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ON THE ENTROPY ESTIMATE FOR THE RICCI FLOW ON COMPACT 2-ORBIFOLDS

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1. Introduction

In [3], Richard Hamilton began the study of the following equation, which we refer to as *Hamilton's Ricci flow*:

$$(*) \qquad \frac{\partial g}{\partial t}(x, t) = (r - R(x, t))g(x, t), \qquad x \in M, \ t > 0.$$

Here g is the metric, R is the scalar curvature of g (=twice the Gaussian curvature K), and r is the average of R. Based on the work of Hamilton [3], and the subsequent extensions by the author [1] and Lang-Fang Wu [5], Wu and the author [2] recently proved the following, which was conjectured by Hamilton.

Theorem 1.1. *If (M, g) is a compact 2-dimensional Riemannian orbifold, then under Hamilton's Ricci flow, g approaches asymptotically a Ricci soliton.*

We say that $\{g_t\}$ is a Ricci soliton if there exist diffeomorphisms $\{\varphi_t\}$ of M such that $g_t = \varphi_t(g_0)$.

The purpose of this note is to give a simple derivation of the entropy estimates which were used in the papers quoted above. This proof is based on an identity, and unlike the previous proofs, it does not use the fact that the solution to Hamilton's Ricci flow exists for all time.

2. The entropy estimates

Let (M, g) be a compact Riemannian 2-orbifold with positive Euler characteristic. In [3, §7], Hamilton showed that, provided the scalar curvature R of g is positive, the entropy $-N(t)$ is increasing under $(*)$, where

$$N(t) = \int_{M_t} R \log R \, dA.$$

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He proved this by showing that if the time derivative of N is ever positive, it will blow up in finite time, which contradicts the fact that the solution exists for all time. Thus $dN/dt \leq 0$.

Here we show that the monotonicity of the entropy actually follows from an identity obtained from integration by parts. Moreover, we can extend this identity to the case where the curvature is negative somewhere and prove that the (modified) entropy is bounded independent of time (see Lemma 2.3). This also gives a new proof of the entropy estimate in [2].

We first consider the case where g has positive curvature. The potential function f of the curvature is defined (up to a constant) by $\Delta f = R - r$ [3, §4]. Let $M_{ij} = \nabla_i \nabla_j f - \frac{1}{2}\Delta f g_{ij}$ [3, §9]. M_{ij} is a symmetric, trace-free 2-tensor. Moreover, the Ricci solitons are characterized by the condition $M_{ij} = 0$ [3, §10]. Now define the 1-form X by $X = \nabla R + R\nabla f$. It is easy to compute that $X_k = 2g^{ij}\nabla_i M_{jk}$. Therefore $X = 0$ for Ricci solitons.

The following identity shows that the entropy $-N$ is increasing.

Lemma 2.1.

$$\frac{dN}{dt} = -2\int |M_{ij}|^2 - \int \frac{|X|^2}{R} \leq 0.$$

Proof. Recall that $X = \nabla R + R\nabla f$. Expanding and integrating by parts gives

$$\int \frac{|X|^2}{R} = \int \frac{|\nabla R|^2}{R} - 2\int (R-r)^2 + \int R|\nabla f|^2.$$

However, a standard integration by parts yields

$$\int (\Delta f)^2 = \int |f_{ij}|^2 + \frac{1}{2}\int R|\nabla f|^2,$$

which implies

$$-2\int |M_{ij}|^2 = -\int (R-r)^2 + \int R|\nabla f|^2.$$

Therefore

$$\int \frac{|X|^2}{R} = \int \frac{|\nabla R|^2}{R} - \int (R-r)^2 - 2\int |M_{ij}|^2,$$

and the lemma follows from the equality

$$\frac{dN}{dt} = -\int \frac{|\nabla R|^2}{R} + \int (R-r)^2. \quad \text{q.e.d.}$$

As a consequence of the lemma, we get another proof of Hamilton's result that Ricci solitons are gradient solitons flowing along ∇f.

Corollary 2.2. *If we have* $dN/dt(t) = 0$ *for any time* $t \in [0, \infty)$, *then* $M_{ij} \equiv 0$.

Now we consider the case where g has negative curvature somewhere. We define $s(t)$ to solve the ordinary differential equation: $ds/dt = s(s-r)$ with initial value $s(0) < \min_{x \in M} R(x, 0)$. We define the (modified) entropy by

$$N(t) = \int_{M_t} (R-s)\log(R-s)\, dA.$$

We can no longer show that the entropy $-N(t)$ is increasing in time, but it is uniformly bounded from below.

Lemma 2.3. $N(t) \le C$.

Proof. Let $L = \log(R-s)$. Using the evolution equations (see [1, §2] and [4])

$$\frac{\partial L}{\partial t} = \Delta L + |\nabla L|^2 + R - r + s$$

and

$$\frac{\partial}{\partial t}[(R-s)dA] = [\Delta R + s(R-s)]dA,$$

we compute

$$\frac{dN}{dt} = \int \left(-\frac{|\nabla R|^2}{R-s} + R(R-r) + s(r-s) - sLR \right).$$

The equality

$$\int \frac{|X|^2}{R-s} + 2|M_{ij}|^2 = \int \left(\frac{|\nabla R|^2}{R-s} - R(R-r) - 2sL(R-r) + s|\nabla f|^2 + s^2 \frac{|\nabla f|^2}{R-s} \right)$$

implies

$$\frac{dN}{dt} = \int -\frac{|X|^2}{R-s} - 2|M_{ij}|^2 + \int \left(s(r-s) - 3sLR + 2srL + s|\nabla f|^2 + s^2 \frac{|\nabla f|^2}{R-s} \right).$$

We have $s|\nabla f|^2 \le 0$ and $|\int s(r-s) - 3sLR + 2srL| \le Cte^{-rt}$. Therefore the only bad (positive) term on the right-hand side of the equation for N is

$$s^2 \int \frac{|\nabla f|^2}{R-s} \le Ce^{-rt} \int |\nabla f|^2,$$

where we used the inequality $R - s \ge ce^{-rt} > 0$.

However, the potential function f satisfies (see [3, 4.2]) $\partial f/\partial t = \Delta f + rf$, and hence

$$\frac{d}{dt}\int f = r\int f + \int |\nabla f|^2.$$

This implies that, provided the solution exists until time T,

$$\int_0^T e^{-rt}\int_{M_t} |\nabla f|^2\, dA\, dt = \left(e^{-rt}\int_{M_t} f\, dA\right)\Big|_0^T \le C,$$

where the constant C is independent of T.

Therefore (minus) the entropy

$$N(T) \le N(0) + \int_0^T Cte^{-rt}\, dt + \int_0^T e^{-rt}\int_{M_t} |\nabla f|^2\, dA\, dt \le C$$

is bounded above independent of T. This completes the proof of 2.3.

References

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