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EMBEDDED HYPERSPHERES WITH PRESCRIBED MEAN CURVATURE

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In [10] Yau raises the nonlinear global problem: is there an embedding $Y: S^n \to \mathbf{R}^{n+1}$ of the n-dimensional sphere into Euclidean $(n+1)$-space, whose mean curvature is a preassigned sufficiently smooth function H defined on $\mathbf{R}^{n+1}$? A theorem of Bakelman and Kantor [4] asserts the existence of such hypersurfaces assuming only natural conditions that H decay faster than the mean curvature of concentric spheres. It is the purpose of this paper to give a new simple geometric treatment of the required a priori estimates and a complete presentation of the existence and uniqueness proof of this result.

A condition that a function H decays in a domain $U \subset \mathbf{R}^{n+1}\text{-}\{0\}$ from an arbitrary point, say zero, faster than $|X|^{-1}$, where $|X|$ is the Euclidean length of X, is given by

$$\begin{gathered} 0 < H \in C^1(\bar{U}), \\ \frac{\partial}{\partial \rho}\rho H(\rho X) \leqslant 0, \quad \text{for all } \rho X \in U. \end{gathered} \tag{1}$$

Theorem. (a) *Suppose that the function H satisfies condition* (1) *in the annular region* $U = \{X \in \mathbf{R}^{n+1} : r_1 < |X| < r_2\}$ *where* $0 < r_1 \leqslant 1 \leqslant r_2$ *and that*

$$\begin{aligned} H(x) &> |X|^{-1} \quad \textit{for } |X| = r_1, \\ H(X) &< |X|^{-1} \quad \textit{for } |X| = r_2. \end{aligned} \tag{2}$$

Then for some $0 < \alpha < 1$ *there exists an embedded hypersphere* $Y \in C^{2,\alpha}(S^n)$ *with mean curvature* $\mathfrak{M}Y = H(Y)$ *which is a radial graph over the unit sphere such that* $r_1 \leqslant |Y| \leqslant r_2$.

(b) *Let Y be a sphere about zero with* $\mathfrak{M}Y = H(Y)$. *If there is a second embedded* C^2 *hypersurface Z about zero that satisfies* $\mathfrak{M}Z = H(Z)$, *and the function H satisfies condition* (1) *in the domain between Y and Z, then the hypersurfaces are homothetic, i.e.,*

$$Z = (1 + t_0)Y, \quad \textit{for some } t_0 > -1,$$

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and all intermediate homotheties satisfy the equation

$$\mathcal{M}((1 + \theta t_0)Y) = H((1 + \theta t_0)Y) \quad \text{for all } 0 \leqslant \theta \leqslant 1.$$

We construct these embeddings, which are radial projections of the standard sphere, by solving the quasilinear elliptic partial differential equation for prescribed mean curvature on the sphere. In the first section we derive the equation by pulling the expression for the mean curvature back from the hypersurface by a homogeneity argument.

We obtain explicit a priori gradient bounds for a class of equations including the mean curvature equation in the second section. We derive the estimates intrinsically, utilizing the maximum principle in much the same way that it is used by Yau in, e.g., [9], but by applying an operator more suited to the mean curvature equation than the Laplacian. The estimate [3, Theorem 4] is more complicated but gives a gradient bound for the mean curvature equation in a general Riemannian space.

In the third section we assemble the a priori estimates and prove the existence of solutions by applying the Leray-Schauder fixed point theorem. Uniqueness up to homothety follows from the maximum principle. In much the same way, Aeppli [1] and A. D. Aleksandrov [2] have shown uniqueness up to homothety for this problem in case H is homogeneous of degree minus one.

Oliker [6] has obtained an analogous result for prescribed Gauss curvature. Related Dirichlet problems for hypersurfaces with prescribed or zero mean curvature, which project centrally to convex domains of the hemisphere, but using different parameterizations, have been considered from the point of view of Schauder theory by Serrin [7] and the direct method by Tausch [8].

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1. Derivation of the equation

We derive an expression for the mean curvature of the radial graph of a function on the unit sphere. We use moving frames and adapt the convention that lower case indices are summed from 1 to n and capitals from 1 to $n + 1$.

