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ON THE DIFFERENTIABILITY OF HOROCYCLES AND HOROCYCLE FOLIATIONS

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Consider a surface S with a complete C^∞-metric of nonpositive curvature and let $\tilde{S}$ be the universal cover of S. Denote by γ_v the geodesic with initial tangent vector v. Unit tangent vectors v and w of $\tilde{S}$ are asymptotic if $\operatorname{dist}(\gamma_v(t), \gamma_w(t))$ is bounded as $t \to \infty$. Unit vectors of S are asymptotic if they have asymptotic lifts to $\tilde{S}$.

For a unit vector $v \in T_1\tilde{S}$ define the Busemann function $b_v\colon \tilde{S} \to \mathbb{R}$ by

$$b_v(q) = \lim_{t \to \infty} \big(\operatorname{dist}(\gamma_v(t), q) - t\big).$$

This function is differentiable and $-(\operatorname{grad} b_v)(q)$ is the unique vector at q asymptotic to v. The horocycle $h(v)$ determined by v is the level set $b_v^{-1}(0)$. Clearly $h(v)$ is the limit as $R \to \infty$ of the geodesic circles of radius R centered at $\gamma_v(R)$. Let $W(v)$ be the set of vectors w asymptotic to v with footpoints on $h(v)$, i.e.

$$W(v) = \{-\operatorname{grad} b_v(q) \colon q \in h(v)\}.$$

The curves $W(v)$, $v \in T_1\tilde{S}$, are the leaves of the horocycle foliation W of $T_1\tilde{S}$. We project the horocycles from $\tilde{S}$ into S to obtain horocycles for vectors in T_1S. Similarly we obtain the horocycle foliation of T_1S again denoted by W.

An important step in E. Hopf's proof of the ergodicity of the geodesic flow on a compact surface S of variable negative curvature was to show that the horocycle foliation of T_1S is C^1. He actually proved [6] that the horocycle foliation is C^1 under the weaker assumption that the curvature of S has bounded derivative and is uniformly bounded away from 0 and $-\infty$. An immediate consequence is that the horocycles and Busemann functions in $\tilde{S}$

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are C^2. In fact, since the sets $W(v)$ are the stable manifolds of the geodesic flow, they and hence also the horocycles and Busemann functions are C^∞. P. Eberlein showed that horocycles and Busemann functions are C^2 on any complete simply connected surface of nonpositive curvature (see [5]). It follows easily that $W(v)$ is a C^1-submanifold of the unit tangent bundle which depends continuously on v in the C^1-topology. In this paper we construct two examples which show that the above results are in a certain sense sharp.

The first example (see §1) is an analytic rotationally invariant metric of nonpositive curvature on the cylinder $S^1 \times \mathbb{R}$ such that $\gamma = S^1 \times \{0\}$ is the only closed geodesic. The curvature along γ vanishes and is negative elsewhere. We show in Theorem 1.1 that any horocycle h perpendicular to γ is not three times differentiable where it crosses γ. Note that $h \cap (S^1 \times (-\varepsilon, \varepsilon))$ is completely determined by the geometry of $S^1 \times (-\varepsilon, \varepsilon) = U$. It is easy to construct a compact smooth surface of nonpositive curvature containing a closed geodesic γ' with a neighborhood isometric to U_ε. We see that a horocycle in this surface is not three times differentiable at a point where it intersects γ' orthogonally. Ya. Pesin (Lemma 2 in [8]) claimed that the horocycles in such a surface would be C^{r-2} if the surface were C^r. The above example shows that this fails for $r \geqslant 5$.

The second example (see §2) provides complete surfaces of finite volume and pinched negative curvature for which the horocycle foliation is not differentiable or even Hölder continuous. More precisely, let $k(v) < 0$ denote the geodesic curvature of the horocycle $h(v)$ at the footpoint of v. For any modulus of continuity $m(\cdot)$ (see Definition 2.1), we construct a smooth family of complete metrics g_ε, $\varepsilon \geqslant 0$, on the torus with one puncture such that the volume of g_ε is finite, the curvature of g_ε is pinched between $-1 - \varepsilon$ and $-1 + \varepsilon$, and $g_\varepsilon = g_0$ outside a fixed neighborhood D of the puncture. In Theorem 2.2 we show that there is a unit vector v_0 with footpoint outside D such that, for every $m(\cdot)$ and $\varepsilon > 0$, the function k has modulus of continuity worse than m at v_0. A similar construction works on a surface with any number of cusps.

