homologous to zero in M. Suppose that Y is not homologous to zero in $M\backslash X$. Suppose that either*

(a) $0 \leq k < n-1$, *or*

(b) $k = n-1$ *and* $H^n(M, \mathbf{C}) = 0$.

Then $\widehat{X}$ has a nonempty intersection with Y.

Remarks. 1. Suppose that X and Y are now linked manifolds in M of dimensions a and k, respectively. Then, as $a+k = 2n-1$, the smaller of a and k is at most $n-1$. Hence the hull of the set corresponding to the smaller of a and k has a nonempty intersection with the other set, unless, in case (b), the smaller is $n-1$ and $H^n(M, \mathbf{C}) \neq 0$.

2. The cohomology condition in (b) is needed. Consider for M the product in $\mathbf{C}^n$ of n copies of C^*, the punctured plane. Let X be the n-torus in M, i.e., the product of n unit circles. Choose Y as a $k = n-1$ sphere in M disjoint from X and such that X and Y are linked in M; for example, Y could be a small sphere in the normal space to X at some point. Then, as $\widehat{X} = X$, the intersection of $\widehat{X}$ and Y is empty. Of course, $H^n(M, \mathbf{C}) \neq 0$.

Corollary 1. *Suppose that $\mathbf{C}^n = S \oplus T$ is an orthogonal decomposition of $\mathbf{C}^n$ into real linear spaces S and T of real dimension s and k respectively with $s > n$ and let $\pi: \mathbf{C}^n \to S$ be the orthogonal projection to S. Let E be a compact subset of S and let $f: E \to T$ be a continuous map and let $\operatorname{Gr}(f)$ be the graph of f in $\mathbf{C}^n$. Let D be a relatively compact component of the complement of E in S. Then $\widehat{\operatorname{Gr}(f)}$, the polynomially convex hull of $\operatorname{Gr}(f)$, covers D, i.e.,*

$$\pi(\widehat{\operatorname{Gr}(f)}) \supseteq D.$$

The special case of the corollary when S is complex linear and D is a ball appeared in [3] with two proofs and a third proof was given by Ahern and Rudin [1]. The second proof in [3], due to J.-P. Rosay, is closest to the methods of this paper. The case $n = 2$ and $s = 3$ where f is a real-valued function on a 2-manifold is of interest. When D is convex with smooth boundary, a very precise description of the hull is due to Bedford and Klingenberg [7]: the hull is a disjoint union of analytic disks. In other cases, the structure of the hull is less well understood, as, for example, when D is a solid torus.

Another phenomenon of linking is the relationship of linking at the boundary of a domain to intersections in the domain. The prototype of such results is the following. Cf. [10, Proposition, p. 383].

Proposition. *Let (V, X) and (W, Y) be oriented submanifolds with boundary in $\mathbf{R}^n$ such that V and W are contained in the open unit ball B and such that their boundaries are contained in the unit sphere bB. Suppose that X and Y are disjoint and that V and W intersect transversally, if at all. Then*

$$I(V, W) = \operatorname{link}(X, Y; bB).$$

Remarks. We are assuming that the linking number is defined. This means that $\dim(V) + \dim(W) = n$. Here $I(V, W)$ denotes the (signed) intersection number of V and W. In the case that V and W are complex manifolds in $\mathbf{C}^n$ with their natural orientations, then the intersection number is just the number of points in the intersection. For example, if V and W are complex linear spaces of complex dimension n meeting transversally at the origin in $\mathbf{C}^{2n}$, it follows that their boundaries X and Y, which are disjoint $2n-1$ spheres in the boundary of the unit ball, satisfy $\operatorname{link}(X, Y; S^{4n-1}) = 1$. With $n = 1$, this fact is used in the standard computations of the Hopf invariant of the Hopf fibration (see [8, pp. 235–239]).

The following is the statement corresponding to the proposition in the case when X is an arbitrary compact set in a Stein manifold and with V replaced by a holomorphic hull of X.

