EXAMPLES OF MANIFOLDS OF
POSITIVE RICCI CURVATURE

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The purpose of this paper is to present some new examples of simply connected Riemannian manifolds of dimension ≥ 7 with positive Ricci curvature, which admit no metric with nonnegative sectional curvature, in both compact and noncompact cases. The noncompact examples presented here are not of finite homotopy type, i.e., they are not homotopy equivalent to the interior of a compact manifold with boundary. This settles a long-standing conjecture (cf. [1], [18]).

It has been one of the major problems in Riemannian geometry to distinguish the topological implications of nonnegative sectional curvature on a manifold M from those of positive Ricci curvature. In the case of $\dim M = 3$, there is actually no difference (all 3-manifolds with positive Ricci curvature admit metric with positive sectional curvature) by the works of R. Hamilton, R. Schoen and S. T. Yau ([14], [15]). The situation of higher dimensions is different. In the compact case, although it is generally believed that a simply connected manifold with positive Ricci curvature may not carry metric with nonnegative sectional curvature, examples were not known ([11], [18]). In the noncompact case the recent progresses are due to L. Berard Bergery, D. Gromoll and W. T. Meyer ([3], [11]). They constructed some examples of manifolds with positive Ricci curvature, which admit no metric with nonnegative sectional curvature. However, their examples are all of finite homotopy type. More recently, U. Abresch and D. Gromoll ([1]) proved that a complete manifold with positive Ricci curvature is of finite homotopy type under some diameter growth condition. This diameter growth condition turns out to be necessary as the examples in this paper show. Therefore the finiteness conjecture for complete Riemannian manifolds with positive Ricci curvature is not true in general. (Compare [6], [7], [10], [12].)

It is a beautiful theorem of M. Gromov [12] that for each positive integer n there is a constant C_n which only depends on n such that the total Betti

number of any n-dimensional complete manifold of nonnegative sectional cur-
vature is less than C_n. Our idea, as one can expect, is to construct complete
positively Ricci curved manifolds with arbitrarily large total Betti numbers.
This also shows that the same conclusion of Gromov's theorem is false for
positively Ricci curved manifolds in general.

Our basic construction, roughly speaking, is as follows. We start with the
product $S^4 \times S^3$ (here S^n, as usual, denotes the standard n-sphere). Pick any
point $q_0 \in S^4$, and let B_{q_0} be an arbitrarily small geodesic ball around q_0 in
S^4. We then remove $B_{q_0} \times S^3$ from $S^4 \times S^3$, and replace it by $S^3 \times D$ where D
is the unit disk bundle of the canonical complex line bundle over $\mathbf{C}P^1$, which
can also be viewed as the 2-dimensional complex projective space $\mathbf{C}P^2$ with
a ball removed (note that the boundary of $S^3 \times D$ is also $S^3 \times S^3$). We show
that, with each factor of $S^4 \times S^3$ properly scaled, the resulting manifold carries
a metric of positive Ricci curvature, and in fact this metric on the complement
of $S^3 \times D$, $(S^4 \backslash B_{q_0}) \times S^3$ is still the original scaled product metric. Since one
can pick as many points on S^4 as one likes, the desired examples are obtained.
These examples can be easily adapted to give complete noncompact examples
of manifolds of positive Ricci curvature with infinite topological type. The
higher dimensional examples will then be gotten, for example, by taking the
product with spheres.

Remarks. (a) It seems that the same sort of examples should exist in
dimension ≥ 4. But it turns out to be technically easier in dimension ≥ 7.

(b) The kind of surgery we are doing is only metrically partial local (we
have to scale the metrics on each factor of $S^4 \times S^3$ and alter the metric
on D substantially). This is an essential difference from the works of M.
Gromov and B. Lawson [13] on positive scalar curvature and Z. Gao and S.
T. Yau [9] on negative Ricci curvature where the surgeries are metrically local.
Metrically local surgery will not work for positively Ricci curved manifolds
due to the diameter theorem of Myers [5]. We will explore positive Ricci
curvature further by surgery in a forthcoming paper [17].

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1. Metrics on $\mathbf{C}P^2$ with a ball removed

In this section, we briefly review some facts concerning metrics on the
complex projective 2-space $\mathbf{C}P^2$. Let $z = (z_1, z_2, z_3)$ be the standard homo-
geneous coordinates of $\mathbf{C}P^2$. The Fubini-Study metric on $\mathbf{C}P^2$ is by definition

$$(1) \qquad\qquad ds^2 = 4\frac{|z \wedge dz|^2}{|z|^4}.$$

It is well known that the metric (1) is of positive sectional curvature, and $\mathbf{C}P^2$ with the metric (1) is a symmetric space. Let $p_0 \in \mathbf{C}P^2$ and $B_{p_0} = \{p \in \mathbf{C}P^2 : d(p, p_0) < \pi/4\}$, where $d(p, p_0)$ denotes the distance between p and p_0 with respect to the metric (1). Then $D \equiv \mathbf{C}P^2 \backslash B_{p_0}$ is a manifold with boundary $\partial D = S^3$. A little computation shows that the metric (1) restricted to D can also be described as follows.

