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A UNIVERSAL CONSTRAINT FOR DONALDSON INVARIANTS

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Abstract

We prove a mod 2 universal constraint and then derive a structure theorem for the SU(2) Donaldson invariants of odd second Chern class on even manifolds.

1. Introduction and the main theorem

Donaldson [5] uses Yang-Mills gauge theory to define infinitely many invariants on smooth 4-manifolds. These invariants are extremely important for studying differentiable structures on 4-manifolds. Because they are difficult to calculate, they remain mysterious. An important open question posed by Donaldson is whether there are any universal relations or constraints on the invariants. This paper is a first step in answering this question. All the Donaldson invariants in this paper are invariants for SU(2).

If ϕ^p and ψ^q are multilinear symmetry functions of degree p and q, define their *symmetric product* as follows:

$$(\phi \circ \psi)^{p+q}(x_1, \cdots, x_{p+q}) = \sum_{i_1<\cdots<i_p} \phi(x_{i_1}, \cdots, x_{i_p})\psi(x_1, \cdots, \widehat{x_{i_1}}, \cdots, \widehat{x_{i_p}}, \cdots, x_{p+q}),$$

where the caret means to omit this term. All addition is done mod 2 in this paper. If, for example, ϕ is of degree 2, then $\phi \circ \phi \equiv 0 \bmod 2$, since the terms cancel in pairs (for example $\phi(x_1, x_2)\phi(x_3, x_4)$ and $\phi(x_3, x_4)\phi(x_1, x_2)$).

Main theorem. *Let X^4 be a simply connected spin manifold with intersection form q and $b_2^+(X) > 1$ odd. If the second Chern number*

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$k > \frac{3}{4}(1 + b_2^+(X))$ *and is odd, then the Donaldson invariants* Φ_k *satisfy*

$$q \circ \Phi_k(\alpha_1, \cdots, \alpha_n) = \sum_{i<j} q(\alpha_i, \alpha_j)\Phi_k(\alpha_1, \cdots, \widehat{\alpha_i}, \cdots, \widehat{\alpha_j}, \cdots, \alpha_n) \equiv 0 \pmod 2.$$

As an application, we can prove a strong structure theorem for even degree Donaldson invariants:

Structure theorem. *Under the assumptions of the main theorem, for any even degree Donaldson polynomial invariant* Φ,

$$\Phi \equiv q \circ H \pmod 2$$

for a symmetric function H *of degree lower by* 2.

This structure theorem implies that many cases of mod 2 Donaldson invariants vanish. For example, under the hypotheses of the structure theorem,

$$\Phi(\alpha, \alpha, \cdots, \alpha) \equiv 0 \pmod 2$$

for any 2-dimensional homology class α.

Note that for algebraic surfaces, Donaldson shows that for large k,

$$\Phi_k(\alpha, \alpha, \cdots, \alpha) \neq 0$$

for the Kähler class α. By our theorem, $\Phi_k(\alpha, \alpha, \cdots, \alpha)$ is even for k odd and $4k - \frac{3}{2}(1 + b_2^+)$ even. The evenness of the degree of the Donaldson invariant depends on b_2^+. If $b_2^+ = 4m + 3$, for example in the case of the Kummer surface, then the degree of the Donaldson invariant is always even. In §3 we will discuss other cases where the invariants vanish. It appears that this universal constraint severely diminishes the value of mod 2 Donaldson invariants. In an upcoming paper of Fintushel and Stern, by combining their own techniques with this universal relation, they are able to show that a lot of mod 2 Donaldson invariants vanish [8].

This paper is based on the work of Donaldson [3] in an essential way and may be regarded as a partial extension of it. In fact, as was pointed out in [3], there is a gap between the situation $b_2^+ = 0, 1, 2$ and the situation $b_2^+ \geq 3$. In the first case it is possible to get constraints on the homology of the manifold as in [3]; in the second case, however, no such direct information seems available. But we can still extract some information about Donaldson's invariants. We note that Donaldson suggested this approach in his paper [3].

