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ON THE NODAL LINE OF THE SECOND EIGENFUNCTION OF THE LAPLACIAN IN $\mathbb{R}^2$

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1. Introduction

A conjecture of L. Payne [8] states that any second eigenfunction of the Laplacian with zero boundary condition for a bounded domain $\Omega \subseteq \mathbb{R}^2$ does not have a closed nodal line. This is also asked by S.-T. Yau [10, Problem 78] for Ω a bounded convex domain in $\mathbb{R}^2$.

L. Payne [9] proved the conjecture provided the domain $\Omega \subseteq \mathbb{R}^2$ is symmetric with respect to one line and convex with respect to the direction vertical to this line. Also. C.-S. Lin [7] proved the conjecture provided the domain $\Omega \subseteq \mathbb{R}^2$ is smooth, convex, and invariant under a rotation with angle $2\pi p/q$, where p and q are positive integers. Recently D. Jerison [5] proved the conjecture for long thin convex sets. Without any assumption on the smoothness of Ω he showed that the nodal line has to intersect $\partial\Omega$ in exactly two points.

In this paper we prove the conjecture when Ω is a bounded convex domain in $\mathbb{R}^2$ with C^∞ boundary.

To fix the notation for a bounded domain $\Omega \subseteq \mathbb{R}^2$ with smooth boundary we let u_2 be a second eigenfunction of Ω, that is, u_2 is a solution of the Dirichlet problem

$$\begin{cases} \Delta u_2 + \lambda_2 u_2 = 0 & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^2 (\partial^2/\partial x_i^2)$ and λ_2 is the second eigenvalue of Ω.

The nodal line N of u_2 is defined by

$$N = \overline{\{x \in \Omega\colon u_2(x) = 0\}}. \tag{1.2}$$

The Courant nodal domain theorem implies that N must divide the domain Ω into exactly two components.

Our main result is the following:

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Theorem 1.1. *If $\Omega \subseteq \mathbb{R}^2$ is a bounded convex domain with C^∞ boundary, then the nodal line N of any second eigenfunction u_2 must intersect the boundary $\partial\Omega$ at exactly two points.*

Theorem 1.1 implies

Corollary 1.2. *If $\Omega \subseteq \mathbb{R}^2$ is a bounded convex domain, then the nodal line N of any second eigenfunction u_2 does not enclose a compact subregion of Ω.*

Proof. Otherwise we can approximate Ω by a convex domain $\tilde{\Omega}$ with C^∞ boundary for which the nodal line of a second eigenfunction of $\tilde{\Omega}$ still encloses a compact subregion of Ω and this contradicts Theorem 1.1.

In §2 we show as in [7] that if Theorem 1.1 is false, then we can find a bounded convex domain Ω with C^∞ boundary such that the nodal line N of a second eigenfunction u_2 intersects $\partial\Omega$ at exactly one point o.

In §3 we prove that the above situation is impossible and this proves Theorem 1.1. To do that we introduce the functions $v_t = \partial u_2/\partial x_1 + tu_2$ for $t \in \mathbb{R}$ where the x_1-direction is tangent to $\partial\Omega$ at o.

We show that for each $t \in \mathbb{R}$ there exists exactly one subdomain Ω_t of Ω such that $v_t = 0$ on $\partial\Omega_t$. Since $\Delta v_t + \lambda_2 v_t = 0$ in Ω, it is concluded that either v_t or $-v_t$ is everywhere positive in Ω_t. Then defining $A = \{t \in \mathbb{R}: v_t > 0 \text{ in } \Omega_t\}$ and $B = \{t \in \mathbb{R}: v_t < 0 \text{ in } \Omega_t\}$ we prove that A, B are closed subsets of $\mathbb{R}$ and finally that both are nonempty. This gives a contradiction since $\mathbb{R} = A \cup B$ and $A \cap B = \varnothing$.

Also we need a technical lemma which allows us to control the singularity of N on $\partial\Omega$. It is because of this that we have to assume Ω has C^∞ boundary. We will state this lemma in §2 but we prove it in §4.

2.