We use the same notation S^3 to denote the S^3 with the metric of constant curvature 1, and view it as a fiber bundle over $\mathbf{C}P^1 = S^2$ under the standard Hopf fibration. Then, topologically, D is the quotient space of $I \times S^3$, where I denotes the unit interval $[0,1]$, by collapsing each fiber in S^3 to a point at the left end, $\{0\} \times S^3$. The metric (1) restricted to $(0,1] \times S^3$, up to a constant scaling, is given by the following, with respect to the natural splitting $T(I \times S^3) = \{R\partial/\partial r\} \oplus TS^3$, where $r \in (0,1]$:

$$(2) \qquad\qquad ds^2 = dr^2 + ds_r^2,$$

where ds_r^2 is a metric on S^3 obtained by rescaling the standard metric on the fiber direction with factor $\sin^2 r \cos^2 r$ and on its orthogonal complement with factor $\cos^2 r$.

It can be easily verified that the two functions $\sin^2 r \cos^2 r$ and $\cos^2 r$ can be replaced by any two functions $g(r)^2$ and $h(r)^2$ provided that $g(0) = 0$, $g'(0) = 1$, $g''(0) = 0$, $h(0) \neq 0$, $h'(0) = 0$ and $g(r) > 0$, $h(r) > 0$ for $r > 0$. Then (2) will also give a well-defined smooth metric on D. One can say nothing about the curvature of this metric in general, of course, but there are some remarkable observations:

(i) With g, h properly chosen, one can show that the connected sum of two $\mathbf{C}P^2$'s admits a metric of nonnegative sectional curvature. This was an example of J. Cheeger in [4].

(ii) With g, h properly chosen, one can show that D carries a metric of positive Ricci curvature and convex boundary. But D does not admit a metric of nonnegative sectional curvature such that the boundary is convex (cf., e.g. [16]).

In fact, it is in some sense a stronger form of the last observation which gives us the basic block to build up the examples of this paper.

2. Surgery on $S^4 \times S^3$

Denote by S_m^4 the 4-sphere with metric of constant curvature m^{-2} ($m > 1$), and by S_δ^3 the 3-sphere with metric of constant curvature δ^{-2}. Let $S_m^4 \times S_\delta^3$ be with the product metric (it is clearly of positive Ricci curvature).

Let $q_0 \in S_m^4$, $B_{q_0} = \{q \in S_m^4 : d(q, q_0) \le 1\}$, and let $r = d(q, q_0)$ be the radius coordinate of the polar coordinates on B_{q_0}. Then, topologically,

$$(B_{q_0} \backslash \{q_0\}) \times S_\delta^3 = (0, 1] \times S^3 \times S^3.$$

We now form a manifold M as

$$M = ((S_m^4 \backslash \{q_0\}) \times S_\delta^3) \cup (D \times S^3) / \sim,$$

where the equivalence relation "$\sim$" identifies the point (r, p, q) on $(0, 1] \times S^3 \times S^3$ in $D \times S^3$ (see discussion in §1) with the point (r, q, p) on $(0, 1] \times S^3 \times S^3$ in $(S_m^4 \backslash \{q_0\}) \times S_\delta^3$.

It is easy to see that M is indeed a smooth manifold. By the discussion in §1, $dr^2 + ds_r^2 + f(r)^2 \, d\bar{s}^2$, where $d\bar{s}^2$ denotes the metric of constant curvature 1 on s^3, gives a well-defined metric on $D \times S^3$ if the function $f(r) > 0$ satisfies $f'(0) = 0$. One sees immediately that this metric gives a well-defined metric on M if $f(r)$, $g(r)$, $h(r)$ satisfy the following additional conditions: $f(r) = m \sin r/m$ and $g(r) = h(r) = \delta$ for $r \ge 1$. Off the set $D \times S^3$, i.e., on $(S_m^4 \backslash B_{q_0}) \times S_\delta^3$, the metric is still the original product metric, in particular, it is of positive Ricci curvature. We will show in the next section that, with δ, $f(r)$, $g(r)$ and $h(r)$ properly chosen, this metric is of positive Ricci curvature on the entire M.