Let $\{e_1, \cdots, e_{n+1}\}$ be a local orthonormal frame field defined on $\mathbf{R}^{n+1}$ such that e_{n+1} is in the outward radial direction. Let $\{\omega^B\}$ denote the dual coframe field. The connection forms are defined as the skew symmetric matrix $\{\omega_A^B\}$ such that

$$d\omega^A = \omega^B \wedge \omega_B^A.$$

Covariant differentiation or $\mathbf{R}^{n+1}$ is given by

$$de_A = \omega_A^B e_B. \tag{3}$$

For hyperspheres $S^n(r)$ of constant radius r, the position vector is

$$X = re_{n+1}. \tag{4}$$

$\{e_i\}$ provide an orthonormal frame on X so we have $dX = \omega^i e_i$, and by substituting (4) and (3),

$$\omega^i = r\omega^i_{n+1}. \tag{5}$$

The graph Y is conveniently represented by $Y = e^u e_{n+1}$, where u is a function on the unit sphere. If u is extended to $\mathbf{R}^{n+1} - \{0\}$ as a constant along radii, the gradient and Hessian of u, given by

$$du = u_i\omega^i, \quad u_{AB}\omega^B = du_A - u_B\omega_A^B,$$

are homogeneous of degrees -1 and -2, respectively. Restricting to Y and using (5) give the Hessian formula

$$u_{ij}\omega^j = du_i - u_j\omega_i^j + e^{-u}u_i\omega^{n+1}. \tag{6}$$

By exterior differentiating the position vector Y, using (5),

$$dY = \left(e_i + e^u u_i e_{n+1}\right)\omega^i,$$

where $E_i = e_i + e^u u_i e_{n+1}$ forms a basis to the tangent space at Y. The induced metric is $ds^2 = g_{ij}\omega^i \otimes \omega^j$ where the coefficients are obtained by taking the Euclidean inner product

$$g_{ij} = \left\langle E_i, E_j\right\rangle = \delta_{ij} + e^{2u}u_iu_j.$$

Hence the inverse matrix is given by $g^{ij} = \delta_{ij} - f^2e^{2u}u_iu_j$, where $f = (1 + e^{2u}|\nabla u|^2)^{-1/2}$. The unit normal vector to Y is $N = f(e_{n+1} - e^u u_i e_i)$. Finally the mean curvature $\mathfrak{M}$ of the hypersurface Y with respect to the inner normal is given by

$$n\mathfrak{M}(Y) = g^{km}\left\langle dN(E_k), E_m\right\rangle.$$

We find using (6),

$$\begin{aligned} dN = {} & f\left(e^{-u}\delta_{ij} - e^u u_iu_j - e^u u_{ij}\right)\omega^i e_j + fu_j\omega^{n+1}e_j \\ & + fu_i\omega^i e_{n+1} + N d\log f. \end{aligned}$$

Hence $\langle dN(E_k), E_m\rangle = fe^{-u}(g_{km} - e^{2u}u_{km})$, so that

$$n\mathfrak{M} = -f^3e^{-u}\left(e^{2u}\left(1 + e^{2u}|\nabla u|^2\right)u_{kk} - e^{4u}u_ku_{km}u_m - n\left(1 + e^{2u}|\nabla u|^2\right)\right).$$

By the homogeneity of the derivatives of u, we can equate their values on Y and $S^n(1)$. Pulling back, we conclude that on $S^n(1)$,

$$\begin{aligned}(7)\qquad &\left((1+|\nabla u|^2)\delta_{km}-u_k u_m\right)u_{km}\\ &\qquad = n\left(1+|\nabla u|^2\right)-ne^u\left(1+|\nabla u|^2\right)^{3/2}\mathfrak{M},\end{aligned}$$

or in divergence form,

$$(8)\qquad \operatorname{div}\left(\left(1+|\nabla u|^2\right)^{-1/2}\nabla u\right)=n\left(1+|\nabla u|^2\right)^{-1/2}-ne^u\mathfrak{M}.$$

2. A gradient estimate for equations of prescribed mean curvature

We prove a gradient estimate for a class of equations, which includes the equation of prescribed mean curvature.

Lemma. *Let $u \in C^2(S^n)$ be a solution to*

$$(9)\qquad a^{ij}u_{ij}=b\left(x,u,|\nabla u|^2\right),$$

where

$$(10)\qquad a^{ij}=\left(1+|\nabla u|^2\right)\delta_{ij}-u_i u_j,$$

and $b=b(x,u,v)\in C^1(S^n\times \mathbf{R}\times\mathbf{R})$. Suppose there are nonnegative constants A_i such that

$$(11)\qquad \begin{gathered}|b_x|\leqslant A_1(1+v)^{3/2},\quad b_u\geqslant -A_2(1+v),\\ (1+3v)b-2(1+v)vb_v\geqslant -A_3(1+v)^{3/2}.\end{gathered}$$