Let (C) be the proposition: "The nodal line of any second eigenfunction of Ω intersects $\partial\Omega$ at exactly two points." Then Theorem 1.1 means that (C) is true for any bounded convex domain Ω with C^∞ boundary. For such domains we have the following technical lemma.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex domain with C^∞ boundary, let $q \in \partial\Omega$, and $r_0 > 0$, and suppose u satisfies*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{in } (\partial\Omega) \cap D(q\,; r_0), \end{cases} \tag{2.1}$$

where λ is a constant, and $D(q, r_0)$ denotes the disc of radius r_0 centered *at q. Also assume that the x_1-axis is in the direction of the tangent of $\partial\Omega$ at q. Then we have the following.*

(i) *u does not vanish of infinite order at q.*

(ii) *If the nodal line of u is N, and $q \in N$, then N approaches q nontangentially with respect to $\partial\Omega$.*

(iii) *There exists an $\varepsilon > 0$ such that*

$$|u| + |\partial u/\partial x_1| > 0 \quad \text{in } \Omega \cap \{x\colon 0 < |x-q| < \varepsilon\}.$$

This lemma will be proved in §4.

The following lemmas are from [7].

Lemma 2.2. *Suppose $p \in \partial\Omega$ and that Ω has C^2 boundary. Then $(\partial u_2/\partial\nu)(0) = 0$ if and only if $p \in N$, where $\partial u_2/\partial\nu$ is the outward normal derivative of u_2 on $\partial\Omega$.*

Lemma 2.3. *Let u_2 be a second eigenfunction of Ω. If $\partial u_2/\partial\nu \geq 0$ on $\partial\Omega$, then up to multiplication by a constant, u_2 is the only second eigenfunction of Ω.*

Theorem 2.1. *Suppose Ω_0 is a bounded convex domain with C^∞ boundary such that (C) fails for Ω_0. Then there exists a convex bounded domain Ω with C^∞ boundary and a second eigenfunction u_2 of Ω such that $\partial u_2/\partial\nu$ has exactly one zero on $\partial\Omega$.*

Proof. This was proved by C.-S. Lin [7]. We sketch the proof here for completeness. Let $\Omega(t)$ be a smooth deformation with $\Omega(0) = \Omega_0$ and $\Omega(1)$ a disc such that $\Omega(t)$ is a bounded convex domain with C^∞ boundary. Since (C) fails for Ω_0 and obviously holds for $\Omega(1)$, we may define $t_0 = \sup\{t \in [0, 1]\colon (C) \text{ fails for } \Omega(t)\}$ and we have $0 \leq t_0 < 1$. Thus there exist sequences t_i, $\tilde{t}_i$ with $t_i \leq t_0 \leq \tilde{t}_i$ and t_i, $\tilde{t}_i \to t_0$ as $i \to +\infty$ and such that there exist normalized eigenfunctions u_i, $\tilde{u}_i$ of $\Omega(t_i)$, $\Omega(\tilde{t}_i)$ respectively such that $\partial u_i/\partial\nu \geq 0$ on $\partial\Omega(t_i)$ and $\partial\Omega\tilde{u}_i/\partial_\nu$ has at least two zeros on $\partial\Omega(\tilde{t}_i)$. Since the second eigenvalue of $\Omega(t)$ is a continuous function of t, [3], we can by elliptic estimates get subsequences of $\{u_i\}$ and $\{\tilde{u}_i\}$ converging to u_0, $\tilde{u}_0$ so that u_0, $\tilde{u}_0$ are second eigenfunctions of $\Omega(t_0)$, and moreover $\partial u_0/\partial\nu \geq 0$ on $\partial\Omega(t_0)$, and $\partial\tilde{u}_0/\partial\nu$ has at least one zero on $\partial\Omega(t_0)$. By Lemma 2.3 we may assume $u_0 = \tilde{u}_0$. Hence $\partial u_0/\partial\nu$ has at least on zero on $\partial\Omega(t_0)$ and does not change sign on $\partial\Omega(t_0)$. But from Lemmas 2.1(ii) and 2.2 it follows that zeros of $\partial u_0/\partial\nu$ on $\partial\Omega(t_0)$ are isolated. Since the nodal line of u_0 divides $\Omega(t_0)$ into exactly two components and u_0 has opposite signs on each of them, the set of zeros of $\partial u_0/\partial\nu$ on $\partial\Omega(t_0)$ is connected. Hence

$\partial u_0/\partial\nu$ has exactly one zero on $\partial\Omega(t_0)$. This proves the theorem with $\Omega = \Omega_0(t_0)$ and $u_2 = u_0$.