From the discussion above, our surgery is actually "half local". We can pick as many disjoint unit balls on S_m^4 as we like to do the same surgery in an obvious way provided m is sufficiently large. A direct argument using the Mayer-Vietoris exact sequence shows that the total Betti number of the resulting manifold M can be as large as we like. Note also that M is clearly still simply connected.

3. Computations of the Ricci curvature

By the discussion in §§1 and 2, we only need to compute the curvature on $D \times S^3$ which takes the form

(3) $$ds^2 = dr^2 + ds_r^r + f(r)^2 \, d\bar{s}^2$$

on $(0, 1] \times S^3 \times S^3$.

The metric (3) is clearly invariant under the action of $U(2) \times SO(4)$. Therefore it suffices to calculate the curvature at one point for each r. We use the coordinates (r, p, q), where

$$p = (\cos \varphi e^{i(\psi + \theta)}, \sin \varphi e^{i(-\psi + \theta)}),$$
$$q = (\cos \alpha e^{i(\beta + \gamma)}, \sin \alpha e^{i(-\beta + \gamma)}).$$

Then

$$ds^2 = dr^2 + h(r)^2\, d\varphi^2 + (h(r)^2 \sin^2 2\varphi + g(r)^2 \cos^2 2\varphi)\, d\psi^2 + g(r)^2\, d\theta^2$$
$$+ 2g(r)^2 \cos 2\varphi\, d\psi\, d\theta + f(r)^2 (d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos 2\alpha\, d\beta\, d\gamma).$$

We have orthogonal vector fields

$$X_0 = \frac{\partial}{\partial r}, \quad X_1 \frac{\partial}{\partial \theta}, \quad X_2 = \frac{\partial}{\partial \varphi}, \quad X_3 = \frac{\partial}{\partial \psi} - \cos 2\varphi \frac{\partial}{\partial \theta},$$

$$X_4 = \frac{\partial}{\partial \alpha}, \quad X_5 = \frac{\partial}{\partial \beta} - \cos 2\alpha \frac{\partial}{\partial \gamma}, \quad X_6 = \frac{\partial}{\partial \gamma}.$$

Set $h(r) \equiv \delta$. The value of δ will be decided later.

Denote by R_{ij} the Ricci tensor valued at $(X_i/\|X_i\|, X_j/\|X_j\|)$. Use formulas

$$\langle R(X,Y)Z,W \rangle = X \langle \nabla_Y Z, W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle - Y \langle \nabla_X Z, W \rangle$$
$$+ \langle \nabla_X Z, \nabla_Y W \rangle - \langle \nabla_{[X,Y]} Z, W \rangle,$$

$$2 \langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle + \langle [X,Y], Z \rangle$$
$$- \langle [X,Z], Y \rangle - \langle [Y,Z], X \rangle.$$

Direct calculation gives

(4)
$$R_{00} = -\frac{g''(r)}{g(r)} - \frac{3f''(r)}{f(r)},$$

(5)
$$R_{11} = -\frac{g''(r)}{g(r)} + \frac{2g(r)^2}{\delta^4} - \frac{3f'(r)g'(r)}{f(r)g(r)},$$

(6)
$$R_{22} = R_{33} = \frac{4\delta^2 - 2g(r)^2}{\delta^4},$$

(7)
$$R_{44} = R_{55} = R_{66} = \frac{2}{f(r)^2} - \frac{2f'(r)^2}{f(r)^2} - \frac{f''(r)}{f(r)} - \frac{f'(r)g'(r)}{f(r)g(r)},$$
$$R_{ij} = 0 \quad \text{for } i \neq j.$$

Therefore, to show that the Ricci curvature is positive, we now show that one can choose $f(r)$, $g(r)$ and $\delta > 0$ properly so that (4)–(7) are positive.

Set $f_0(r) = m \sin(r/m)$. When $r > \frac{1}{2}$, we have

$$f_0''(r) \leq -\frac{1}{m} \sin \frac{1}{2m} < 0,$$

$$2 - 2f_0'(r)^2 - f_0(r)f_0''(r) \geq 3 \sin^2 \frac{1}{2m} > 0.$$

There exists f with

$$f(r) = f_0(r) \quad \text{for } r \geq 1,$$
$$f(r)f'(r) = f_0(r)f_0'(r) \quad \text{for } r \leq \tfrac{3}{4},$$
$$f(r)f'(r) \leq f_0(r)f_0'(r) \quad \text{for } \tfrac{3}{4} < r < 1,$$

where the strict inequality holds at least somewhere such that, when $f \geq \frac{1}{2}$,

$$f''(r) < 0 \quad \text{and} \quad 2 - 2f'(r)^2 - f(r)f''(r) > 0.$$

Then there exists $\delta_0 > 0$ (we can assume $\delta_0^2 < \frac{1}{6}$) such that, when $r \leq \frac{3}{4}$,

$$f(r)^2 = m^2 \in \sin^2 r/m + \delta_0^2,$$

$$\frac{f''(r)}{f(r)} = \frac{\delta_0^2 \cos^2(r/m) - \delta_0^2 \sin^2(r/m) - m^2 \sin^4(r/m)}{[m^2 \sin^2(r/m) + \delta_0^2]^2} \leq \frac{\delta_0^2}{(r^2 + \delta_0^2)^2}.$$