Assume that $\sup|u|\leqslant M$. *Then there exist constants $C_1(n,A_1,A_2,A_3)$ and $C_2(A_1,A_2)$ so that*

$$(12)\qquad e^{C_2u(x)}|\nabla u(x)|^2\leqslant C_1e^{C_2M}.$$

Proof. Assuming initially that $u\in C^3$, we consider the function $\varphi=e^{2Cu}$ where C is a constant to be chosen later, and $v=|\nabla u|^2$. Computing at the point x_0 where φ attains its maximum,

$$(13)\qquad 0=\nabla\varphi(x_0),$$

$$(14)\qquad 0\geqslant a^{ij}\varphi_{ij}(x_0).$$

(13) becomes

$$(15)\qquad 0=u_ju_{ji}+Cvu_i,\qquad i=1,\cdots,n,$$

which implies

$$(16)\qquad u_iu_{ij}u_j=-Cv^2,$$

$$u_i u_{ij} u_{jk} u_k = C^2 v^3. \tag{17}$$

We may choose a coordinate frame at x_0 satisfying $\delta_{1i} v^{1/2} = u_i$. Either $v(x_0) = 0$ in which case (12) holds or in these coordinates we have from (15),

$$u_{11} = -Cv, \tag{18}$$

$$u_{ij} u_{ij} \geqslant C^2 v^2. \tag{19}$$

From (9) it follows

$$(1 + v) u_{ii} - v u_{11} = b.$$

We solve for the Laplacian using (18),

$$u_{ii} = -Cv + C + (b - C)(1 + v)^{-1}. \tag{20}$$

We also differentiate the equation,

$$(2 u_m u_{mk} \delta_{ij} - u_{ik} u_j - u_{jk} u_i) u_{ij} + a^{ij} u_{ijk} = b_k. \tag{21}$$

Since the Ricci curvature on the standard sphere is $R_{ij} = (n - 1)\delta_{ij}$, the Ricci formula (e.g., [9]) becomes

$$u_i u_{ijj} = u_i u_{jji} i + (n - 1) v. \tag{22}$$

Contracting (21) with u_k and using (16), (17), (20) and (22) we obtain

$$\begin{aligned} &(1 + v) u_i u_{ijj} - u_i u_j u_k u_{ijk} \\ &\quad = (n - 1) v (1 + v) + 2C^2 v^2 + 2C(b - C) v^2 (1 + v)^{-1} + b_k u_k. \end{aligned} \tag{23}$$

We differentiate the right member of (9) and obtain from (16),

$$b_k u_k \geqslant -|b_x| v^{1/2} - 2C b_v v^2 + b_u v.$$

At the maximum point, (14) is

$$\begin{aligned} 0 \geqslant\ & a^{ij} u_{ij} C v + 2C u_i u_{ij} u_j + (1 + v) u_{ij} u_{ij} \\ & - u_i u_{ij} u_{jk} u_k + (1 + v) u_i u_{ijj} - u_i u_j u_k u_{ijk}. \end{aligned} \tag{24}$$

Thus we may substitute in (9), (16), (17), (19) and (23), multiply by $(1 + v)$, and group terms involving b to obtain

$$\begin{aligned} 0 \geqslant\ & C^2 v^2 (v - 1) + (n - 1) v (v + 1)^2 + (v + 1)\big(b_u v - |b_x| v^{1/2}\big) \\ & + Cv\big((1 + 3v) b - 2 b_v v (v + 1)\big). \end{aligned} \tag{25}$$

Applying the structure hypotheses (11), estimating the binomials and multiplying by $\exp(6Cu)$ we arrive at

$$\begin{aligned} 0 \geqslant\ & \big(C^2 + n - 1 - 3(A_1 + A_2)\big)\varphi^3 - 2CA_3 e^{Cu} \varphi^{5/2} \\ & - (C^2 - 2n + 2) e^{2Cu} \varphi^2 - (2A_2 + 2CA_3 - n + 1) e^{4Cu} \varphi \\ & - 3A_1 e^{5Cu} \varphi^{1/2}. \end{aligned}$$

By setting $C^2 = 3(A_1 + A_2)$ and comparing each term to the first, we have at x_0

$$(n-1)\varphi^3 \leqslant 4\max\{2CA_3e^{Cu}\varphi^{5/2}, (C^2 - 2n + 2)e^{2Cu}\varphi^2, \\ (2A_2 + 2CA_3 - n + 1)e^{4Cu}\varphi, 3A_1e^{5Cu}\varphi^{1/2}\}.$$

By neglecting terms which are negative for small C and interpolating, we find that there is a constant k depending only on n such that

$$\varphi(x) \leqslant ke^{2CM}\max\{(A_1 + A_2)(1 + A_3^2), A_1^{2/5}\}.$$

Hence we have found explicitly C_1 and C_2 which tend toward zero as $A_1 + A_2$ does.