3. Proof of Theorem 1.1

By Theorem 2.1 we conclude that Theorem 1.1 follows from:

Theorem 3.1. *If $\Omega \subseteq \mathbb{R}^2$ is a bounded convex domain with C^∞ boundary, and u_2 is a second eigenfunction of Ω, then $\partial u_2/\partial\nu$ cannot have exactly one zero on $\partial\Omega$.*

Proof. Assume $\partial u_2/\partial\nu$ vanishes at exactly one point $o \in \partial\Omega$. We choose the coordinate axes so that o is the origin, the x_1-axis is tangent to $\partial\Omega$ at o, and the x_2-axis is in the direction of the inward normal to $\partial\Omega$ at o. Let $\Omega^+ = \{x \in \Omega\colon u_2(x) > 0\}$ and $\Omega^- = \{x \in \Omega\colon u_2(x) < 0\}$. By Lemma 2.2, $N \cap \partial\Omega = \{o\}$. Also since by the maximum principle u_2 has to change sign near any point in $N \cap \Omega$, we may assume that $\partial\Omega^- = N$, and it is easy to see that Ω^- is simply connected.

Since Ω is convex, the set of points p on $\partial\Omega$, where the tangent to $\partial\Omega$ at p is parallel to the x_1-axis, consists exactly of two closed line segments I, J (which may be points) parallel to the x_1-axis. Assume $o \in I$ and let Γ^+ and Γ^- be the two open subarcs of $\partial\Omega$, which form $\partial\Omega \setminus (I \cup J)$. Since u_2 is nonnegative near $\partial\Omega \setminus (I \cup J)$ by Hopf's boundary point lemma we may assume that

$$\partial u_2/\partial x_1 > 0 \quad \text{in } \Gamma^+ \qquad \text{and} \qquad \partial u_2/\partial x_1 < 0 \quad \text{in } \Gamma^-. \tag{3.1}$$

Also it is obvious that

$$\partial u_2/\partial x_1 = 0 \quad \text{on } I \cup J. \tag{3.2}$$

For any $t \in \mathbb{R}$ we define v_t on $\overline{\Omega}$ by

$$v_t(x) = e^{-tx_1}\frac{\partial}{\partial x_1}(e^{tx_1}u_2(x)) = \frac{\partial u_2}{\partial x_1}(x) + tu_2(x). \tag{3.3}$$

Then we have

$$\Delta v_t + \lambda_2 v_t = 0 \quad \text{in} \quad \Omega, \tag{3.4}$$

where λ_2 is the second eigenvalue of Ω.

Let $N_t = \overline{\{x \in \Omega\colon v_t(x) = 0\}}$ be the nodal line of v_t. By (3.1) we have $N_t \cap \partial\Omega \subseteq I \cup J$. Also $v_t = 0$ on $I \cup J$. The basic lemma about N_t is the following:

Lemma 3.1. *For every $t \in \mathbb{R}$ there exists at least one subdomain Ω_t of Ω such that $\partial\Omega_t \subseteq N_t \cup I \cup J$.*