Set $g_0(r) = \delta_0 r/(r^2 + \delta_0^2)^{1/2}$. We have

$$g_0(0) = 0, \quad g_0'(0) = 1 \quad \text{and} \quad g_0''(0) = 0.$$

There exists g with

$$g(r) = g_0(r) \quad \text{for } r \leq \tfrac{1}{2},$$
$$g'(0) = 0 \quad \text{for } r \geq \tfrac{3}{4},$$
$$g''(0) \leq 0 \quad \text{and} \quad \frac{g'(r)}{g(r)} \leq \frac{g_0'(r)}{g_0(r)} \quad \text{for all } r.$$

Now set $\delta = g(1)$.

Clearly, the f, g, h defined above satisfy the conditions at $r = 0$, $r \geq 1$ mentioned in §§1 and 2. Note also that $\delta < \delta_0$.

Obviously, $4\delta^2 - 2g(r)^2 > 0$ always. When $r \leq \frac{1}{2}$,

$$-\frac{g''(r)}{g(r)} - \frac{3f''(r)}{f(r)} \geq \frac{3\delta_0^2}{(r^2 + \delta_0^2)^2} - \frac{3\delta_0^2}{(r^2 + \delta_0^2)^2} = 0,$$

$$-\frac{g''(r)}{g(r)} - \frac{3f'(r)g'(r)}{f(r)g(r)} = \frac{3\delta_0^2}{(r^2 + \delta_0^2)^2} - \frac{3m\sin(r/m)\cos(r/m)}{m^2 \sin^2(r/m) + \delta_0^2} \cdot \frac{\delta_0^2}{(r^2 + \delta_0^2)r}$$

$$= \frac{3\delta_0^2}{r(r^2 + \delta_0^2)^2[(m^2 \sin^2(r/m) + \delta_0^2]}$$
$$\times [r(m^2 \sin^2 \frac{r}{m} + \delta_0^2) - m(r^2 + \delta_0^2)\sin \frac{r}{m} \cos \frac{r}{m}] \geq 0.$$

When $r \leq \frac{3}{4}$,

$$2 - 2f'(r)^2 - f(r)f''(r) - f(r)f'(r)\frac{g'(r)}{g(r)}$$

$$\geq 4\sin^2 \frac{r}{m} + \cos^2 \frac{r}{m} - \sin^2 \frac{r}{m} - \frac{m^2 \sin(r/m)\cos(r/m)}{m^2 \sin^2(r/m) + \delta_0^2}$$

$$- m\sin \frac{r}{m} \cos \frac{r}{m} \cdot \frac{\delta_0^2}{(r^2 + \delta_0^2)r}$$

$$\geq 2\sin^2 \frac{r}{m} + \frac{\delta_0^2}{r^2 + \delta_0^2} - \frac{\delta_0^2}{r^2 + \delta_0^2} \geq 0.$$

When $\frac{1}{2} \leq r \leq \frac{3}{4}$,

$$-\frac{g''(r)}{g(r)} - \frac{3f''(r)}{f(r)} > 0,$$

$$-\frac{g''(r)}{g(r)} + \frac{2g(r)^2}{\delta^4} - \frac{3f'(r)g'(r)}{f(r)g(r)} > \frac{2g_0(1/2)^2}{\delta_0^4} - \frac{3\delta_0^2}{[(1/2)^2 + \delta_0^2]^2}$$

$$> \frac{1}{(1/2)^2 + \delta_0^2}\left(\frac{1}{2\delta_0^2} - 3\right) > 0.$$

When $r \geq \frac{3}{4}$, it is easy to see that (4)–(7) are positive. We are done.

Note. In the calculation above, the Ricci curvature appears to be 0 somewhere (only at $r = 0$). It does not matter due to a theorem of T. Aubin and P. Ehrlich ([2], [8]) which states that if a complete metric is of nonnegative Ricci curvature and the Ricci curvature is positive at least at one point, then the metric can be deformed to a complete metric of positive Ricci curvature.