By manipulating all expressions involving third derivatives in weak form, it is possible to use the Ricci formula and the differential equation to eliminate third derivatives from $a^{ij}\varphi_{ij}$ first, then use (13)–(18) as before and show that (25) and (12) hold assuming only that $u \in C^2$.

3. Proof of the Theorem

Let the radial graph be given by

$$Y = e^{u(x)}e_{n+1}, \tag{26}$$

where u is an unknown function of the unit vector $x \in S^n$. We show that the equation of the prescribed mean curvature derived in §1,

$$\begin{aligned} &\left((1 + |\nabla u|^2)\delta_{ij} - u_iu_j\right)u_{ij} \\ &\quad = n\left(1 + |\nabla u|^2\right) - ne^uH(e^ux)\left(1 + |\nabla u|^2\right)^{3/2}, \end{aligned} \tag{27}$$

can be solved by using the Leray-Schauder fixed point theroem [5, Theorem 10.6]. For simplicity we extend the definition of H to $U = \mathbf{R}^{n+1} - \{0\}$ so that it equals the original on the annulus $r_1 \leqslant |X| \leqslant r_2$ and so that (1) holds. Hence (2) holds with equalities replaced by $|X| \leqslant r_1$ or $|X| \geqslant r_2$. We will show that a solution for the extended problem lies in the original annulus and so solves the original problem. Let B be the Banach space $C^{1,\alpha}(S^n)$. For the parameter $0 \leqslant t \leqslant 1$, we construct a family of solution operators T_t on B given by sending $w \in B$ to the solution u_t of

$$\begin{aligned} L[w]u_t &= \operatorname{div}\left(\left(1 + |\nabla w|^2\right)^{-1/2}\nabla u_t\right) - u_t \\ &= t\left[n\left(1 + |\nabla w|^2\right)^{-1/2} - ne^wH(e^wx) - w\right]. \end{aligned} \tag{28}$$

This is well defined since $L[w]$ is a selfadjoint linear elliptic operator on $L^2(S^n)$ with trivial kernel. If $L[w]u = 0$, then

$$0 = \int_{S^n} uL(u)\,du = -\int_{S^n} \left[\left(1 + |\nabla w|^2\right)^{-1/2} |\nabla u|^2 + u^2\right]$$

implies $u = 0$. By the Fredholm alternative and elliptic regularity, (28) can be solved by $u_t \in C^{2,\alpha}(S^n)$, hence T_t is a compact operator on B. Also $T_0 w = 0$ for all $w \in B$. Since a solution of (27) is a fixed point of T_1 by the Leray-Schauder theorem, it suffices to find an a priori estimated $\|u\|_B < \overline{M}$ for any $u \in B$ such that $T_t u = u$ and any $0 \leqslant t \leqslant 1$.

Supremum estimates follow from the maximum principle and assumption (2). To obtain an upper bound, let $u \in B$ satisfy $T_t u = u$. Let x_1 be the point where $u(x_1) = \sup_{S^n}(u)$, and $\bar{u}$ the constant function $\bar{u} = u(x_1)$. If $\bar{u} > \log r_2 (> 0)$, by assumptions (1) and (2) we then have

$$L[u]u(x_1) = t(n - ne^u H - u)|_{x=x_1} > -tu(x_1) \geqslant -u(x_1) = L[\bar{u}]\bar{u},$$

which is a contradiction. The lower bound $u \geqslant \log r_1$ is similar.

A fixed point of T_t satisfies

$$\begin{aligned} a^{ij}u_{ij} = b_t &= tn\left(1 + |\nabla u|^2\right) \\ &\quad + \left[-tne^u H(e^u x) + (1-t)u\right]\left(1 + |\nabla u|^2\right)^{3/2}, \end{aligned} \tag{29}$$

where a^{ij} is given by (10). By differentiating b_t we find first that conditions (11) of the Lemma are satisfied for all $0 \leqslant t \leqslant 1$ by taking

$$\begin{aligned} A_1 &= \sup_{r_1 \leqslant |X| \leqslant r_2} |X|^2 |\nabla^T H(X)|, \\ A_2 &= 0, \\ A_3 &= n \sup_{r_1 \leqslant |X| \leqslant r_2} |H(X)X| + \log r_2, \\ M &= \max\{\log r_2, -\log r_1\}, \end{aligned}$$