For other results related to the differentiability of horocycles and horocycle foliations see [1], [4], [7], [9].

1. Let $S = S^1 \times \mathbb{R}$ be the cylinder with the natural coordinates $s \in S^1$, $t \in \mathbb{R}$. For any $a > 0$ set $Y(t) = 1 + at^4$. Equip S with the analytic metric

$$g(s,t) = \begin{pmatrix} Y^2(t) & 0 \\ 0 & 1 \end{pmatrix}.$$

Then (S, g) is a surface of revolution with the curves $s = \text{const}$ as meridian geodesics. The curve γ: $s \to (s, 0)$ is a closed unit speed geodesic. The Gaussian curvature is given by

$$K(s,t) = -\frac{Y''(t)}{Y(t)} = -\frac{12at^2}{1+at^4}.$$

Note that the curvature is negative except on γ, where it vanishes. Fix an orientation for γ and let V be the field of unit vectors negatively asymptotic to γ.

1.1. Theorem. (i) *The vector field V has no second derivatives in the t-direction at any point on γ.*

(ii) *Let $b(\cdot)$ be a Busemann function in the universal cover of S determined by the lift $\tilde{\gamma}$ of γ. Then b has no third derivative in the t-direction at any point of $\tilde{\gamma}$.*

(iii) *Any horocycle in S orthogonal to the geodesic γ has no third derivative at the point where it intersects γ.*

Proof. Assertions (ii) and (iii) follow easily from (i) which we now prove.

By the rotational symmetry, the oriented angle between $(\partial/\partial s)(s,t)$ and $V(s,t)$ does not depend on s. We denote it by $\alpha(t)$. Since $K < 0$ except on γ we have with proper orientation that $t \cdot \alpha(t) > 0$ for $t \neq 0$.

Let $\sigma(\tau) = (s(\tau), t(\tau))$ be a geodesic in S negatively asymptotic to γ. The Killing field $Y(t) \cdot \partial/\partial s$ gives rise to the Clairaut integral

$$\langle \dot{\sigma}(\tau), Y(t(\tau)) \cdot \partial/\partial s \rangle = Y(t) \cdot \cos\alpha(t) \equiv \text{const}.$$

However,

$$\lim_{\tau \to -\infty} \operatorname{dist}(\dot{\sigma}(t), \dot{\gamma}(\tau)) = 0,$$

for otherwise γ would bound a flat strip (cf. Proposition 5.1 in [3]). Hence

$$Y(t) \cdot \cos\alpha(t) \equiv 1.$$

Since the function α is odd,

$$\alpha(t) = \operatorname{sign} t \cdot \arccos\frac{1}{1+at^4}, \qquad t \neq 0,$$

and $\alpha(0) = 0$. A simple calculation shows that $\alpha'(0) = 0$ and

$$\lim_{t \searrow 0} \frac{\alpha'(t)}{t} = \sqrt{8a} \quad \text{and} \quad \lim_{t \nearrow 0} \frac{\alpha'(t)}{t} = -\sqrt{8a}.$$

In particular, α has no second derivative at 0, which proves assertion (i).

2. We start with an explicit construction of a hyperbolic metric on the punctured torus. Consider the region R in the hyperbolic plane H shown shaded in Figure 1. It is bounded by the vertical geodesics passing through the points 0 and $1/2$ and by the circles of radius $\sqrt{2}/8$ centered at the points 0,

$1/4$, and $1/2$. Let R' be the reflection of R with respect to the imaginary axis. Identify the geodesics bounding $R \cup R'$ as indicated in Figure 1 to obtain a hyperbolic surface with one cusp and two boundary circles. Now glue together the boundary circles. This produces a hyperbolic surface (S, g) diffeomorphic to a punctured torus. The horizontal line passing through the point i in Figure 1 gives rise to a horocycle h of length 1 in S which bounds the cusp D (see Figure 2).

