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ON THE SLOPE AND KODAIRA DIMENSION OF $\overline{M}_g$ FOR SMALL g

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Let M_g denote the moduli space of smooth curves of genus g and let

$$\overline{M}_g = M_g \cup \bigcup_{i=0}^{[g/2]} \Delta_i$$

be its stable compactification; here $\Delta_i = \Delta_{i,g}$ for $0 < i < g$ parametrizes stable curves of the form $C_1 \cup_P C_2$, where C_1 and C_2 have genus i and $g-i$, respectively, and meet at P, and Δ_0 parametrizes irreducible nodal curves of geometric genus $g-1$. Let λ denote the class of the Hodge line bundle on $\overline{M}_g$, i.e., the line bundle whose fiber over $[C] \in \overline{M}_g$ is $\bigwedge^g H^0(C, \omega_c)$, and put

$$\delta_i = [\Delta_i], \qquad \delta = \sum_{i=2}^{[g/2]} \delta_i + \delta_0 + \frac{1}{2}\delta_1.$$

Many important divisor classes on $\overline{M}_g$ have the form $a\lambda - b\delta$, $a, b > 0$, for example the canonical class

$$K_{\overline{M}_g} \sim 13\lambda - 2\delta$$

(cf. [7]). With Harris, one defines the *slope* s_g of M_g as

$$s_g = \inf\{a/b | a\lambda - b\delta \text{ is effective}, a, b > 0\}.$$

This number carries some important information about $\overline{M}_g$. For example, as λ is birationally ample, $\overline{M}_g$ is of general type (resp. has nonnegative Kodaira dimension) whenever $s_g < 6\frac{1}{2}$ (resp. $s_g \le 6\frac{1}{2}$), and this, in fact, is how these statements were proven by Harris, Mumford, and Eisenbud for $g \ge 24$ (resp. $g = 23$). It is proved in [4] that $s_g \le 6 + 12/(g+1)$ whenever $g+1$ is composite. On the other hand, the lower bound $s_g > 6\frac{1}{2}$

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is obviously equivalent to $\overline{M}_g$ having Kodaira dimension $-\infty$. For additional motivation for computing or estimating s_g, see [6], [2].

The problem of bounding s_g from below was considered by Harris and Morrison [5], who conjectured in general that

$$s_g \geq 6 + 12/(g+1)$$

with equality when $g+1$ is composite. They give an asymptotic lower bound ($\sim 576/5g$) for s_g, as well as some explicit lower bounds for small g $(g \leq 6)$. Additional lower bounds on s_g for $g \leq 5$ and $g = 15$ were given in [2], [3].

The purpose of this paper is to give further lower bounds on s_g for $6 \leq g \leq 9$ and $g = 16$. These suffice, in particular, to prove that M_{16} has Kodaira dimension $-\infty$. We summarize our results as follows.

Theorem 0.1. *For $g = 2, \cdots, 9, 15, 16$ we have $s_g \geq \sigma_g$, where σ_g is given in Table 1, in which $(\mathrm{HM})_g$ denotes the Harris-Morrison lower bound in genus g.*

g	σ_g	$(\mathrm{HM})_g$	$6+12/(g+1)$
2	10	10	10
3	9	9	9
4	8.5	~ 8.4242	8.4
5	7.8	8	8
6	7.8333	7.328	7.7142
7	7.4285		7.5
8	7		7.3333
9	7		7.2
15	6.667		6.75
16	6.56		6.7058

TABLE 1

Remark. We have $s_4 \leq 8\frac{5}{6}$ by [5].

Our method of proof, in analogy with [5], is to show the noneffectiveness of a divisor class A on $\overline{M}_g$ by "testing" against suitable pencils $F \subset \overline{M}_g$, i.e., by showing that $F \cdot A < 0$. We will use three kinds of pencils: the first kind "fill up" $\overline{M}_g$, the second kind fill up a boundary component $\Delta_i \subset \overline{M}_g$, while a third and more subtle kind fill up a divisor in a boundary component. As for the first kind of pencils, these are mostly classical, coming from pencils of smooth or nodal curves on rational surfaces (mostly $\mathbb{P}^2$); the one nonclassical exception is a pencil of genus 15 constructed in [1]. As for the second and third kinds, these are obtained by starting with a pencil of the first kind, base-changing, if necessary to get sections, and then applying one of two possible types of "lifts": type α consists in attaching a fixed curve along a section, while type β consists in gluing together two (disjoint) sections.