Theorem 2. *Let M be a Stein manifold of complex dimension at least 2, and D a smoothly bounded relatively compact strictly pseudoconvex domain in M. Let X be a compact subset of bD. Let Y be a k-dimensional compact oriented smooth submanifold of bD with $0 \le k \le n-2$ which is homologous to zero in bD and which is disjoint from X, i.e., $Y \subseteq G :\equiv bD \backslash X$, and suppose that there is a $(k+1)$-dimensional submanifold W of D such that $Y = bW$. Let $\widehat{X}$ be the $\mathscr{O}_{\overline{D}}$ hull of X. Suppose that Y links X in bD in the sense that Y is not homologous to zero in G. Then $\widehat{X}$ has a nonempty intersection with W.*

As a consequence we obtain the following corollary originally obtained by the author with E. L. Stout [4] by a different method, extending the Euclidean space case of [2]; also see [6]. The corollary was also proved by Lupaccioulu [9] who obtained more general results related to Theorem 2 in the case of pseudoconcave manifolds. Our approach is perhaps more geometric. With more elaborate hypotheses, the strict pseudoconvexity of D in Theorem 2 could be relaxed.

Corollary 2. *Let M, D, X and $\widehat{X}$ be as in Theorem 2. Each component of $D\backslash\widehat{X}$ contains in its boundary exactly one component of $bD\backslash X$.*

Proof. Without loss of generality we can suppose that D is connected. Then bD is connected, since D is Stein and $n \geq 2$. It suffices to prove the following. If p and q are points in distinct components of $bD\backslash X$ and if W is a simple smooth curve in D joining p to q, then W has a nonempty intersection with $\widehat{X}$. Let Y be $bW = \{q, -p\}$, a 0-dimensional submanifold of dD. The connectedness of bD implies that Y is homologous to zero in bD. Since p and q lie in different components of $bD\backslash X$, Y is not homologous to 0 in $bD\backslash X$. Thus we can apply Theorem 2 to conclude that $\widehat{X}$ meets W.

2. Proof of Corollary 1

Set $X = \mathrm{Gr}(f)$. We argue by contradiction and suppose that there exists $p \in \pi(\widehat{X})\backslash D$. Set $Q = \pi^{-1}(\{p\})$, a real k-plane in $\mathbf{C}^n$. Then $\widehat{X} \cap Q$ is empty. Hence $\widehat{X} \cap Y$ is empty for all geometric k-spheres Y in $\mathbf{C}^n$ of sufficiently large radius R, which are tangent to Q at $(p, 0) \in S \times T = \mathbf{C}^n$. It is evident and straightforward to check that Y "links" X, i.e., Y does not bound in $\mathbf{C}^n\backslash X$, if R is sufficiently large. Since $k = 2n - s < n$, Theorem 1 implies that $\widehat{X}$ meets Y. Contradiction.

3. Poincaré duals and linking

We next recall some of the basic facts needed about Poincaré duals and linking. A very nice reference for all of this is the book of Bott and Tu [8]. Our manifolds will be smooth and oriented; for such a manifold M the qth de Rham cohomology group will be denoted by $H^q(M)$, and the de Rham cohomology with compact support by $H_c^q(M)$. For a noncompact oriented manifold M of dimension m, Poincaré duality states that

$$H^k(M) = (H_c^{m-k}(M))^*,$$

and also, if M is of finite type,

$$(H^k(M))^* = H_c^{m-k}(M).$$

If Y is a closed oriented submanifold of M of dimension k, then its Poincaré dual is a closed $m-k$ form η_Y on M with the property that

$$(*) \qquad \int_Y \alpha = \int_M \alpha \wedge \eta_Y$$

for all closed k forms α with compact support in M. Sometimes to avoid ambiguity we denote the Poincaré dual by η_Y^M. The form is not uniquely determined, but its cohomology class $[\eta_Y] \in H^{m-k}(M)$ is unique and is also referred to as the Poincaré dual.

Three basic properties of the Poincaré duals are:

(i) Localization. For any tubular neighborhood of Y in M there is a Poincaré dual η_Y with support in that neighborhood.