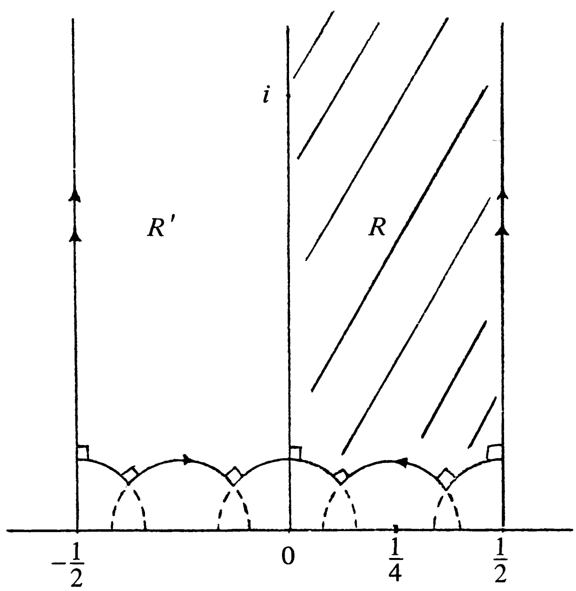

FIGURE 1

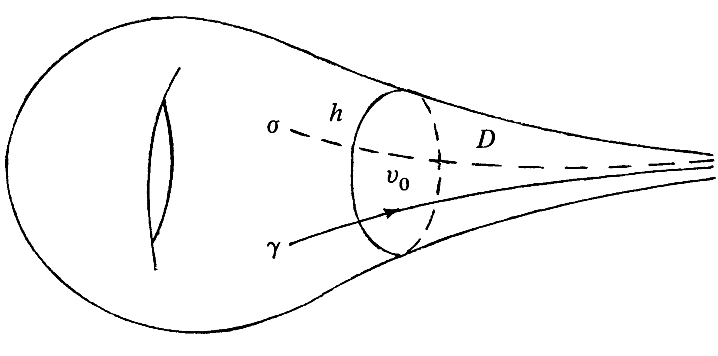

FIGURE 2

2.1. Definition. If X is a metric space and $f\colon X \to \mathbb{R}$ is a continuous function, then the modulus of continuity of f at $x \in X$ is defined by

$$m_{f,x}(\delta) = \sup\{|f(x) - f(x')|: \operatorname{dist}(x, x') < \delta\}.$$

2.2. Theorem. *Let (S, g) be the hyperbolic surface constructed above and let v_0 be a unit normal to the horocycle h that points into D. Suppose $m\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone function for which $m(\delta) \to 0$ as $\delta \to 0$.*

Then there is a smooth 1-parameter family of C^∞-metrics g_ε, $0 \leqslant \varepsilon < 1/10$, on S such that $g_0 = g$ and for each ε:

(i) *$g_\varepsilon = g$ on the l-neighborhood of $S \setminus D$;*

(ii) *the curvature K_ε of g_ε satisfies*

$$-1 - \varepsilon \leqslant K_\varepsilon \leqslant -1 + \varepsilon;$$

(iii) *there is a smooth curve of unit vectors v_δ starting at v_0 such that $v_\delta \neq v_0$ for $\delta > 0$ and*

$$|k(v_\delta) - k(v_0)| \geqslant m(\operatorname{dist}(v_\delta, v_0)),$$

where $k(v)$ is the geodesic curvature of the horocycle $h(v)$ defined in the introduction.

Proof. Let γ be the geodesic ray with $\dot\gamma(-2) = v_0$ and σ be the geodesic ray opposite to γ in D (see Figure 2). Cut the cusp D along σ to obtain the region in the hyperbolic plane shown in Figure 3. The geodesic rays σ^- and σ^+ are asymptotic. Consider Fermi coordinates (s, t) along γ so that $(0,0) = \gamma(0)$, t is