The paper is organized as follows. §1 consists of generalities, notably some trivial but useful "principles" for proving noneffectiveness of divisor classes by testing on suitable curves, and some formulas useful for doing intersection theory on $\overline{M}_g$; in particular, for computing intersection numbers related to the lifts of types α and β. In §2 we describe explicitly our pencils and prove Theorem 0.1.

1. General principles and formulas

We begin by formulating the general principles which we will use in proving the noneffectiveness of divisor classes. Let X be a $\mathbb{Q}$-factorial projective variety, such as $\overline{M}_g$. Then for any curve F (say irreducible) and divisor D on X, the intersection number $F \cdot D \in \mathbb{Q}$ is defined. If $B \subset X$ is a subvariety, we will say that F *fills up* B if F moves in an irreducible algebraic family $\{F_t | t \in T\}$ of curves on X such that $\bigcup_{t\in T} F_t$ is dense in B. Recall that F is said to be nef if $F \cdot D \geq 0$ for every effective divisor D. More generally, a collection $\{F_1, \cdots, F_N\}$ of curves is said to be nef if for every effective divisor D, we have

$$\max_{1\leq i\leq N} F_i \cdot D \geq 0.$$

Principle 1.0. *If F fills up X, then F is nef.*

Principle 1.1. *If F fills up a prime divisor B such that $F \cdot B \geq 0$, then F is nef.*

Principle 1.2. *Suppose that for each i the curve F_i lies on a prime divisor B_i and is nef as a curve on B_i with respect to $\mathbb{Q}$-Cartier divisors, and suppose moreover that*

$$F_i \cdot \sum_{j=1}^{n} B_j \geq 0, \qquad i = 1, \cdots, N.$$

Then the collection $\{F_1, \cdots, F_N\}$ is nef on X.

The proofs of these are essentially trivial. For illustration, let us prove 1.2. Suppose D is an effective divisor on X such that $D \cdot F_i < 0$ for

$i = 1, \cdots, N$. Then

$$D|_{B_i} \cdot_{\overline{B}_i} F_i = D \cdot_X F_i < 0.$$

So by assumption $D|_{B_i}$ cannot be effective on B_i and, in particular, $D \geq B_i$, $i = 1, \cdots, N$. But then we have $(D - \sum_{j=1}^{N} B_j) \cdot F_i < 0$, $i = 1, \cdots, N$.

Applying the same argument to $D - \sum B_j$ in place of D, we conclude inductively that D contains arbitrarily high multiples of the B_j, which is impossible. q.e.d.

We now specialize to the case $X = \overline{M}_g$. For a "pencil," i.e., an irreducible curve $F \subset \overline{M}_g$, it will be convenient to define the *slope*

$$s(F) = \frac{F \cdot \delta}{F \cdot \lambda},$$

provided both numerator and denominator are positive. Note that, essentially by definition, whenever F is nef we have

$$s_g \geq s(F).$$

More generally, we have

Principle 1.3. *For any nef collection $\{F_1, \cdots, F_N\}$ of pencils on $\overline{M}_g$, we have*

$$s_g \geq \min_{1 \leq i \leq N} s(F_i).$$

Proof. If $D \sim a\lambda - b\delta$ is effective, $a, b > 0$, then for some $1 \leq i \leq N$,

$$0 \leq D \cdot F_i = aF_i \cdot \lambda - bF_i \cdot \delta,$$

i.e., $\min s(F_i) \leq \frac{a}{b}$, hence the assertion. q.e.d.