4. Noncompact examples

We first summarize the basic results of the construction in the previous sections into the following lemma. Denote by S_R^4 the 4-sphere of constant curvature R^{-2}, and by S^3 the 3-sphere of constant curvature 1.

Lemma.   *For any $N > 1$, there exists $R_0 = R_0(N) > 0$ such that for each $R \geq R_0$ the following is true. If one removes $B(q_0, R/N) \times S^3$ from $S_R^4 \times S^3$ and replaces it by $S^3 \times D$, where $q_0 \in S_0^4$, $B(q_0, R/N)$ is the geodesic ball of radius R/N around q_0 and D is as in §1, then the resulting manifold carries a metric of nonnegative Ricci curvature. This metric, on the complement of $S^3 \times D$, i.e., $(S_R^4 \backslash B(q_0, R/N)) \times S^3$, is the product metric induced from $S_R^4 \times S^3$.*

Proof.   This can be easily seen from §§2 and 3 by taking $m = N$ and then $R_0 = m/\delta$, where δ is from the computations in §3.   q.e.d.

The construction of the noncompact examples is roughly as follows. We first construct a 4-dimensional parabola such that there are a sequence of points $\{q_k \colon k = 1, 2, \cdots\}$, which goes to infinity, and a corresponding sequence of disjoint geodesic balls $\{B_{q_k} \colon k = 1, 2, \cdots\}$ where B_{q_k} is of radius r_k around q_k with the following property: B_{q_k} is isomorphic to a geodesic ball on the 4-sphere of constant curvature R_k^{-2} where $R_k \geq R_0$ is as in the Lemma with respect to $N = R_k/r_k$. We then take the product of this parabola with S^3. By the Lemma, we can do the surgery as we did in the previous sections around $\{q_k\} \times S^3$; the desired example then follows.

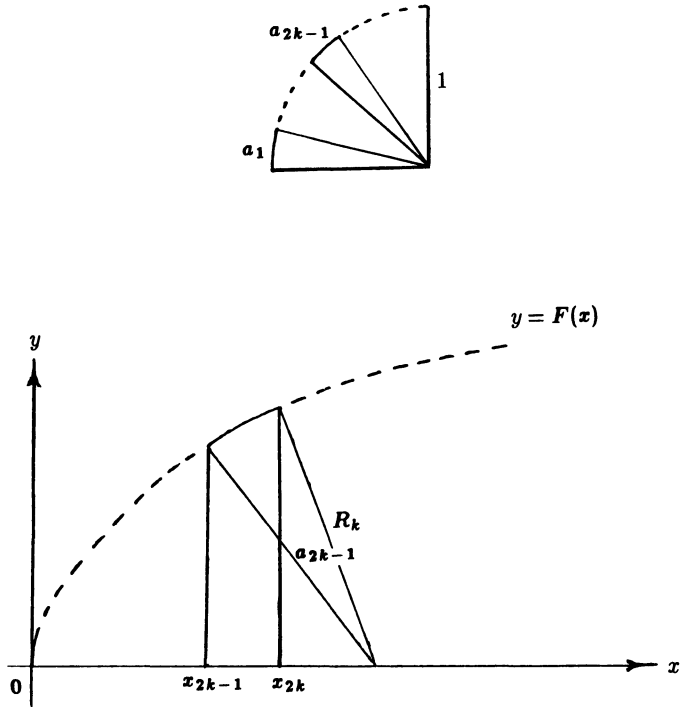This process can really go through as we show in detail for example as follows.

FIGURE 1

Take a sequence of positive numbers $\{a_k > 0\colon k = 1, 2, \cdots\}$ such that

$$\sum_{k=1}^{\infty} a_k = \frac{\pi}{2}.$$

Let $\{R_k > 0, k = 1, 2, \cdots, \}$ be a sequence of positive numbers where $R_k > R_0$ is as in the Lemma with respect to $N = N_k = 1/a_{2k-1}$.

It is easy to see that, with the R_k's properly chosen, one can construct a parabolic curve $y = F(x)$ $(F''(x) < 0)$ in the xy-plane such that there is a sequence of numbers $0 = x_1 < x_2 < \cdots$, when restricted to $x_{2k-1} \leq x \leq x_{2k}$, the curve is a portion of a circle of radius R_k centered on x-axis with length $> R_k/N_k$ (see Figure 1).

Put this curve canonically into 5-dimensional euclidian space and rotate it around the x-axis. We then get a parabola which clearly satisfies the property mentioned above.

Remark. It can be estimated that the example constructed above has diameter growth of order $O(x^{2/3})$ and bounded sectional curvature. (Compare [1]).

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