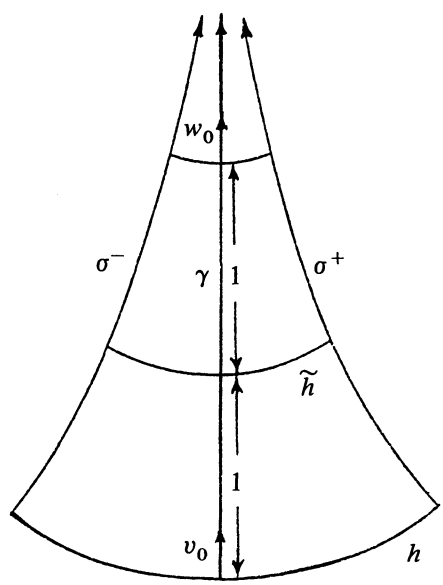

FIGURE 3

the arclength along γ, and the curves $t = \text{const}$ are unit speed geodesics perpendicular to γ.

Suppose $m(\delta)$ is defined if $0 \leqslant \delta < \delta_0$, $\delta_0 > 0$. By increasing $m(\delta)$ if necessary, we can assume without loss of generality that $m(\cdot)$ is C^∞ except at 0 and that $m(\delta) \geqslant \delta$. Let $M(\delta) = \sqrt{m(10\delta)}$. For $\delta \in (0, \delta_0]$ set

$$\varphi(\delta) = -\tfrac{1}{4}\ln M(\delta). \tag{2.1}$$

Note that φ is monotone and $\varphi(\delta) \to \infty$ as $\delta \to 0$. For $t \geqslant -\frac{1}{8}\ln M(\delta_0)$ define

$$f_0(t) = \tfrac{1}{3}\varphi^{-1}(2t)e^{t/2}.$$

Since $M(\delta) \geqslant \delta$, (2.1) implies that $f_0(t) \leqslant e^{-16t}/30$, $t \geqslant 0$, and hence f_0 decreases faster than the distance between γ and σ^+ or σ^-. Therefore there exists a monotone C^∞-function $f\colon \mathbb{R} \to (0,1)$ with the following properties:

(2.2) there is $t_0 > 0$ such that $f(t) = \frac{1}{3}\varphi^{-1}(2t)e^{t/2}$ for $t \geqslant t_0$:

(2.3) the points $(\pm f(t), t)$, $t \geqslant -1/2$, are in the region bounded by σ^-, σ^+ and the horocycle $\tilde{h}$ shown in Figure 3.

Let $q\colon \mathbb{R} \to [0,1]$ be a monotone C^∞-function such that $q(t) = 0$ if $t \leqslant -1/2$, and $q(t) = 1$ for $t \geqslant 0$. Choose an even C^∞-function $a\colon \mathbb{R} \to [0, 1/2]$ such that $a(0) = 0$, $a''(0) = 1$, $-1/2 \leqslant a''(x) \leqslant 1$ for all x, and $a(x) = 0$ if $|x| \geqslant 1$.

In the (s, t)-coordinates, the hyperbolic metric g is given by

$$g(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & \cosh^2 s \end{pmatrix}.$$

Consider the one-parameter family of metrics

$$g_\varepsilon(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & Y^2(s,t) \end{pmatrix},$$

where

$$Y(s,t) = \cosh s + \varepsilon \cdot q(t) \cdot a\left(\frac{s}{f(t)}\right) \cdot f^2(t).$$

Note that $g_0 = g$ and the curves $s \to (s, t)$ are unit speed geodesics with variation field Y for any $\varepsilon \geqslant 0$. By our choice of a, q, and f,

$$g_\varepsilon(s,t) = g_0(s,t) \quad \text{for } t \leqslant -1/2 \text{ or } |s| \geqslant f(t). \tag{2.4}$$

In particular, by (2.3), σ^- and σ^+ remain asymptotic geodesics and the metrics g_ε give rise to a one-parameter family of metrics on S which satisfies statement (i) of the theorem. Part (a) of the following lemma shows that (ii) holds.

2.3. Lemma. (a) $-1 - \varepsilon \leqslant K_\varepsilon(s,t) \leqslant -1 + \varepsilon$;

(b) $K_\varepsilon(0,t) = -1 - \varepsilon$ *for* $t \geqslant 0$.