Next, we will recall some formulas which we will use in computing intersection numbers on $\overline{M}_g$. Let $\pi: \mathcal{Y} \to B$ be a proper morphism from a locally complete intersection surface to a smooth complete curve of genus h, whose fibers $Y_b = \pi^{-1}(b)$ are stable curves of genus i, and let $f: B \to \overline{M}_i$ be the natural map, with image cycle F. Then we have (cf. [1])

$$(1.1) \qquad \begin{aligned} F \cdot \delta &= c_2(Y) + 4(1-h)(i-1), \\ F \cdot \lambda &= \chi(\mathcal{O}_Y) + (1-h)(i-1). \end{aligned}$$

Another standard device which we will use is that of "lifting" a pencil in $\overline{M}_i$, $i < g$, to $\overline{M}_g$. We will use two types of such lifts:

Type α. With notation as above, let $A \subset \mathcal{Y}$ be a *multisection*, i.e., a smooth irreducible curve, not containing any singular points of fibers, and mapping to B with degree $m > 0$, and let $\alpha(F, A, g)$ or $\alpha(F, A)$ be the pencil in $\overline{M}_g$ parametrized by A, where $a \in A$ corresponds to the curve $Y_{\pi(a)} \cup C$ obtained by gluing $Y_{\pi(a)}$ to a fixed general curve of genus $g - i$ by identifying a with a fixed general point $P \in C$. Then we have

$$
(1.2)\qquad
\begin{gathered}
\alpha(F, A) \cdot_{\overline{M}_g} \lambda = mF \cdot \lambda, \qquad \alpha(F, A) \cdot_{\overline{M}_g} \delta_j = mF \cdot \delta_j, \\
j \neq i, g-i, \\
\alpha(F, A) \cdot_{\overline{M}_g} \delta_i = m(2h-2) - 2(g(A) - 2) + A \cdot_y A.
\end{gathered}
$$

Proof. The first two formulas are evident. To verify the third, it is easy to see by base-change that we may assume A is a section of π. In this case the formulas of [7] yield

$$
\begin{aligned}
\alpha(F, A) \cdot_{\overline{M}_g} \delta_i &= -A \cdot \omega_{\mathscr{Y}/B} = m(2h-2) - A \cdot \omega_{\mathscr{Y}} \\
&= m(2h-2) - (2g(A) - 2) + A \cdot A
\end{aligned}
$$

by adjunction. What we are using from [7] is essentially just the fact that the normal bundle to Δ at a curve C_0 with a unique node P is essentially $T_1 \otimes T_2$, where T_1 and T_2 are the tangent spaces to the branches of C_0 at P.

Type β. Now in the above situation suppose there are given two transverse multisections A_1, A_2. Base-changing with respect to $A_1 \to B$, then $A_1 \times_B A_2 \to A_1$, we obtain a family with two sections meeting transversely; blowing up the intersection of the two sections, we obtain a family, say $\tilde{\pi}: \tilde{\mathscr{Y}} \to \tilde{B}$, with two *disjoint* sections $\tilde{A}_1, \tilde{A}_2$. Gluing these together, we obtain a pencil

$$\beta(F, A_1, A_2) \subset \Delta_0 \subset \overline{M}_{i+1}.$$

Putting $m_i = \deg(A_i \to B)$, $i = 1, 2$, we have, again by [7],

$$
\begin{aligned}
&\beta(F, A_1, A_2) \cdot_{\overline{M}_{i+1}} \lambda = m_1 m_2 F \cdot_{\overline{M}_i} \lambda, \\
&\beta(F, A_1, A_2) \cdot_{\overline{M}_{i+1}} \delta_j = F \cdot \delta_j, \qquad j \neq 0, 1, i, \\
&\beta(F, A_1, A_2) \cdot \delta_1 = A_1 \cdot A_2, \\
(1.3)\quad &\beta(F, A_1, A_2) \cdot_{\overline{M}_{i+1}} \delta_0 = m_1 m_2 F \cdot \delta_0 \\
&\qquad + \left[\sum_{l=1}^{2} m_l(2h-2) - (2g(A_l) - 2) + A_l \cdot A_l\right] \\
&\qquad - A_1 \cdot A_2.
\end{aligned}
$$

Remark 1.4. Note that by construction, if F is nef on $\overline{M}$ and A contains a general point of a general fiber Y_b, then $\alpha(F, A, g)$ is nef on $\Delta_{i,g}$, and similarly for β.

2. Construction

In this section we will give the specific construction which, together with the methods of §1, will prove Theorem 0.1. For simplicity of exposition, we will concentrate on the case of genus 16; in fact, as regards proof, the other cases are essentially special cases of this.