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2. Proof of the main theorem

Let X be a compact, connected, oriented 4-manifold, $P \to X$ a principal SU(2)-bundle, and E and g_P the vector bundles associated to P by the fundamental representation on $\mathbb{C}^2$ and the adjoint representation, respectively. Let $\mathscr{A}$ be the space of connections on P, and let $\mathscr{G}$ be the group of gauge transformations such that

$$\mathscr{B} = \mathscr{A}/\mathscr{G}(\mathscr{B}(X))$$

is an infinite-dimensional space of gauge equivalence classes of connections on P. Let $\mathscr{G}_0$ be the group of gauge transformations which act as the identity on the fiber of P over the base point x_0. Then $\mathscr{G}_0$ acts freely on $\mathscr{A}$. Let $\widetilde{\mathscr{B}} = \mathscr{A}/\mathscr{G}_0$. We can form the universal bundle:

$$\begin{array}{c} \mathscr{A} \times_{\mathscr{G}_0} P \quad (= \widetilde{\mathbb{P}}) \\ \downarrow \\ \widetilde{\mathscr{B}} \times X \end{array}$$

Denote by $\tilde{\mu}\colon H_2(X, \mathbb{Z}) \to H^2(\widetilde{\mathscr{B}}, \mathbb{Z})$ the slant product $\tilde{\mu}(\alpha) = c_2(\widetilde{\mathbb{P}})/\alpha$, where $c_2(\widetilde{\mathbb{P}})$ is the usual second Chern class in $H^4(\widetilde{\mathscr{B}} \times X, \mathbb{Z})$. By choosing a specific geometric representation of $\tilde{\mu}$, Donaldson shows that $\tilde{\mu}(\alpha)$ descends to $\mu(\alpha)$ on $\mathscr{B}^* = \mathscr{A}^*/\mathscr{G}$, where $\mathscr{A}^*$ is the space of irreducible connections. There is a geometric way to describe this map which is important for us. We quickly sketch Donaldson's construction.

For $\alpha \in H_2(X, \mathbb{Z})$, choose a surface Σ representing α. There is a restriction map $r_\Sigma\colon \widetilde{\mathscr{B}}(X) \to \widetilde{\mathscr{B}}(\Sigma)$. Now Σ carries a spin structure, so we have a twisted Dirac operator ∂_Σ Over Σ which has numerical index 0, and the class

$$\operatorname{ind}(\partial_\Sigma) \in K(\widetilde{\mathscr{B}}(\Sigma)).$$

Any complex virtual bundle defines a complex line bundle, the *determinant line bundle*,

$$\det([V] - [W]) = (\wedge^{\dim V} V) \otimes_{\mathbb{C}} (\wedge^{\dim W} W)^*.$$

Thus for every surface Σ in X we get a complex line bundle $L_\Sigma = (\det \operatorname{ind} \partial_\Sigma)^{-1}$ over $\widetilde{\mathscr{B}}(\Sigma)$. Then it can be proved that $\tilde{\mu}(\alpha) = c_1(L_\Sigma)$.

Note that $\widetilde{\mathscr{B}}^* = \mathscr{A}^*/\mathscr{G}_0 \to \mathscr{B}^*$ is a principal SO(3)-bundle. Donaldson showed that we can push L_Σ down to $\mathscr{B}^*(\Sigma)$. An index theorem argument shows that L_Σ can be extended over the zero divisor. In other words, L_Σ is defined on $\mathscr{B}$-{nonzero reductions}. The restriction of an irreducible connection may not be irreducible, but this can be taken care of by working on a tubular neighborhood N of Σ instead of Σ itself [5]. Then we can suppose r_{N_Σ} maps the moduli space $\mathscr{M}$ of irreducible anti-self-dual connections to irreducible connections, i.e.,

$$r_N(\mathscr{M}_l) \subset \mathscr{B}^*(N) \quad \text{for } l \leq k.$$

Now we are in a position to give the definition of Donaldson's polynomial invariant. For any k, the moduli space $\mathscr{M}_k$ for the bundle with $c_2 = k$ is a finite-dimensional manifold; $\dim \mathscr{M}_k = 8k - 3(1 + b_2^+)$. For b_2^+ odd, $\dim \mathscr{M}_k = 2b$ for $b = 4k - \frac{3}{2}(1 + b_2^+)$. For any $\alpha_1, \cdots, \alpha_b \in H_2(X, \mathbb{Z})$, choose $\Sigma_1, \cdots, \Sigma_b$ in general position representing $\alpha_1, \cdots, \alpha_b$. Furthermore, we can choose the small tubular neighborhood N_i of Σ_i in general position, i.e., $N_i \cap N_j \cap N_k = \emptyset$ for distinct i, j, k, and $N_i \cap N_j$ is exactly a tubular neighborhood of points $\Sigma_i \cap \Sigma_j$. Then we have complex line bundles L_{Σ_i} over $\mathscr{B}_{N_i}$-{nonzero reductions}. Choose sections s_{Σ_i} of L_{Σ_i} such that the divisors V_{Σ_i} are transverse to each other and to $\mathscr{M}_l$ for $l \leq k$. In particular, the trivial connection θ is a zero reduction, so we can suppose $s_{\Sigma_i}([\theta]) \neq 0$. Consider the zero-dimensional manifold $V_1 \cap \cdots \cap V_b \cap \mathscr{M}_k$. By Uhlenbeck's compactness theorem, there is a natural compactification of $\mathscr{M}_k$ as

$$\overline{\mathscr{M}_k} \subset \mathscr{M}_k \cup \cdots \cup \mathscr{M}_{k-i} \times S^i(X) \cup \cdots \cup S^k(X).$$