Proof. By the Jacobi equation,

$$-K_\varepsilon(s,t) = \frac{1}{Y}\frac{\partial^2 Y}{\partial s^2} = \frac{\cosh s + \varepsilon \cdot q(t) \cdot a''(s/f(t))}{\cosh s + \varepsilon \cdot q(t) \cdot a(s/f(t)) \cdot f^2(t)}.$$

The left inequality in (a) holds, since

$$-K_\varepsilon(s,t) \leqslant \frac{\cosh s + \varepsilon \cdot q(t) \cdot a''(s/f(t))}{\cosh s}$$
$$\leqslant 1 + \varepsilon \cdot q(t) \cdot a''\left(\frac{s}{f(t)}\right).$$

Since $f(t) < 1$,

$$-K_\varepsilon(s,t) \geqslant \frac{\cosh s - \varepsilon/2}{\cosh s + \varepsilon/2} \geqslant 1 - \varepsilon.$$

This proves (a). To prove (b) note that $a(0) = 0$ and $a''(0) = 1$. q.e.d.

Let w_δ be the unit vector with footpoint at $\gamma(0) = (0,0)$ which makes the angle $\delta > 0$ with $\dot\gamma(0) = w_0$ (see Figure 4). Denote by γ_δ the geodesic with initial velocity w_δ. Let $\gamma_\delta(\tau(\delta))$ and $\gamma_\delta(T(\delta))$ be the points where γ_δ intersects the curves $s = f(t)$ and σ^+ respectively.

2.4. Lemma. *Suppose the right triangle shown in Figure 5 lies in a simply connected surface with curvature pinched between $-1-\varepsilon$ and $-1+\varepsilon$. If d and δ are small enough and t is large enough, then*

$$\tfrac{1}{3}e^{\sqrt{1-\varepsilon}\,t}\delta \leqslant d \leqslant 2e^{\sqrt{1+\varepsilon}\,t}\delta.$$

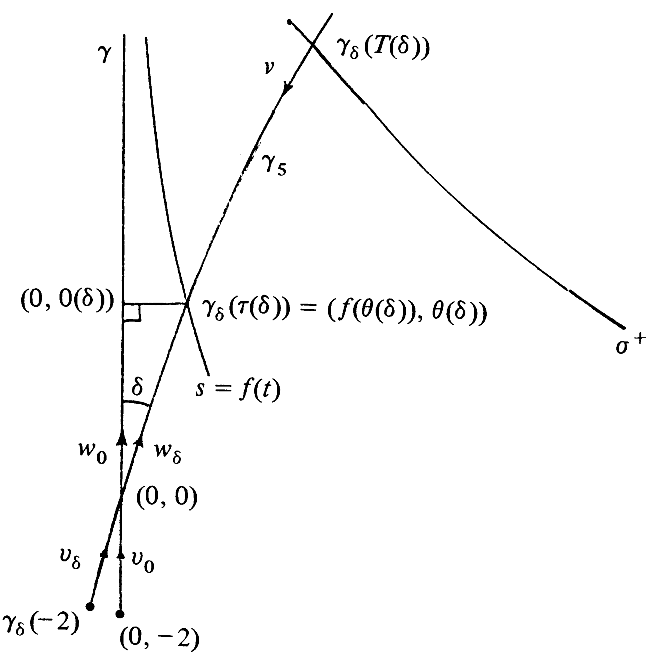

FIGURE 4

Proof. By comparing with the surfaces of constant curvature $-1 - \varepsilon$ and $-1 + \varepsilon$, we get

$$\tanh \sqrt{1-\varepsilon}\, d \geqslant \sinh \sqrt{1-\varepsilon}\, t \cdot \tan \delta,$$
$$\tanh \sqrt{1+\varepsilon}\, d \leqslant \sinh \sqrt{1+\varepsilon}\, t \cdot \tan \delta.$$