Proof. Clearly the gradient of u_2 vanishes at every point of $N \cap \Omega$, where N is not C^1. By Lemma 2.1(ii), (iii) (for $u = u_2$, $q = 0$) N approaches o nontangentially with respect to $\partial\Omega$, and there exists $\varepsilon > 0$ such that $\partial u_2/\partial x_1$ has no zeros in $N \cap \{x\colon 0 < |x| < \varepsilon\}$. Hence there exists $\eta > 0$ such that if $0 < \overline{\eta} \leq \eta$ and $\Sigma_{\overline{\eta}} = \{(x_1, x_2)\colon 0 < x_2 \leq \overline{\eta}\}$, then $N \cap \Sigma_{\overline{\eta}}$ is a C^1 one-manifold with boundary on the line $x_2 = \overline{\eta}$, and such that its tangent is never parallel to the x_1-axis. Hence $N \cap \Sigma_{\overline{\eta}}$ does not contain embedded circles or arcs with both endpoints on $x_2 = \overline{\eta}$. Also $N \cap \partial\Omega = \{o\}$, and N divides Ω into exactly two components. Thus $N \cap \Sigma_{\overline{\eta}}$ consists of two simple arcs each of which has one endpoint on $x_2 = \overline{\eta}$ and approaches o. Hence $N \cap \Sigma_\eta$ consists of two simple arcs which approach o, and every line $x_2 = \overline{\eta}$ for $0 < \overline{\eta} \leq \eta$ intersects each of them exactly once. Let γ_1 be one of these arcs.

Now suppose there exists $t_0 \in \mathbb{R}$ such that there exists no subdomain Ω' of Ω with $\partial\Omega' \subseteq N_{t_0} \cup I \cup J$. Then from (3.1), (3.2) and the fact that Γ^+, Γ^- are connected, it follows that $\overline{\Omega} \setminus (N_{t_0} \cup I \cup J)$ has exactly two components so that the sets $U^+ = \{x \in \overline{\Omega}\colon v_{t_0}(x) > 0\}$ and $U^- = \{x \in \overline{\Omega}\colon v_{t_0}(x) < 0\}$ are both connected. We may assume that γ_1 lies in the interior of U^+. Since $\overline{\Gamma}_1 \cap \partial U^+ = \{o\}$ and γ_1 is a simple arc, $U^+ \setminus \gamma_1$ is connected. So if $x = (x_1, \eta) \in \gamma_1$ and $y = (y_1, \eta) \in \Gamma^+$, we may join x and y by a simple polygonal path γ_2 such that $\gamma_2 \setminus \{x, y\}$ lies in the interior of $U^+ \setminus \gamma_1$. Let γ_3 be the subarc of $\Gamma^+ \cup I$ with endpoints o and y. Then $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ is a simple closed curve, hence by the Jordan curve theorem there exists a unique bounded domain W with $\partial W = \gamma$. Since Ω is simply connected, we have $W \subseteq \Omega$. Since $v_{t_0} \geq 0$ on γ, $\Gamma^- \cap \overline{W} = \varnothing$, and U^- is connected, we have $v_{t_0} \geq 0$ in W, so by the maximum principle $v_{t_0} > 0$ in W.

Since γ_2 does not meet the line $x_2 = 0$, there exists η_1 with $0 < \eta_1 \leq \eta$ such that γ_2 does not meet the line $x_2 = \eta_1$. This line meets γ_1 at exactly one point $\overline{x}$ and Γ^+ at exactly one point $\overline{y}$. Hence the line segment $[\overline{x}, \overline{y}]$ intersects the curve $\gamma = \partial W$ at exactly $\overline{x}$ and $\overline{y}$. Since points on $[\overline{x}, \overline{y}]$ which are near Γ^+ must be in W, we conclude that $[\overline{x}, \overline{y}]$ lies in $\overline{W}$. On the other hand, $e^{t_0 x_1} u_2(x)$ vanishes on $\overline{x}$ and $\overline{y}$, hence by (3.3) there exists z in the open segment with endpoints $\overline{x}$ and $\overline{y}$ such that $v_{t_0}(z) = 0$. But then $z \in W$ and this is a contradiction since $v_{t_0} > 0$ in W.

Lemma 3.2. *For every $t \in \mathbb{R}$ we have the following:*

(i) *There exists exactly one subdomain* Ω_t *of* Ω *such that* $\partial\Omega_t \subseteq N_t \cup I \cup J$.

(ii) *Either* $v_t > 0$ *in* Ω_t *or* $v_t > 0$ *in* Ω_t.

(iii) $\lambda_1(\Omega_t) = \lambda_2$, *where* $\lambda_1(\Omega_t)$ *is the first eigenvalue of* Ω_t *with respect to the Dirichlet boundary condition.*

(iv) Ω_t *is simply connected.*

(v) *There exists* $\rho > 0$ *depending only on* λ_2 *such that* Ω_t *contains a disc of radius* ρ.