From our choice of section s_{Σ_i}, if there is a sequence $[A_n] \in V_{\Sigma_i}$ weakly convergent to $([A], x_1, \cdots, x_n)$, then either some $x_j \in N_i$ or $[A] \in V_{\Sigma_i}$. A dimension counting argument shows that if $k > \frac{3}{4}(1 + b_2^+)$, then $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap \mathscr{M}_k$ is compact, hence finite. Counting these points, using the orientation on $\mathscr{M}_k$, we define the Donaldson polynomial invariant

$$\Phi_k(\alpha_1, \cdots, \alpha_b) = \#(V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap \mathscr{M}_k).$$

Donaldson shows that Phi_k is independent of the choice of metric, and hence is an invariant of the smooth structure of X.

From now on assume $k > \frac{3}{4}(1 + b_2^+)$, and $b_2^+ > 1$ is odd. Then the Donaldson invariant is well defined. Consider the moduli space $\mathscr{M}_{k+1}$. It has dimension

$$\dim \mathscr{M}_{k+1} = 8k - 3(1 + b_2^+) + 8.$$

Let $b = 4k - \frac{3}{2}(1 + b_2^+) + 2$. For any $\alpha_1, \cdots, \alpha_b \in H_2(X, \mathbb{Z})$, choose representing surfaces $\Sigma_1, \cdots, \Sigma_b$ in general position and such that their tubular neighborhoods $N_1, \cdots, N_b$ are in general position as well. As above, consider

$$N = V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap \mathscr{M}_{k+1}.$$

Then $\dim N = 4$, and a dimension counting argument shows that $\overline{N} \subset \overline{\mathscr{M}_{k+1}}$ can only touch the first intermediate stratum $\mathscr{M}_k \times X$ at $([A], x)$ for

$$(*) \qquad x \in N_i \cap N_j, \qquad [A] \in V_{\Sigma_1} \cap \cdots \cap \widehat{V_{\Sigma_i}} \cap \cdots \cap \widehat{V_{\Sigma_j}} \cap \cdots \cap V_{\Sigma_b}.$$

Note that

$$[A] \in V_{\Sigma_1} \cap \cdots \cap \widehat{V_{\Sigma_i}} \cap \cdots \cap \widehat{V_{\Sigma_j}} \cap \cdots \cap V_{\Sigma_b} \cap \mathscr{M}_k,$$

and its dimension is zero. Since we will only consider the mod 2 invariants, we will ignore the orientation in the rest of the argument. Therefore,

$$\begin{aligned}\#(V_{\Sigma_1} \cap \cdots \cap \widehat{V_{\Sigma_i}} \cap \cdots \cap \widehat{V_{\Sigma_j}} \cap \cdots \cap V_{\Sigma_b} \cap \mathscr{M}_k) \\ = \Phi_k(\alpha_1, \cdots, \widehat{\alpha_i}, \cdots, \widehat{\alpha_j}, \cdots, \alpha_b).\end{aligned}$$

On the other hand, $N_i \cap N_j$ is just a small neighborhood of the points of $\Sigma_i \cap \Sigma_j$. Its number of components is measures by $q(\alpha_i, \alpha_j)$. Hence the number of ends of the moduli space is

$$\sum_{i<j} q(\alpha_i, \alpha_j)\Phi_k(\alpha_1, \cdots, \widehat{\alpha_i}, \cdots, \widehat{\alpha_j}, \cdots, \alpha_n) = q \circ \Phi_k(\alpha_1, \cdots, \alpha_b).$$

For the readers familiar with Donaldson's work [3], our case is exactly the case causing the trouble in [3]. For the case of $b_2^+ = 1, 2$, Donaldson cut the moduli space in a way to avoid all the intermediate strata. Then the intersection with the bottom stratum will give a constraint on the homology of the base manifold. If $b_2^+ \geq 3$, the submanifold sliced out by divisors in [3] must meet the first intermediate stratum.