In outline, the argument goes as follows. We construct for each $i = 1, \cdots, 8, 15$ a pencil $F_i \subset \overline{M}_i$ and a section A_i (for $i = 15$, we construct two sections $A_{1,15}$ and $A_{2,15}$) and compute the appropriate intersection numbers. Then we may define

$$
\begin{aligned}
F_{i,16} &= \alpha(F_i, A_i, 16), \qquad i = 1, \cdots, 8, \\
F_{0,16} &= \beta(F_{15}, A_{1,15}, A_{2,15}).
\end{aligned} \tag{2.1}
$$

By construction, the F_i are nef; they even fill up their $\overline{M}_i$ for $i \leq 8$. Using Remark 1.4, the fact that each $F_{i,16} \cdot \delta > 0$ (by construction), and Principle 1.2, we may conclude that $\{F_{0,16}, \cdots, F_{8,15}\}$ forms a nef collection of pencils on $\overline{M}_{16}$. We may then apply Principle 1.3 to get a lower bound on s_{16}.

i	Type of pencil F_i	# of base points	$\lambda \cdot F_i$	$\delta \cdot F_i$	$s(F_i)$	$s(F_{i,16})$
1	Plane cubic	9	1	12	12	11
2	Type $(2, 3)$ on quadric	12	2	20	10	9.5
3	Plane quartic	16	3	27	9	8.6
4	Type (3.3) on quadric	18	4	34	8.5	8.25
5	Quadric section on del Pezzo quartic	16	5	39	7.8	7.6

TABLE 2

Next, we construct $F_6, \cdots, F_9$ (where F_9 is not needed for the bound on s_{16} but yields the bound on s_9). Here we take in each case a pencil of nodal plan curves of degree d with assigned nodes in generic position, plus the appropriate number of assigned simple base points in generic position to make a pencil, and then some unassigned base points. It is classical in these cases that by blowing up the base points we get a pencil F_i of stable curves filling up $\overline{M}_i$, and we take as A_i the section corresponding to a simple base point.

i	d	# of nodes	# of simple base points	$F_i \cdot \delta$	$F_i \cdot \lambda$	$s(F_i)$	$s(F_{i,16})$
6	6	4	20	47	6	7.83	7.66
7	7	8	17	52	7	7.43	7.28
8	8	13	12	56	8	7	6.875
9	8	12	16	63	9	7	6.88

TABLE 3

Finally we consider the one nonclassical case $i = 15$. Here the appropriate pencil F_{15} was constructed in [1], [3], and its α-lift was considered in [2]; incidentally, this α-lift $\alpha(F_{15}, A, 16)$ may be used in place of $F_{1,16}$ constructed above. The case of the β-lift is quite similar and so we will just sketch it. There is a family

$$\begin{array}{ll} \mathscr{Y} & \subset \mathbb{P}^1 \times \mathbb{P}^3 \\ \pi \downarrow & \\ \mathbb{P}^1 & \end{array}$$

of stable, irreducible automorphism-free curves of genus 15, whose degree in $\mathbb{P}^3$ is 14. For the corresponding pencil F_{15} in $\overline{M}_{15}$ we have

$$F_{15} \cdot \lambda = 318, \qquad F_{15} \cdot \delta = F_{15} \cdot \delta_0 = 2120,$$

and moreover F_{15} fills up a divisor $D \subset \overline{M}_{15}$ such that $F_{15} \cdot D > 0$, so that F_{15} is nef in $\overline{M}_{15}$.

As our multisections A_1, A_2 we take the pullbacks of two generic planes $H_1, H_2 \subset \mathbb{P}^3$. As $\mathscr{Y}$ projects birationally to a surface of degree 16 in $\mathbb{P}^3$, we have

$$\begin{aligned} &A_1^2 = A_2^2 = A_1 \cdot A_2 = 16, \\ &g(A_i) \le \frac{15 \cdot 14}{2} = 105, \qquad i = 1, 2, \\ &A_i \cdot Y_6 = 14, \qquad i = 1, 2. \end{aligned}$$

Setting $F_{0,15} = \beta(F_{15}, A_1, A_2)$ and plugging into the formulas, we compute that

$$\begin{aligned} &F_{0,16} \cdot \lambda = 14^2 \cdot 318, \\ &F_{0,16} \cdot \delta = 2120 \cdot 14^2 - 2(14 \cdot 220 + 16) + 16, \\ &s(F_{0,16}) = 6.5670, \end{aligned}$$

yielding the bound on s_{16}.

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