Proof. Fix $t \in \mathbb{R}$. (i) Assume there exist two distinct subdomains of Ω whose boundary is contained in $N_t \cup I \cup J$. Then there are two disjoint subdomains Ω_1, Ω_2, of Ω such that $v_t = 0$ on $\partial\Omega_1 \cup \partial\Omega_2$. Since $\Delta v_t + \lambda_2 v_t = 0$ in $\Omega_1 \cup \Omega_2$ and λ_2 is the second eigenvalue of Ω, this yields a contradiction as in the proof of Courant's nodal domain theorem. Hence (i) follows from Lemma 3.1.

(ii) follows from (i).

(iii) follows from (i) and (ii) since $\Delta v_t + \lambda_2 v_t = 0$ in Ω_t.

(iv) follows from (i) and the fact that Ω is simply connected.

(v) follows from (iii) and (iv) by using Hayman's inner radius theorem [4].

Lemma 3.3. *There exist* $C_1, C_2 > 0$ *such that* $v_t > 0$ *in* Ω_t *for every* $t > C_1$ *and* $v_t > 0$ *in* Ω_t *for every* $t < -C_2$.

Proof. Let $z \in \Gamma^+$. Since $\partial u_2/\partial x_1(z) > 0$, there exists $\delta > 0$ sufficiently small such that $\partial u_2/\partial x_1 > 0$ in $\overline{\Omega} \cap D(z;\delta)$, and moreover since $z \in \partial\Omega^+$, we may assume that $D(z;\delta) \cap \overline{\Omega^-} = \varnothing$. Let $K = \overline{\Omega} \cap D(z;\delta)$. Then using $\lambda_1(\Omega^+) = \lambda_2$ and the monotonicity principle for eigenvalues, we have $\lambda_1(\Omega^+ \setminus K) > \lambda_2$. Hence by the continuity of the first eigenvalue under continuous deformations of the domain [3] we can choose a sufficiently large compact subset E of Ω^- such that $\lambda_1(\Omega \setminus (K \cup E)) > \lambda_2$.

Let $M = \sup\{|\partial u_2/\partial x_1(x)| : x \in \Omega\}$ and $\alpha = \inf\{|u_2(x)| : x \in E\} > 0$. Now fix any t such that $t > \alpha^{-1}M$. Then

$$v_t > 0 \quad \text{in } K \qquad \text{and} \qquad v_t < 0 \quad \text{in } E.$$

Hence $N_t \cup I \cup J \subseteq \overline{\Omega} \setminus (K \cup E)$, and since $\overline{\Omega} \setminus K$ is simply connected we have $\Omega_t \subseteq \overline{\Omega} \setminus K$. Also from Lemma 3.2(iii) it follows that $\lambda_1(\Omega_t) = \lambda_2 < \lambda_1(\Omega \setminus (K \cup E))$ so that $\Omega_t \subseteq \Omega \setminus (K \cup E)$. Hence $\Omega_t \cap E \neq \varnothing$, and since $v_t < 0$ in E by Lemma 3.2(ii) we must have $v_t < 0$ in Ω_t if $t > \alpha^{-1}M = C_1$. Similarly we can find $C_2 > 0$ such that $v_t < 0$ in Ω_t if $t < -C_2$.

Lemma 3.4. *The sets $A = \{t \in \mathbb{R}: v_t > 0$ in $\Omega_t\}$ and $B = \{t \in \mathbb{R}: v_t < 0$ in $\Omega_t\}$ are closed.*