The idea in this paper is to intersect enough divisors to cut out a 4-dimensional submanifold. Then it meets the first intermediate stratum in a discrete set of points. If k is large enough, it also avoids the bottom stratum. Therefore it only meets the first intermediate stratum and this intersection gives our constraint on the Donaldson invariants.

Next we study the behavior of N near pairs $([A], x])$ satisfying $(*)$. Let $([A_0], x)$ be such a pair.** Since every component of $N_i \cap N_j$ is a neighborhood of x for $x \in \Sigma_i \cap \Sigma_j$, it is equivalent to study the behavior near $([A_0], x)$ for $x \in \Sigma_i \cap \Sigma_j$. We will see that after suitable cutting of N near the ends we get a compact 4-manifold N' with boundary. The part of the boundary corresponding to each end is homologous to $SO(3)$. Here $SO(3)$ is regarded as a gluing parameter of $[A_0]$ with a standard instanton of S^4 at x.

By Taubes' work [12], except for the bottom stratum, $\overline{\mathscr{M}_{k+1}}$ is a stratified space with a local cone bundle structure, and the ith stratum is $\mathscr{M}_{k+1-i} \times S^i(X)$. In particular, there is a neighborhood of the first stratum $\mathscr{M}_k \times X$ which is a local cone bundle with fibre $=$ cone(SO(3)). Therefore there is a neighborhood $W_{([A_0],x)}$ of $([A_0], x)$ diffeomorphic to $W_{[A_0]} \times \Omega_x \times SO(3) \times (0, \lambda)$ and there is a well-defined cone projection $p: W_{([A_0],x)} \to W_{[A_0]} \times \Omega_x$, where $W_{[A_0]}$ is a neighborhood of $[A_0]$ in $\mathscr{M}_k$, and Ω_x is a neighborhood of $x \in X$ containing a component of $N_i \cap N_j$. Denote by $W^{\lambda}_{(x,[A_0])}$ the subset of $W_{(x,[A_0])}$ with fixed scalar λ, which is diffeomorphic to $W_{[A_0]} \times \Omega_x \times SO(3) \times \{\lambda\}$. Without loss of generality, let $x \in \Sigma_1 \cap \Sigma_2$ and $[A_0] \in V_{\Sigma_3} \cap \cdots \cap V_{\Sigma_b} \cap \mathscr{M}_k$. Choose Ω_x such that $\Omega_x \cap N_i = \varnothing$ for $i \geq 3$. Then for $i \geq 3$ and $[A] \in W_{(x,[A_0])}$, $[A]|_{N_i}$ is close to $p([A])|_{N_i}$. Hence $s_{\Sigma_i}([A])$ gets arbitrarily close to $s_{\Sigma_i}(p([A]))$ for small λ. Let $\bar{\nu}_i$ be a tubular neighborhood of V_{Σ_i} in $\mathscr{M}_k$ and let $\bar{v}'_i$ be the complement of a smaller tubular neighborhood. Then $s_{\Sigma_i}(\bar{\nu}'_i) \neq 0$, and $s_{\Sigma_i}(p^{-1}(\bar{\nu}'_i \times \Omega_x)) \neq 0$ for λ small. Then both s_{Σ_i} and $s_{\Sigma_i} \circ p$ define cohomology classes of $H^2(W_{(x,[A_0])}, p^{-1}(\bar{\nu}'_i \cap W_{[A_0]} \times \Omega_x))$, and they are the same since those two maps are close to each other. But clearly the class defined by $s_{\Sigma_i} \circ p$ is the pullback of the generator of $H^2(W_{[A_0]} \times \Omega_x, (\bar{\nu}'_i \cap W_{[A_0]}) \times \Omega_x)$.

For s_{Σ_1} and s_{Σ_2}, we quote a theorem of Donaldson. In this paragraph, $j = 1, 2$. First of all, $s_j([A_0]) \neq 0$, where $s_j = s_{\Sigma_j}$, so we can suppose $s_j(W_{[A_0]}) \neq 0$. Let ν_j be a tubular neighborhood of Σ_j in X and let $\nu'_j \subset \nu_j$ be the complement of a smaller closed tubular neighborhood. Without loss of generality, we can assume that $N_j \subset \nu_j$. Consider the set $T_j \subset \mathscr{M}_{k+1}$ consisting of pairs (x, A), where $x \in \nu_j$ and A is a connection satisfying the following two conditions:

(i) Away from x, A is close to an element of $W_{[A_0]}$;

(ii) $c_2(A) = k+1$, so that a small ball around x contributes $\cong 8\pi^2$ to the Chern-Weil integral.