2.5. Lemma. *If δ is small enough, then $T(\delta) \geqslant -\frac{1}{3}\ln \delta - 2$.*

Proof. Parametrize σ^+ by arclength so that $\operatorname{dist}(\gamma(t), \sigma^+(t)) \to 0$ as $t \to \infty$. Recall that h has length 1. By comparing with a surface of constant curvature $-1 - \varepsilon$, we see that the arc of the horocycle connecting $\gamma(t)$ and $\sigma^+(t)$ has length at least $\frac{1}{2}e^{-\sqrt{1+\varepsilon}(2+t)}$. Let $b(t)$ be the length of the geodesic segment $s \to (s, t)$ between γ and σ^+. Since the curvature is uniformly bounded, small enough pieces of horocycles are uniformly C^1-approximated by geodesic segments. Hence for t large enough

$$b(t) \geqslant \tfrac{1}{4}e^{-\sqrt{1+\varepsilon}(2+t)}. \tag{2.5}$$

Denote by $t(\delta)$ the t-coordinate of $\gamma_\delta(T(\delta))$ in the (s, t)-coordinates. By Lemma 2.4 and (2.5), we have

$$t(\delta) \geqslant -\frac{\ln\left(8\delta e^{2\sqrt{1+\varepsilon}}\right)}{2\sqrt{1+\varepsilon}} \geqslant -\tfrac{1}{3}\ln \delta - 2.$$

Since the curvature is negative, $T(\delta) \geqslant t(\delta)$.

2.6. Lemma. *If δ is small enough, then $\tau(\delta) \leqslant \varphi(\delta)$.*

Proof. Let $\theta(\delta)$ be the t-coordinate of $\gamma_\delta(\tau(\delta))$ (see Figure 4). By Lemma 2.4,

$$f(\theta(\delta)) \geqslant \tfrac{1}{3}e^{\sqrt{1-\varepsilon}\,\theta(\delta)}\delta,$$

and so by (2.2),

$$\delta \leqslant 3f(\theta(\delta)) \cdot e^{-\sqrt{1-\varepsilon}\,\theta(\delta)} \leqslant \varphi^{-1}(2\theta(\delta)).$$

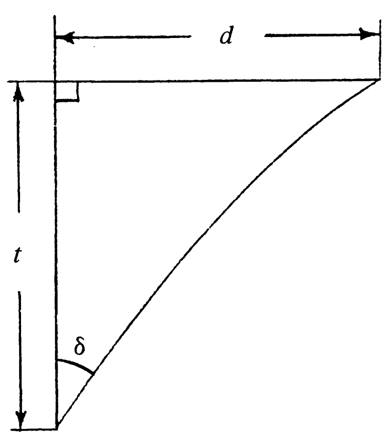

FIGURE 5

Since φ is decreasing, $\varphi(\delta) \geqslant 2\theta(\delta)$. Note now that $\tau(\delta) \leqslant \theta(\delta) + f(\theta(\delta)) \leqslant 2\theta(\delta)$ for δ small enough. q.e.d.

Now we are ready to prove (iii). Consider the Riccati equation

$$u' + u^2 + K(\gamma_v(t)) = 0.$$

Since the curvature is negative, there are solutions of this equation that are defined for all t. Let $u^-(v, \cdot)$ be the smallest and $u^+(v, \cdot)$ the largest such solutions. Recall that the curvature $k(v)$ of the horocycle $h(v)$ is $u^-(v,0)$ (cf. [2]).

The solution of the initial value problem

$$u_\kappa' + u_\kappa^2 - \kappa^2 = 0, \qquad u_\kappa(0) = \lambda, \tag{2.6}$$

is given by

$$u_\kappa(t) = \kappa \frac{\lambda \cdot \cosh(\kappa t) + \kappa \cdot \sinh(\kappa t)}{\lambda \cdot \sinh(\kappa t) + \kappa \cdot \cosh(\kappa t)}. \tag{2.7}$$

By assumption, $M(\delta) \geqslant \delta$. Therefore, by Lemmas 2.5 and 2.6,

$$\begin{aligned} T(\delta) - \tau(\delta) &\geqslant -\tfrac{1}{3}\ln\delta - 2 - \varphi(\delta) \\ &\geqslant -\tfrac{1}{3}\ln\delta - 2 + \tfrac{1}{4}\ln M(\delta) \geqslant -\tfrac{1}{12}\ln\delta - 2. \end{aligned} \tag{2.8}$$

Let $v = -\dot\gamma_\delta(T(\delta))$ (see Figure 4). Since $K_\varepsilon \geqslant -1 - \varepsilon$, we have

$$u^+(v,0) \leqslant \sqrt{1+\varepsilon}. \tag{2.9}$$

Indeed, if u is a solution of the Riccati equation $u' + u^2 + K_\varepsilon(\gamma_v(t)) = 0$ with $u(t_0) > \sqrt{1+\varepsilon}$ for some t_0, then $u(t) \to \infty$ in finite times as t decreases from t_0.

