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A FIRST EIGENVALUE ESTIMATE FOR MINIMAL HYPERSURFACES

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0. Introduction

In this paper we obtain a lower bound for the first eigenvalue of a compact orientable hypersurface embedded minimally in a compact orientable manifold with positive Ricci curvature. The proof of our result utilizes Reilly's formula [4] which is actually an integrated version of Bochner's formula, as it becomes clear in the proof of Theorem 1. It should be mentioned that the idea of integrating Bochner's formula was used before by Lichnerowicz [3].

Combining our theorem with the result of Yang and Yau [5] we obtain an upper bound for the area of an embedded minimal surface of S^3 solely in terms of its genus. See Proposition 4 for a more general statement. We also estimate an upper bound of the length of closed geodesics in a convex surface.

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1. Reilly's formula

Let Ω be a Riemannian manifold of dimension n with smooth boundary $\partial\Omega$. Let f be a function defined on Ω which is smooth up to $\partial\Omega$. We denote $z = f|_{\partial\Omega}$ and $u = (\partial f/\partial n)|_{\partial\Omega}$, where $\partial f/\partial n$ is the outward normal derivative of f. $\bar{\Delta} f$ and $\bar{\nabla} f$ denote the Laplacian and the gradient of f with respect to the Riemannian metric of Ω, whereas Δg and ∇g denote the Laplacian and the gradient of g (defined on $\partial\Omega$) with respect to the induced Riemannian metric on $\partial\Omega$. Let $X, Y \in T_p\Omega$. Then we define the Hessian tensor $(\bar{D}^2 f)(X, Y) = X(Yf) - (\bar{D}_X Y)f$ where X and Y are extended arbitrarily to a vector field near

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p and $\bar{D}_X Y$ is the covariant derivative of the Riemannian connection on Ω. It is easy to see that the definition of $\bar{D}^2 f$ does not depend on the extensions. In the local computation of this section, $\{e_1, \cdots, e_{n-1}, e_n\}$ is a local orthonormal frame such that at $q \in \partial\Omega$, $e_1, \cdots, e_{n-1}$ are tangent to $\partial\Omega$, and e_n is the outward normal vector. We denote $f_{ij} = \bar{D}^2 f(e_i, e_j)$ and $|\bar{D}^2 f|^2 = \sum_{i,j=1}^n f_{ij}^2$. We define the second fundamental form $\mathrm{II}(v, w) = \langle \bar{D}_v e_n, w \rangle$, where v and w are vectors tangent to $\partial\Omega$, and the mean curvature $K = \sum_{i=1}^{n-1} \mathrm{II}(e_i, e_i)$. The sign of K is chosen so that the sphere in the Euclidean space has a positive mean curvature.

We state the Reilly formula:

Theorem 1 (*Reilly* [4]).

$$\int_\Omega (\bar{\Delta} f)^2 - |\bar{D}^2 f|^2 = \int_\Omega \mathrm{Ric}(\bar{\nabla} f, \bar{\nabla} f) + \int_{\partial\Omega} (\Delta z + Ku)u - \langle \nabla z, \nabla u \rangle + \mathrm{II}(\nabla z, \nabla z),$$

where Ric(,) *is the Ricci tensor of* Ω.

Proof. Integrate Bochner's formula to get

$$\int_\Omega \frac{1}{2} \bar{\Delta} |\bar{\nabla} f|^2 - \sum_{i=1}^n f_i (\bar{\Delta} f)_i = \int_\Omega |\bar{D}^2 f|^2 + \mathrm{Ric}(\bar{\nabla} f, \bar{\nabla} f).$$

Using the Stokes theorem, we obtain

$$(*) \quad \begin{aligned} \int_\Omega \frac{1}{2} \bar{\Delta} |\bar{\nabla} f|^2 - \sum_{i=1}^n f_i (\bar{\Delta} f)_i &= \int_{\partial\Omega} \sum_{i=1}^n f_i f_{in} - f_n \sum_{i=1}^n f_{ii} + \int_\Omega (\bar{\Delta} f)^2 \\ &= \int_{\partial\Omega} \sum_{i=1}^{n-1} f_i f_{in} - f_n \sum_{i=1}^{n-1} f_{ii} + \int_\Omega (\bar{\Delta} f)^2. \end{aligned}$$

Now for $i \neq n$,

$$f_{in} = (\bar{D}^2 f)(e_i, e_n) = e_i(e_n f) - (\bar{D}_{e_i} e_n) f = u_i - \sum_{j=1}^{n-1} h_{ij} z_j,$$

where $h_{ij} = \mathrm{II}(e_i, e_j)$. Also

$$\begin{aligned} \sum_{i=1}^{n-1} f_{ii} &\doteq \sum_{i=1}^{n-1} \left[e_i(e_i f) - (\bar{D}_{e_i} e_i) f \right] \\ &= \sum_{i=1}^{n-1} \left[e_i(e_i f) - (D_{e_i} e_i) f - \left[\bar{D}_{e_i} e_i - D_{e_i} e_i \right] f \right] = \Delta z + Ku, \end{aligned}$$

where $D_{e_i} e_i = (\bar{D}_{e_i} e_i)^T$. Putting these expressions for f_{in} and $\sum_{i=1}^{n-1} f_{ii}$ back into $(*)$, the proof is complete.