Let $T'_j \subset T_j$ be the set corresponding to ν'_j. Donaldson showed in [6] that the projection $p: (T_j, T'_j) \to (\nu_j, \nu'_j)$ is a fibration and the class induced by s_j is the pullback of the generator of $H^2(\nu_j, \nu_j')$. Obviously we can choose small enough λ so that $W_{(x,[A_0])} \subset T_j$. Then s_j induces a class in $H^2(W_{(x,[A_0])}, W_{(x,[A_0])} \cap T_j')$ which is the pullback of the generator of $H^2(W_{[A_0]} \times \Omega_x, W_{[A_0]} \times \Omega_x \cap \nu')$.

It is easy to see that $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap p^{-1}(\partial(W_{[A_0]} \times \Omega_x)) = \varnothing$. Furthermore we can perturb s_i slightly such that $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b}$ is transverse to $W^\lambda_{(x,[A_0])}$. So if we cut N along $W^\lambda_{(x,[A_0])}$, we get a compact manifold with boundary $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap W^\lambda_{(x,[A_0])}$ near $(x, [A_0])$. Taking the intersection of V_{Σ_i} corresponds to taking the cup product with the dual cohomology class. The previous argument implies that $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap W_{(x,[A_0])}$ is dual to a cohomology class of $H^{2b}(W_{(x,[A_0])}, p^{-1}(\partial(W_{[A_0]} \times \Omega_x)))$ which is the pullback of the generator of $H^{2b}(W_{[A_0]} \times \Omega_x, \partial(W_{[A_0]} \times \Omega_x))$. If we fix a small λ, the same argument shows that $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap W^\lambda_{(x,[A_0])}$ is dual to the cohomology class of $H^{2b}(W^\lambda_{(x,[A_0])}, p_\lambda^{-1}(\partial(W_{[A_0]} \times \Omega_x)))$ which is the pullback of the generator of $H^{2b}(W_{[A_0]} \times \Omega_x, \partial(W_{[A_0]} \times \Omega_x))$. But the generator of $H^{2b}(W_{[A_0]} \times \Omega_x, \partial(W_{[A_0]} \times \Omega_x))$ is dual to $(x, [A_0])$ and hence $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_b} \cap W^\lambda_{(x,[A_0])}$ is homologous to $p^{-1}((x, [A_0])) = \mathrm{SO}(3)$ since $W^\lambda_{(x,[A_0])}$ is a compact manifold with boundary $p_\lambda^{-1}(\partial(W_{[A_0]} \times \Omega_x))$.

Next we want to show that the ends are homologically nonzero. Donaldson uses the spin structure of the manifolds to define certain mod 2 cohomology classes as the Steifel-Whitney classes of the index of a family of twisted Dirac operators. We simply state the results and refer the reader to Donaldson's paper for details.

The index bundle for c_2 even, written as Donaldson's notation, is $\det(\mathrm{ind}(D_A))$. An excision argument shows that over the gluing parameter $\mathrm{SO}(3)$, $\mathrm{ind}(D_A)$ is $m_0 + m_1\eta$, where η is a Hopf line bundle over $\mathrm{SO}(3)$. When we glue A_0 together with an instanton on S^4, $m_1 = 1$. In this case $\det(\mathrm{ind}(D_A)) = \eta$.

Let $k+1$ be even. Choose the first Steifel-Whitney class $u_1 = w_1(\det(\mathrm{ind}(D_A)))$. The argument above shows that $u_1|_{\mathrm{SO}(3)}$ is the generator of $H^*(\mathrm{SO}(3), \mathbb{Z}_2)$, and hence $u_1^3(\mathrm{SO}(3)) = 1$. Finally

$$q \circ \Phi_k(\alpha_1, \cdots, \alpha_b) = u_1^3(\partial N) \equiv 0 \pmod 2.$$

This finishes the proof.

Remark. The proof breaks down for the case k even because $u_1 = 0$. In fact, it has been shown that $\mathscr{B}^*$ is simply connected for c_2 odd corresponding to our case k even (see [11]). Hence for this case $H^1(\mathscr{B}^*, \mathbb{Z}) = 0$.

3. Structure of the mod 2 Donaldson invariant

In this section, we study the constraints imposed by the main theorem on the structure of the Donaldson invariants. We will make use of several basic facts from linear algebra over the field $\mathbb{Z}_2$. As the author could not find them in the literature, they are included here. Throughout this section, all operations are mod 2 unless we specify otherwise.