By construction, $K_\varepsilon(\gamma_\delta(t)) = -1$ for $\tau(\delta) \leqslant t \leqslant T(\delta)$. Hence, using (2.7) with $\kappa = 1$, (2.8), and (2.9) we get

$$u^+(v, T(\delta) - \tau(\delta)) \leqslant \frac{\sqrt{1+\varepsilon} \cdot \cosh\theta + \sinh\theta}{\sqrt{1+\varepsilon} \cdot \sinh\theta + \cosh\theta},$$

where $\theta = -\frac{1}{12}\ln\delta - 2$. Therefore

$$\begin{aligned} u^+(v, T(\delta) - \tau(\delta)) - 1 &\leqslant \frac{(\sqrt{1+\varepsilon} - 1)e^{-\theta}}{\sqrt{1+\varepsilon} \cdot \sinh\theta + \cosh\theta} \\ &\leqslant (\sqrt{1+\varepsilon} - 1)e^{-\theta} \leqslant \frac{\varepsilon}{10}, \end{aligned} \tag{2.10}$$

if $\delta > 0$ is sufficiently small.

Our estimates on the solutions of the Riccati equation will use the following lemma.

2.7. Comparison Lemma. *Let $u_i(t)$, $i = 0,1$, be the solutions of the initial value problems*

$$u_i' + u_i^2 + K_i(t) = 0, \quad u_i(0) = \lambda_i, \qquad i = 0,1.$$

Suppose $\lambda_1 \geqslant \lambda_0$, $K_1(t) \leqslant K_0(t)$ *for* $t \in [0, t_0]$, *and* $u_0(t_0)$ *is defined. Then*

$$u_1(t) \geqslant u_0(t) \quad \textit{for } t \in [0, t_0].$$

Proof. The difference $\Delta u(t) = u_1(t) - u_0(t)$ satisfies the linear equation

$$\Delta u' = -(u_0 + u_1)\Delta u + K_0(t) - K_1(t). \qquad \text{q.e.d.}$$

Now we estimate $u^+(v, T(\delta)) = -u^-(w_\delta, 0)$. Since $K_\varepsilon \geqslant -1 - \varepsilon$ everywhere, we can use Lemma 2.7 to compare $u^+(v,t)$, $T(\delta) - \tau(\delta) \leqslant t \leqslant T(\delta)$, with the solution u_κ of (2.6) with $\kappa = \sqrt{1+\varepsilon}$ and $\lambda = 1 + \varepsilon/10$. By Lemma (2.6), (2.7), and (2.10) we obtain

$$\begin{aligned} -u^-(w_\delta, 0) &= u^+(v, T(\delta)) \leqslant u_\kappa(\tau(\delta)) \leqslant u_\kappa(\varphi(\delta)) \\ &= \sqrt{1+\varepsilon}\,\frac{(1+\varepsilon/10)\cosh\eta + \sqrt{1+\varepsilon}\sinh\eta}{(1+\varepsilon/10)\sinh\eta + \sqrt{1+\varepsilon}\cosh\eta}, \end{aligned}$$

where $\eta = \sqrt{1+\varepsilon} \cdot \varphi(\delta)$. Note that $u^-(w_0, 0) = -\sqrt{1+\varepsilon}$, by Lemma 2.3(b). Therefore

$$\begin{aligned} |u^-(w_\delta,0) - u^-(w_0,0)| &= \left|\sqrt{1+\varepsilon} + u^-(w_\delta,0)\right| \\ &\geqslant \sqrt{1+\varepsilon}\,\frac{\left(\sqrt{1+\varepsilon} - 1 - \varepsilon/10\right)e^{-\eta}}{\sqrt{1+\varepsilon}\,e^{\eta}} \\ &= \left(\sqrt{1+\varepsilon} - 1 - \frac{\varepsilon}{10}\right)e^{-2\sqrt{1+\varepsilon}\varphi(\delta)} \\ &\geqslant \frac{\varepsilon}{5}e^{\sqrt{1+\varepsilon}/2\cdot\ln M(\delta)} \geqslant \frac{\varepsilon}{5}M(\delta) \end{aligned} \tag{2.11}$$