Proof. Assume $t_j \in A$, $t_j \to t_0$ as $j \to \infty$ but $t_0 \notin A$. Then $v_{t_j} > 0$ in Ω_{t_j} and by Lemma 3.2(ii) $v_{t_0} < 0$ in Ω_{t_0}. By Lemma 3.3(v) we can find discs $D(x_j; \rho) \subseteq \Omega_{t_j}$. Taking a subsequence we may assume $x_j \to x_0 \in \overline{\Omega}$. It follows then that $D(x_0; \rho) \subseteq \overline{\Omega}$ and $v_{t_0} \geq 0$ in $D(x_0; \rho)$. By the maximum principle, $v_{t_0}(x_0) > 0$. Since $v_{t_0} < 0$ in Ω_{t_0}, Lemma 3.2(i) implies that N_{t_0} does not separate x_0 from $\partial\Omega \setminus (I \cup J)$. Hence there exists a curve $\zeta: [0, 1] \to \overline{\Omega}$ such that $\zeta(0) = x_0$, $\zeta(1) \in \partial\Omega$, and $\zeta^* \cap (N_{t_0} \cup I \cup J) = \varnothing$, where ζ^* is the image of ζ. Since $x_0 \in D(x_j; \rho) \subseteq \Omega_{t_j}$ for sufficiently large j, $\zeta^* \cap (N_{t_j} \cup I \cup J) \neq \varnothing$. Therefore there exist $y_j \in \zeta^*$ such that $v_{t_j}(y_j) = 0$. Again by taking a subsequence we may assume that $y_j \to y_0 \in \zeta^*$. Then $v_{t_0}(y_0) = \lim_{j\to\infty} v_{t_j}(y_i) = 0$, hence $y_0 \in N_{t_0} \cup I \cup J$ which is a contradiction since $\zeta^* \cap (N_{t_0} \cup I \cup J) = \varnothing$. Thus A is closed. The proof for B is similar. q.e.d.

Now we can finish the proof of Theorem 3.1. By Lemma 3.4, A and B are closed subsets of $\mathbb{R}$. By Lemma 3.2 we have $A \cap B = \varnothing$ and $A \cup B = \mathbb{R}$, and by Lemma 3.3 both A and B are nonempty. This is a contradiction since $\mathbb{R}$ is connected. Hence Theorem 3.1 is proved, and now Theorem 1.1 follows from Theorems 2.1 and 3.1.

4.

Proof of Lemma 2.1. (i) From the boundary regularity of elliptic differential equations it follows that u is C^∞ up to the boundary near q. Let $H = \{(x_1, x_2): x_2 > 0\}$ be the upper halfplane. Let $f: H \to \Omega$ be a conformal mapping of H onto Ω. By a theorem of Kellogg [6] f extends C^∞ to the boundary of H, and we may assume $f(0) = q$. Let $v = u \circ f$ in $\overline{H}$. Then there exists sufficiently small $\bar{r}_0 > 0$ such that v is C^∞ up to the boundary in $D(0; \bar{r}_0) \cap \overline{H}$ and $v = 0$ in $D(0; \bar{r}_0) \cap \partial H$. By (2.1) we have $|\Delta v| = |\Delta u| \circ f ||f'|^2 \leq C|v|$ in $D(0; \bar{r}_0) \cap \overline{H}$ for some C. Define $\tilde{v}$ on $D(0; \bar{r}_0)$ by

$$\tilde{v}(x_1, x_2) = \begin{cases} v(x_1, x_2) & \text{if } x_2 \geq 0, \\ -v(x_1, -x_2) & \text{if } x_2 \leq 0. \end{cases} \tag{4.1}$$

Then it is easy to check that $\partial\tilde{v}/\partial x_1$, $\partial\tilde{v}/\partial x_2$, $\partial^2\tilde{v}/\partial x_1 \partial x_2$, and $\partial^2\tilde{v}/\partial x_1^2$ are continuous on $D(0; \bar{r}_0)$, and moreover $\partial^2\tilde{v}/\partial x_1^2 = 0$ in

$D(0;\bar{r}_0)\cap\partial H$ since $v=0$ on $D(0;r\bar{r}_0)\cap\partial H$. From the inequality $|\Delta v|\le C|v|$ in $D(0;\bar{r}_0)\cap\overline{H}$, it follows that $\partial^2\tilde{v}/\partial x_2^2$ is also continuous in $D(0;\bar{r}_0)$. Since v is C^∞ up to the boundary in $D(0;\bar{r}_0)\cap\overline{H}$, we conclude that $\tilde{v}$ is $C^{2,1}$ in $D(0;\bar{r}_0)$, and moreover $|\Delta\tilde{v}|\le C|\tilde{v}|$ in $D(0;\bar{r}_0)$. Thus by Aronzajn's unique continuation theorem [1] $\tilde{v}$ does not vanish of infinite order in L^1-mean at 0. Since v is C^∞ up to the boundary in $D(0;\bar{r}_0)\cap\overline{H}$ v does not vanish of infinite order at 0. Hence u does not vanish of infinite order at q.