(ii) If the oriented submanifolds Y and W of M meet transversally, then

$$\eta_Y \wedge \eta_W = \eta_{Y \cap W}.$$

(iii) If $f: M' \to M$ is an orientation-preserving map, and Y is an oriented submanifold of M, then, assuming appropriate transversality,

$$f^*(\eta_Y) = \eta_{f^{-1}(Y)}.$$

In particular, if A and Y are oriented submanifolds of M intersecting transversally, and f is an inclusion map $i: A \hookrightarrow M$, then (iii) gives

$$\eta_Y|_A = i^*(\eta_Y) = \eta^A_{A \cap Y}.$$

Let Y be a compact oriented submanifold of M. By localization, we can take η_Y with compact support in M. We can then ask whether $(*)$ remains valid if we drop the hypothesis that α have compact support in M. By Poincaré duality, this is so, provided that M has finite type. However, even if M does not have finite type, we can find a particular η_Y such that

$$(**) \qquad \int_Y \alpha = \int_M \alpha \wedge \eta_Y \quad \text{for all closed } k\text{-forms } \alpha \text{ on } M.$$

To see this we choose a tubular neighborhood N of Y in M. Then N is of finite type and so there is a "compact Poincaré dual" (see [8, p. 51]) η'^N_Y of Y in N such that

$$\int_Y \beta = \int_N \beta \wedge \eta'^N_Y$$

for all closed k forms β in N; $\eta_Y'^N$ is a closed $(m-k)$-form with compact support in N. Now define η_Y as the extension to M of $\eta_Y'^N$ by 0 outside of N. Then for any closed k-form α on M we have

$$\int_Y \alpha = \int_N \alpha \wedge \eta_Y'^N = \int_M \alpha \wedge \eta_Y.$$

Thus $(**)$ holds.

Suppose furthermore that Y is homologous to zero in M and let η_Y be chosen so that $(**)$ holds. Then we claim that $[\eta_Y] = 0$ in $H_c^{m-k}(M)$. By Poincaré duality it suffices to show that

$$\int_M \alpha \wedge \eta_Y = 0$$

for all closed forms α on M. This follows from $(**)$ because the integral over Y is zero by Stokes' theorem, since Y is homologous to zero in M. Thus there exists a $(m-k-1)$-form ω_Y with compact support in M such that $\eta_Y = d\omega_Y$.

Suppose that X and Y are disjoint oriented compact submanifolds of M, which are homologous to zero and satisfying $s+k = m-1$ with dimensions s and k respectively. Then $\operatorname{link}(X, Y)$ is defined and can be computed as follows. Choose η_X and η_Y with compact and disjoint supports. By the last paragraph we have ω_X with compact support in M such that $d\omega_X = \eta_X$. Then

$$\operatorname{link}(X, Y) = \int_M \omega_X \wedge \eta_Y.$$

4. Proof of Theorem 1

We argue by contradiction and suppose that $\widehat{X}$ is disjoint from Y. Then there exists a relatively compact $\mathscr{O}_M$-convex domain Ω in M containing $\widehat{X}$ and such that $\overline{\Omega}$ is disjoint from Y. Let η_Y be a Poincaré dual of Y in $M\backslash X$ such that $\operatorname{spt}(\eta_Y)$ is disjoint from $\overline{\Omega}$ and $(**)$ holds for k-forms α in $M\backslash X$. Extending by 0 we can view η_Y as a closed form in M. As in §3, since Y is homologous to zero in M, there exists a $(2n-k-1)$-form ω_Y with compact support in M such that $d\omega_Y = \eta_Y$. Let D_1 be a relatively compact subdomain in M containing $\overline{\Omega} \cup \operatorname{spt}(\omega_Y)$ such that bD_1 is smooth. Choose a relatively compact subdomain D_2 of

Ω such that X is contained in D_2 and bD_2 is smooth. Set $D = D_1\backslash\overline{D}_2$. Then $\operatorname{spt}(\eta_Y) \subseteq D$ and $bD = bD_1 \cup (-bD_2)$. As Y is not homologous to zero in $M\backslash X$ there exists, by de Rham's theorem, a closed k-form α on $M\backslash X$ such that $0 \neq \int_Y \alpha$.