The key observation is the analogy between the symmetric product by the even intersection form and wedge product by the Kähler form for complex manifolds. It turns out that we can establish a partial theory in analogy with the Lefschetz decomposition of the Dolbeault cohomology. An interesting question is where the Donaldson invariants "lie" under this decomposition.

Let us recall that for simply connected even 4-manifolds, every intersection form is equivalent over $\mathbb{Z}$ to $tE_8 \oplus s\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular, the rank of $H_2(X, \mathbb{Z})$ is even. By Poincaré duality, q is a nondegenerate bilinear, symmetric, unimodular form. It is well known that q is equivalent to $n\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ over $\mathbb{Z}_2$. In other words, there is a basis $x_1, y_1, x_2, y_2, \cdots, x_n, y_n$ such that

$$q(x_i, y_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us use $e_1, \theta_1, \cdots, e_n, \theta_n$ to denote the dual basis. Over $\mathbb{Z}_2$, since symmetry is equivalent to skew-symmetry, we adopt the following notation:

Definition. $\phi \in \mathrm{Sym}^p(H_2^*(X))$ if $\phi(z_1, \cdots, z_p) = \phi(z_{i_1}, \cdots, z_{i_p})$ for every permutation $(i_1, \cdots, i_p)$ of $(1, 2, \cdots, p)$.

$\phi \in \bigwedge^p(H_2^*(X))$ will be called a p-form if $\phi \in \mathrm{Sym}^p$ and $\phi(z_1, \cdots, z_n) = 0$ whenever $z_i = z_j$ for some $i \neq j$. Clearly $e_i, \theta_i \in \mathrm{Sym}^1(H_2^*)$. Naturally we define the operation of symmetric product as in the introduction. Therefore, the intersection form q can be written as

$$q = \sum_{i=1}^{n} e_i \circ \theta_i.$$

One special feature of the mod 2 symmetric product is demonstrated in the following proposition.

Proposition. *For any* $\phi \in \mathrm{Sym}^p$, $\phi \circ \phi = 0$.

The proof follows easily from the fact that our addition is modulo 2.

It is easy to check that $\bigwedge^*$ is generated by

$$e_{i_1} \circ \cdots \circ e_{i_k} \circ \theta_{j_1} \cdots \circ \theta_{j_t},$$

where e_{i_k}, θ_{j_k} are mutually distinct. But the symmetric product of degree one symmetric functions does not generate all the symmetric functions, as demonstrated by the preceding proposition. In fact they only generate the forms. Thus we are led to make the following definition.

Definition. The power $e_i^k \in \mathrm{Sym}^k$ is the element of Sym^k which satisfies

$$e_i^k(z_1, \cdots, z_k) = \begin{cases} 1 & \text{if } z_1 = \cdots = z_k = x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We can define θ_i^k similarly and extend by the binomial formula. An easy calculation shows that

$$e_i \circ e_i^k = \begin{cases} e^{k+1} & \text{if } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Under this notation, Sym^p is generated by

$$e_{i_1}^{n_1} \circ \cdots \circ e_{i_k}^{n_k} \circ \theta_{j_1}^{m_1} \cdots \circ \theta_{j_t}^{m_t},$$

where e_{i_k}, θ_{j_k} are mutually distinct. When there is no confusion, we will omit the $\circ$ sign. Define

$$L_q\colon \mathrm{Sym}^p \to \mathrm{Sym}^{p+2}$$

to be $L_q(c) = q \circ (c) = \sum_{i=1}^n e_i \circ \theta_i \circ (c)$. One can observe that the L_q induce the following sequence of homomorphisms of vector spaces:

$$\mathbb{Z}_2 \xrightarrow{L_q} \mathrm{Sym}^2 \xrightarrow{L_q} \mathrm{Sym}^4 \xrightarrow{L_q} \cdots \xrightarrow{L_q} \cdots \to \mathrm{Sym}^{2m} \xrightarrow{L_q} \cdots, \tag{2}$$

$$\mathrm{Sym}^1 \xrightarrow{L_q} \mathrm{Sym}^3 \xrightarrow{L_q} \mathrm{Sym}^5 \xrightarrow{L_q} \cdots \xrightarrow{L_q} \cdots \to \mathrm{Sym}^{2m+1} \xrightarrow{L_q} \cdots. \tag{3}$$

By the proposition, $L_q \circ L_q = 0$. Hence $\mathrm{Im}\, L_q \subset \mathrm{Ker}\, L_q$. So we can define the cohomology $H^p = \mathrm{Ker}\, L_q / \mathrm{Im}\, L_q$. Note that our constraint on the Donaldson invariant just says that $\Phi_k \in \mathrm{Ker}\, L_q$ for k odd. The purpose of this section is to calculate the H^p.