provided $M(\delta) < 1$. This shows that $k(\cdot) = u^-(\cdot, 0)$ fails to have modulus of continuity m at w_0. However the footpoint of w_0 lies in the region where the metric g was changed to obtain g_ε.

Let $v_\delta = \dot\gamma_\delta(-2)$. Since $-1 - \varepsilon \leqslant K_\varepsilon < 0$, the norm of the differential of the time 2 map for the geodesic flow of g_ε is bounded by $e^{2+\varepsilon} \leqslant 10$ (see e.g. Lemma 5.1 in [2]). Therefore

$$\operatorname{dist}(v_\delta, v_0) \leqslant 10\operatorname{dist}(w_\delta, w_0) = 10\delta. \tag{2.12}$$

For a unit vector w of the metric g_ε and a number $\lambda \leqslant 0$ denote by $\psi(w,\lambda)$ the value at $t = -2$ of the solution of the initial value problem

$$u' + u^2 + K(\gamma_w(t)) = 0, \qquad u(0) = \lambda.$$

Consider the map Ψ: $T_1S \times [-2, 0] \to T_1S \times [-2, 0]$ given by

$$\Psi(w, \lambda) = (\dot\gamma_w(-2), \psi(w, \lambda)).$$

Equip $T_1S \times [-2, 0]$ with the product metric. Since Ψ is a diffeomorphism onto its image and all of the vectors w_δ are in the same compact fiber of T_1S, there is a constant $c > 0$ such that for all λ, $\lambda_0 \in [-2, 0]$

$$\operatorname{dist}(\Psi(w_\delta, \lambda), \Psi(w_0, \lambda_0)) \geqslant c \cdot \operatorname{dist}((w_\delta, \lambda), (w_0, \lambda_0)). \tag{2.13}$$

Now by the triangle inequality,

$$\operatorname{dist}((w_\delta, \lambda), (w_0, \lambda_0)) \geqslant \operatorname{dist}(w_\delta, w_0)$$

and

$$\begin{aligned}\operatorname{dist}(\Psi(w_\delta, \lambda), \Psi(w_0, \lambda_0)) &\leqslant \operatorname{dist}(\dot\gamma_\delta(-2), \dot\gamma_0(-2)) + |\psi(w_\delta, \lambda) - \psi(w_0, \lambda_0)| \\ &\leqslant 10 \operatorname{dist}(w_\delta, w_0) + |\psi(w_\delta, \lambda) - \psi(w_0, \lambda_0)|\end{aligned}$$

by (2.12). Since $u^-(v_\delta, 0) = \psi(w_\delta, u^-(w_\delta, 0))$, it follows from (2.13) and (2.11) that

$$\begin{aligned}&|u^-(v_\delta, 0) - u^-(v_0, 0)| \\ &\quad\geqslant c \cdot \operatorname{dist}((w_\delta, u^-(w_\delta, 0)), (w_0, u^-(w_0, 0))) - 10 \operatorname{dist}(w_\delta, w_0) \\ &\quad\geqslant c|u^-(w_\delta, 0) - u^-(w_0, 0)| - 10 \operatorname{dist}(w_\delta, w_0) \geqslant c\frac{\varepsilon}{5} M(\delta) - 10\delta.\end{aligned}$$

Recall that $M(\delta) = \sqrt{m(10\delta)} \geqslant \sqrt{10\delta}$, and so for any small enough δ,

$$\begin{aligned}|u^-(v_\delta, 0) - u^-(v_0, 0)| &\geqslant c\frac{\varepsilon}{5}\sqrt{m(10\delta)} - 10\delta \geqslant m(10\delta) \\ &\geqslant m(\operatorname{dist}(v_\delta, v_0))\end{aligned}$$

by (2.12). This completes the proof of the theorem.

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