We have

$$\begin{aligned} 0 \neq \int_Y \alpha &= \int_{M\backslash X} \alpha \wedge \eta_Y \quad \text{(by (**))} \\ &= \int_D \alpha \wedge \eta_Y = \int_D \alpha \wedge d\omega_Y \\ &= (-1)^k \int_D d(\alpha \wedge \omega_Y) = (-1)^k \int_{bD} \alpha \wedge \omega_y \quad \text{(Stokes)} \\ &= (-1)^k \int_{bD_1} \alpha \wedge \omega_Y + (-1)^k \int_{-bD_2} \alpha \wedge \omega_Y \\ &= (-1)^k \int_{-bD_2} \alpha \wedge \omega_Y \quad (\operatorname{spt}(\omega_Y) \cap bD_1 = \varnothing). \end{aligned}$$

Now in case (a), $k < n-1$ and so $2n-k-1 > n$. Hence $H^{2n-k-1}(\Omega) = 0$ since Ω is Stein [5]. On Ω, $d\omega_Y = \eta_Y = 0$. Hence there is a $(n-k-2)$-form σ on Ω such that $\omega_Y = d\sigma$ on Ω. Thus

$$\int_{-bD_2} \alpha \wedge \omega_Y = \int_{-bD_2} \alpha \wedge d\sigma = (-1)^k \int_{bD_2} d(\alpha \wedge \sigma) = 0$$

by Stokes. This contradicts the choice of α.

In case (b), $2n - k - 1 = n$. Since (M, Ω) is a Runge pair, it follows from [5] that the natural restriction map $H^n(M) \to H^n(\Omega)$ is surjective. As $H^n(M) = 0$, we have $H^n(\Omega) = 0$, and the argument of case (a) can be applied to arrive at the same contradiction.

5. Proof of the Proposition

Extend V and W to a neighborhood N of $\overline{B}$ and choose Poincaré duals η_V and η_W in N such that $\operatorname{spt}(\eta_V) \cap \operatorname{spt}(\eta_W)$ is a compact subset of B. Then $\eta_V \wedge \eta_W = \eta^B_{V\cap W}$ is a Poincaré dual of $V \cap W$. In particular, $\int_B \eta^B_{V\cap W} = I(V, W)$.

Let $j: bB \to N$ be the inclusion map. We may assume that N is a ball. Hence there exists an $(n-k-1)$-form ω_V in N such that $d\omega_V = \eta_V$. Set $\eta^{bB}_X = j^*(\eta_V)$ and $\eta^{bB}_Y = j^*(\eta_W)$. These are Poincaré duals on bB

with disjoint supports. Set $\omega_X^{bB} = j^*(\omega_V)$. Then on bB, we have

$$d(\omega_X^{bB}) = d(j^*(\omega_V)) = j^*(d\omega_V) = j^*(\eta_V) = \eta_X^{bB}.$$

Thus

$$\begin{aligned}
\operatorname{link}(X, Y; bB) &= \int_{bB} \omega_X^{bB} \wedge \eta_Y^{bB} \\
&= \int_{bB} j^*(\omega_V) \wedge j^*(\eta_W) \\
&= \int_{bB} j^*(\omega_V \wedge \eta_W) = \int_{bB} \omega_V \wedge \eta_W \\
&= \int_B d(\omega_V \cap \eta_W) \quad \text{(Stokes)} \\
&= \int_B d\omega_V \wedge \eta_W \quad (\eta_W \text{ is closed}) \\
&= \int_B \eta_V \wedge \eta_W = \int_B \eta_{V\cap W} \\
&= I(V, W).
\end{aligned}$$

6. Proof of Theorem 2

By replacing M by an appropriate Stein neighborhood of $\overline{D}$ in M we can assume that (M, D) is a Runge pair, that $\widehat{X}$ is the $\mathscr{O}_M$-convex hull of X and that W extends to be a submanifold of M which intersects bD transversally in Y.

We argue by contradiction and suppose that $\widehat{X}$ is disjoint from W. Then there is a relatively compact $\mathscr{O}_M$ convex domain Ω containing $\widehat{X}$ such that $\overline{\Omega}$ is disjoint from W. Let η_W be a Poincaré dual on M with support disjoint from $\overline{\Omega}$. Since $2n - k - 1 > n$ and M is Stein, $H^{2n-k-1}(M) = 0$. Hence there exists a $(2n - k - 2)$-form ω_W on M such that $d\omega_W = \eta_W$. (η_W is a closed $(2n - (k + 1))$-form on M.)

Let $j: G \to M$ be the inclusion map. Set $\eta_Y^G = j^*(\eta_W)$ and $\omega_Y^G = j^*(\omega_W)$. Then η_Y^G is a Poincaré of Y in G with compact support in G such that (**) holds on G, at least if we choose the support of η_W close to Y.