Remark. It should be mentioned that Reilly's original formula is more general than Theorem 1.

2. Main Theorem

We state our main result.

Theorem 2. *Let M be a compact orientable embedded minimal hypersurface of a compact orientable Riemannian manifold N. Suppose the Ricci curvature of N is bounded below by a positive constant k. Then $\lambda_1(M) \geqslant k/2$, where $\lambda_1(M)$ is the first Neumann eigenvalue of the Laplacian of M.*

Proof. Since the Ricci curvature of N is strictly positive, the first Betti number of N is zero. Combining this with the fact that both M and N are orientable, and chasing through the exact sequences of homology groups, it is easy to see that M divides N into two components Ω_1 and Ω_2 such that $\partial\Omega_1 = M = \partial\Omega_2$.

Let z be a first eigenfunction of M, i.e., $\Delta z + \lambda_1 z = 0$, where $\lambda_1 = \lambda_1(M)$. Let f be the solution of the Dirichlet problem such that

$$\bar{\Delta} f = 0, \quad \text{on } \Omega_1, \qquad f|_{\partial\Omega_1} = z.$$

By the Cauchy-Schwarz inequality, $(\bar{\Delta} f)^2 \leqslant n|\bar{D}^2 f|^2$. Putting this into Theorem 1, we get

$$\int_{\Omega_1} \frac{n-1}{n}(\bar{\Delta} f)^2 \geqslant \int_{\Omega_1} k|\bar{\nabla} f|^2 + \int_{\partial\Omega_1} (\Delta z)u - \langle \nabla z, \nabla u\rangle + \mathrm{II}(\nabla z, \nabla z).$$

Integrate by parts and use the fact that $\bar{\Delta} f = 0$ and $\Delta z + \lambda_1 z = 0$. We then get

$$0 \geqslant \int_{\Omega_1} k|\bar{\nabla} f|^2 + \int_{\partial\Omega_1} -2\lambda_1 z u + \int_{\partial\Omega_1} \mathrm{II}(\nabla z, \nabla z).$$

We can assume $\int_{\partial\Omega_1} \mathrm{II}(\nabla z, \nabla z) \geqslant 0$, otherwise we can work with Ω_2 rather than with Ω_1. Also

$$\int_{\partial\Omega_1} zu = \int_{\partial\Omega_1} f f_n = \int_{\Omega_1} |\bar{\nabla} f|^2 + f\bar{\Delta} f = \int_{\Omega_1} |\bar{\nabla} f|^2.$$

Thus we have $0 \geqslant (k - 2\lambda_1)\int_{\Omega_1} |\nabla f|^2$. Since f is not a constant function, we have $k - 2\lambda_1 \leqslant 0$.

Corollary 3. *Let M be a compact embedded minimal hypersurface of S^n, the standard sphere of sectional curvature 1. Then $\lambda_1(M) \geqslant (n-1)/2$.*

Proof. A compact embedded hypersurface of S^n is already orientable. This corollary follows from Theorem 2, since the Ricci curvature of S^n is $n - 1$.

Remark. Our result is closely related to Yau's conjecture [6]. It is well known that the coordinate functions are eigenfunctions of a minimal hypersurface M of S^n with eigenvalue $n-1$. Yau conjectured that the first eigenvalue of M is actually $n-1$. Our result can be regarded as some evidence that Yau's conjecture may be true.

3. Applications

Yang and Yau [5] proved that if M is an orientable Riemannian surface of genus g with area A, then $\lambda_1 A \leqslant 8\pi(g+1)$. Combining this with Theorem 2, we get the following.

Proposition 4. *Let M and N be the same as in Theorem 2. Assume that* $\dim M = 2$. *Then* $A \leqslant 16\pi(g+1)/k$.

Corollary 5. *Let M be a compact embedded minimal surface of S^3. Then* $A \leqslant 8\pi(g+1)$.

The Reilly formula (Theorem 1) and the eigenvalue estimate (Theorem 2) are still valid even when M is a closed geodesic. But $\lambda_1(M) = 4\pi/l^2$, where l is the length of M. Thus if the Gaussian curvature of $N \geqslant k > 0$, then the length of any closed geodesic $\leqslant \sqrt{8\pi/k}$. Combining this with the Bumpy Metric Theorem of Abraham [1], we obtain the following result: There are only finitely many closed geodesics on S^2 equipped with generic metric with positive Gaussian curvature.

Remark. Using the above result, one can prove the following smooth compactness theorem: For any sequence of embedded minimal surface of fixed genus g in a compact simply connected 3-manifold with strictly positive Ricci curvature, one can extract a subsequence which converges together with all derivatives to an embedded minimal surface of the same genus g. For more details, see the forthcoming paper by H. I. Choi and R. Schoen [2].

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