(ii) Assume $q=0$. By (i) we may write

$$u(x)=p_m(x)+O(|x|^{m+1}),\qquad |x|\text{ small},\ x\in\Omega,\tag{4.2}$$

where $p_m\neq 0$ is a homogeneous polynomial of degree m. Since $\Delta u+\lambda u=0$ in Ω we have

$$\Delta p_m(x)=O(|x|^{m-1})\quad\text{in }\Omega\text{ for small }x.\tag{4.3}$$

Since $\Delta p_m(x)$ is homogeneous of degree $m-2$ and the x_1-axis is tangent to $\partial\Omega$ at 0, we conclude that $\Delta p_m(x)=0$ in $H=\{(x_1,x_2):x_2>0\}$, provided that the x_2-axis is in the direction of the inward normal to $\partial\Omega$ at 0. Since $u=0$ on $D(0;r_0)\cap\partial\Omega$, by (4.2) we also have that $p_m(x)=0$ on ∂H, so that reflection p_m is a harmonic polynomial vanishing on ∂H. Thus introducing polar coordinates, we have

$$p_m(r,\theta)=cr^m\sin m\theta,\qquad\text{where } c\neq 0.\tag{4.4}$$

For $r>0$ sufficiently small let $0\le\theta^-(r)<\theta^+(r)\le\pi$ be the unique angles with $(r,\theta^{\pm}(r))\in\partial\Omega$. If $(r,\theta)\in\Omega\cap N$ and $r<r_0$, then

$$u(r,\theta^-(r))=u(r,\theta)=u(r,\theta^+(r))=0\quad\text{and}\quad\overline{\theta}<\theta<\theta^-(r).$$

Therefore there exist θ_1 and θ_2, $\theta^-(r)<\theta_1<\theta<\theta_2<\theta^+(r)$, such that $\partial u/\partial\theta((r,\theta_i))=0$ for $i=1,2$. By (4.2) and (4.4) we conclude that

$$|\cos m\theta_i|\le C_1 r\quad\text{for some constant } C_1 \text{ and } i=1,2.$$

If $r<(2C_1)^{-1}$, $c_1=\pi/4m>0$, and $c_2=\pi-\pi/4m<\pi$, then $\theta_1>c_1$ and $\theta_2<c_2$, and thus we have the following: If $(r,\theta)\in\Omega\cap N$ and $r<(2C_1)^{-1}$, then $0<c_1<\theta<c_2<\pi$. Hence if $0\in N$, then N approaches 0 nontangentially with respect to $\partial\Omega$.

(iii) Clearly there is nothing to prove if $0\notin N$. So we assume that $0\in N$. Then by (ii) there exist c_1, c_2, $0<c_1<c_2<\pi$, and $\delta>0$ such that if (r,θ) is in $\Omega\cap N$ and $0<r<\delta$, then $c_1<\theta<c_1$ and therefore

$\sin\theta > c_3$, where $c_3 = \min\{\sin\theta_1, \sin\theta_2\} > 0$. Also from (4.2) and (4.4) it follows that

$$\frac{\partial u}{\partial x_1}(r, \theta) = mcr^{m-1}\sin(m-1)\theta + O(r^m). \tag{4.5}$$

If (r, θ) is in Ω and $\partial u/\partial x_1(r, \theta) = u(r, \theta) = 0$, then $|\sin m\theta| \leq C_2 r$ and $|\sin(m-1)\theta| \leq C_2 r$ for some constant C_2. These imply that $|\sin\theta| \leq 2C_2 r$.

On the other hand, if $0 < r < \delta$ and $u(r, \theta) = 0$, then $\sin\theta > c_3$. Hence if $\varepsilon = \min\{\delta, (2C_2)^{-1}c_3\} > 0$, we have $|u| + |\partial u/\partial x_1| > 0$ in $\Omega \cup \{x : 0 < |x| < \varepsilon\}$.

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