As Y does not bound in G there exists a closed k-form α on G such that $\int_Y \alpha \neq 0$, by de Rham.

Choose a relatively compact domain E_1 of $\Omega \cap bD$ such that bE_1 is smooth and $X \subseteq E_1$. Set $E = bD \backslash E_1$. Then $Y \subseteq E \subseteq G$ and

$bE = -bE_1 \subseteq \Omega \cap G$. Thus we have

$$\begin{aligned} 0 \neq \int_Y \alpha &= \int_E \alpha \wedge \eta_Y^G \quad (\text{by } (**);\ \operatorname{spt}(\eta_Y^G) \subseteq E) \\ &= \int_E \alpha \wedge d\omega_Y^G = (-1)^k \int_E d(\alpha \wedge \omega_Y^G) \\ &= (-1)^k \int_{bE} \alpha \wedge \omega_Y^G \quad (\text{Stokes}). \end{aligned}$$

We now consider two cases. First suppose $k < n-2$. Then $2n-k-2 > n$ and therefore $H^{2n-k-2}(\Omega) = 0$, as Ω is Stein. Since $d\omega_Y = \eta_Y = 0$ on Ω, there exists a $(2n-k-3)$-form σ on Ω such that $d\sigma = \omega_Y$ on Ω. Set the inclusion map $i: \Omega \cap bD \to \Omega$ and set $\sigma' = i^*(\sigma)$. Then, on bE, $d\sigma' = i^*(\omega_Y) = \omega_Y^G$ and so

$$\begin{aligned} \int_{bE} \alpha \wedge \omega_Y^G &= \int_{bE} \alpha \wedge d\sigma' \\ &= (-1)^k \int_{bE} d(\alpha \wedge \sigma') = 0 \quad (\text{Stokes}), \end{aligned}$$

this contradicts the choice of α.

In the second case $k = n-2$ and $2n-k-2 = n$. Since (M, Ω) is a Runge pair, the natural restriction $H^n(M) \to H^n(\Omega)$ is surjective [5]. Since ω_W is closed on Ω, we conclude there exists a closed n-form ϕ on M and an $(n-1)$-form θ on Ω such that

$$\omega_W = \phi + d\theta$$

on Ω. Hence

$$\begin{aligned} \int_{bE} \alpha \wedge \omega_Y^G &= \int_{bE} \alpha \wedge \phi + \int_{bE} \alpha \wedge d\theta \\ &= \int_E d(\alpha \wedge \phi) + (-1)^k \int_{bE} d(\alpha \wedge \theta) \end{aligned}$$

by Stokes' theorem. Again by Stokes the last integral vanishes. Also the integral over E vanishes since $\alpha \wedge \phi$ is closed because α and ϕ are closed (and defined on E). This again contradicts the choice of α and completes the proof.

References

[1] P. Ahern & W. Rudin, *Hulls of 3-spheres in* C^3, Contemporary Math., Vol. 137, Amer. Math. Soc., Providence, RI, 1992, 1–28.

[2] H. Alexander, *A note on polynomial hulls*, Proc. Amer. Math. Soc. **33** (1972) 389–391.

[3] ——, *Polynomial hulls of graphs*, Pacific J. Math. **147** (1991) 201–212.

[4] H. Alexander & E. L. Stout, *A note on hulls*, Bull. London Math. Soc. **22** (1990) 258–260.

[5] A. Andreotti & R. Narasimhan, *A topological property of Runge pairs*, Ann. of Math. (2) **76** (1962) 499–509.

[6] R. F. Basener, *Complementary components of polynomials hulls*, Proc. Amer. Math. Math. Soc. **69** (1978) 230–232.

[7] E. Bedford & W. Klingenberg, *On the envelope of holomorphy of a 2-sphere in* C^2, J. Amer. Math. Soc. **4** (1991) 623–646.

[8] R. Bott & L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Math., Vol. 52, Springer, New York, 1982.

[9] G. Lupacciolu, *Topological properties of* q*-convex sets*, Trans. Amer. Math. Soc., to appear.

[10] W. Fulton, *Intersection theory* Ergeb. Math. Grenzgeb., Vol. 2, Springer, New York, 1984.

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