Definition. For any $x \in H_2(X)$, we define the *contraction* i_x: $\mathrm{Sym}^{p+1} \to \mathrm{Sym}^p$ as $i_x\alpha(z_1, \cdots, z_p) = \alpha(x, z_1, \cdots, z_p)$.

Then we can define the *contraction by* q, I_q: $\mathrm{Sym}^{p+2} \to \mathrm{Sym}^p$, as $I_q = \sum_{i=1}^n i_{x_i} i_{y_i}$.

Lemma.

$$i_{x_i}L_{e_j} + L_{e_j}i_{x_i} = \delta_{ij}, \qquad i_{y_i}L_{\theta_j} + L_{\theta_j}i_{y_i} = \delta_{ij},$$
$$i_{x_i}L_{\theta_j} + L_{\theta_j}i_{x_i} = 0, \qquad i_{y_i}L_{e_j} + L_{e_j}i_{y_i} = 0.$$

Proof. We only prove the first equality, as the others follow similarly:

$$\begin{aligned} i_{x_i}L_{e_j}(\alpha)(z_1, \cdots, z_p) &= L_{e_j}(\alpha)(x_i, z_1, \cdots, z_p) \\ &= e_j(x_i) + \sum e_j(z_k)\alpha(x_i, z_1, \cdots, \tilde{z}_k, \cdots, z_p) \\ &= \delta_{ij} + L_{e_j}i_{x_i}(\alpha)(z_1, \cdots, z_p). \end{aligned}$$

Proposition. $[L_p, I_q] = n + p$ *as a homomorphism* $\mathrm{Sym}^p \to \mathrm{Sym}^p$.

Proof.

$$\begin{aligned} L_qI_q &= \sum_{i,j} L_{e_i}L_{\theta_i}i_{x_j}i_{y_j} = \sum_{i,j} L_{e_i}i_{x_j}L_{\theta_i}i_{y_j} \\ &= \sum_{i,j} i_{x_j}L_{e_i}L_{\theta_i}i_{y_j} + \sum_i L_{\theta_i}i_{y_i} \\ &= \sum_{i,j} i_{x_j}L_{e_i}i_{y_j}L_{\theta_i} + \sum_i L_{\theta_i}i_{y_i} + \sum_i i_{x_i}L_{e_i} \\ &= I_qL_q + n + \sum_i (L_{e_i}i_{x_i} + L_{\theta_i}i_{y_i}). \end{aligned}$$

Let $z_j = \sum_i (a_i^j x_i + b_i^j y_i)$. Then

$$\begin{aligned} \sum_i L_{e_i}i_{x_i}(\alpha)(z_1, \cdots, z_p) &= \sum_{i,j} e_i(z_j)\alpha(x_i, z_1, \cdots, \tilde{z}_j, \cdots, z_p) \\ &= \sum_{i,j} a_i^j \alpha(x_i, z_1, \cdots, \tilde{z}_j, \cdots, z_p). \end{aligned}$$

In the same way,

$$\begin{aligned} \sum_i L_{\theta_i}i_{y_i}(\alpha)(z_1, \cdots, z_p) &= \sum_{i,j} \theta_i(z_j)\alpha(y_i, z_1, \cdots, \tilde{z}_j, \cdots, z_p) \\ &= \sum_{i,j} b_i^j \alpha(y_i, z_1, \cdots, \tilde{z}_j, \cdots, z_p), \end{aligned}$$

and therefore

$$\begin{aligned}
&\sum_i (L_{e_i} i_{x_i} + L_{\theta_i} i_{y_i})(\alpha)(z_1, \cdots, z_p) \\
&\quad = \sum_{i,j} (\alpha(a_i^j x_i, z_1, \cdots, \tilde{z}_j, \cdots, z_p) + \alpha(b_i^j y_i, z_1, \cdots, \tilde{z}_j, \cdots, z_p)) \\
&\quad = \sum_j \alpha(z_j, z_1, \cdots, \tilde{z}_j, \cdots, z_p) = p\alpha(z_1, \cdots, z_p).
\end{aligned}$$

Hence $[L_q, I_q] = n + p$. When $n + p$ is odd, $[L_q, I_q] = 1$. Then $L_q I_q(\alpha) = I_q L_q(\alpha) + \alpha$. If $L_q(\alpha) = 0$, $\alpha = L_q I_q(\alpha)$. Therefore, $\alpha \in \operatorname{Im} L_q$, which implies the following theorem.

Theorem. *When $n + p$ is odd, $H^p = 0$.*

Structure theorem. *Under the assumption of the main theorem, for even degree Donaldson polynomial invariants Φ,*

$$\Phi \equiv q \circ H \pmod 2$$

for some symmetric function H of degree lower by 2.

Proof. For even manifolds, the intersection form is $tE_8 \oplus s\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. So the rank of $H_2(X) = 2n = 2(4t + s)$. Note that $b_2^+ = s$. Thus when b_2^+ is odd, n is also odd. Therefore if degree p is also even, $n + p$ is odd. By the theorem, $H^p = 0$. But from the main theorem $\Phi \in \operatorname{Ker} L_q$, which implies that $\Phi \in \operatorname{Im} L_q$. So $\Phi = q \circ H$. Moreover, $H = I_q(\Phi)$.

For degree p odd, H^p may be nonzero. But we still have $[L_q, I_q] = 0$. In fact we get a $\mathbb{Z}_2$-representation of the Lie algebra sl_2. In particular, it is easy to check that

$$I_q(\operatorname{Ker} L_q) \subset \operatorname{Ker} L_q, \qquad I_q(\operatorname{Im} L_q) \subset \operatorname{Im} L_q.$$

Therefore, I_q induces the maps

$$H^p \to H^{p-2} \to H^{p-4} \to \cdots.$$

For our case, we do not quite have the decomposition of H^p as the case of complex manifolds. Instead we have a filtration

$$(*) \qquad 0 \subset \operatorname{Ker} I_q \subset \operatorname{Ker} I_q^2 \subset \cdots \subset H^p.$$

Note that Φ induces an element $\widehat{\Phi} \in H^p$.

On the other hand, $q \in \bigwedge^2$. In fact, L_q induces a map

$$L_q' \colon \bigwedge^p \to \bigwedge^{p+2}.$$

Thus we obtain a subspace $K^p = \operatorname{Ker} L'_q / \operatorname{Im} L'_q \subset H^p$. Furthermore we have a similar filtration as $(*)$ for K^p.

The structure theorem implies that in many cases mod 2 Donaldson invariants vanish.

Corollary 2. *Under the assumption of the structure theorem, let* $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_k, \beta_k \in H_2(X)$ *satisfy* $q(\alpha_i, \beta_j) = 1$ *if* $i = j$, *and* 0 *otherwise. Then*

$$\Phi(\alpha_1^{t_1}, \beta_1^{s_1}, \cdots, \alpha_k^{t_k}, \beta_k^{s_k}) = 0 \pmod 2,$$

if $t_i s_i$ *is even for every* i. *Here,* $\alpha_i^{t_i}$ *means plugging in* α_i t_i *times.*

The proof is trivial.

A special case is that $\Phi_k(\alpha, \alpha, \cdots, \alpha) = 0 \pmod 2$ for even degree Donaldson invariants Φ_k with k odd on even manifolds.

References

[1] M. F. Atiyah, N. J. Hitchin & I. M. Singer, *Self duality in four dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.

[2] S. K. Donaldson, *An application of gauge theory to four dimensional topology*, J. Differential Geometry **18** (1983) 279–315.

[3] ——, *Connections, cohomology and the intersection forms of* 4-*manifolds*, J. Differential Geometry **24** (1986) 275–341.

[4] ——, *The orientation of Yang-Mills moduli spaces and* 4-*manifold topology*, J. Differential Geometry **26** (1987) 387–428.

[5] ——, *Polynomial invariants for smooth* 4-*manifolds* to appear.

[6] ——, Lecture Notes.

[7] R. Fintushel & R. Stern, SO(3)-*connections and the topology of* 4-*manifolds*, J. Differential Geometry **20** (1984) 523–539.

[8] ——, 2-*torsion instanton invariants*, preprint.

[9] D. S. Freed & K. K. Uhlenbeck, *Instantons and* 4-*manifolds*, Math. Sci. Res. Inst. Publications, Vol. 1. Springer, New York, 1984.

[10] R. Kirby, *The topology of* 4-*manifolds*, Lecture Notes in Math., Vol. 1374, Springer, Berlin, 1989.

[11] J. Morgan, Lecture notes.

[12] C. H. Taubes, *Self-dual connections on manifolds with indefinite intersection matrix*, J. Differential Geometry **19** (1984) 517–560.

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