where the radial projection ∇^T is the Euclidean gradient minus the radial derivative. Applying the lemma to solutions of (29) gives that there are constants $C_1 = C_1(n, r_1, r_2, \sup|H|, \sup|\nabla^T H|)$ and $C_2 = C_2(r_1, r_2, \sup|\nabla^T H|)$ such that for all $0 \leqslant t \leqslant 1$,

$$|\nabla u|^2 \leqslant C_1 \exp C_2\left(\sup_{S^n} u - \inf_{S^n} u\right) \leqslant C_1 (r_2/r_1)^{C_2},$$

where the C_i are functions which go to zero as $\sup|\nabla^T H|$ does.

That there exist an $0 < \alpha < 1$ depending only on n and $\sup|\nabla u|$, and $\overline{M}$ depending on n, $\sup u$, $\sup|\nabla u|$, $\sup|H|$, which in turn depends only on n, r_1, r_2, $\sup|H|$, $\sup|\nabla^T H|$, but not on t, such that

$$|u|_{C^{1,\alpha}(S^n)}l < \overline{M},$$

follows from [5, Theorem 12.6], partitioning and the compactness of S^n. This completes the proof of the existence theorem.

To obtain the uniqueness result, consider the case where there are points of Z outside Y. The case where there are points of Z inside Y is handled similarly. Let Y and the outer surfaces of Z be given locally by the functions u and z, respectively, which satisfy (8) written $Pu = Q$ for short. Consider the surface $\tilde{u} = u + C$, which is a homothetic dilation of u by (26), where $C > 0$ is the constant for which $\tilde{u}(x) \geqslant z(x)$ for all x while $\tilde{u}(x_2) = z(x_2)$ at some x_2. Using assumption (1) and definition (8) we see that

$$\begin{aligned} P\tilde{u}(x_2) &= Pu(x_2) = Q(x_2, u, \nabla u) \\ &\leqslant Q(x_2, \tilde{u}, \nabla \tilde{u}) = Q(x_2, z, \nabla z) = Pz(x_2). \end{aligned}$$

By the strong maximum principle, $z \equiv \tilde{u}$, hence by (1),

$$P(u + C')(x) = Pu(x) = Q(x, u, \nabla u) = Q(x, u + C', \nabla(u + C')),$$

for $0 \leqslant C' \leqslant C$ and all x. Thus we have shown that the only way several solutions of the equation occur is as an interval of dilations of one another.

We remark that in case the function H is radially symmetric, then $\nabla^T H = 0$, and the solutions are also radially symmetric. Although this follows from the uniqueness statement, it can also be derived from the fact that the gradient bound obtained here tends to zero as $\sup|\nabla^T H|$ vanishes.

References

[1] A. Aeppli, *On the uniqueness of compact solutions for certain elliptic differential equations*, Proc. Amer. Math. Soc. **11** (1960) 826–832.

[2] A. D. Aleksandrov, *Uniqueness theorems for surfaces in the large*. I, Vestnik Leningrad Univ. **11** (1956) 5–17 = Amer. Math. Soc. Transl. Ser. 2, **21** (1962) 341–354.

[3] I. Bakelman & B. Kantor, *Estimates for solutions of quasilinear elliptic equations connected with problems of geometry in the large*, Mat. Sb. **91** (**133**) (1973) No. 3 = Math. USSR-Sb. **20** (173) 348–363.

[4] ______, *Existence of spherically homeomorphic hypersurfaces in Euclidean space with prescribed mean curvature*, Geometry and Topology, Leningrad, **1** (1974) 3–10.

[5] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of the second order*, Springer, New York, 1977.

[6] V. Oliker, *Hypersurfaces in* $\mathbf{R}^{n+1}$ *with prescribed Gaussian curvature and related equations of Monge-Ampere type*, preprint, 1982.

[7] J. Serrin, *The problem of Dirichlet for quasilinear elliptic equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A, **264** (1969) 413–496.

[8] E. Tausch, *The n-dimensional least area problem for boundary on a convex cone*, Arch. Rational Mech. Anal. **75** (1981) 407–416.

[9] S. T. Yau, *Harmonic functions on a complete Riemannian manifold*, Comm. Pure Appl. Math. **28** (1975) 201–228.

[10] ______, *Problem section*, Seminar on Differential Geometry, S. T. Yau, ed., Annals of Math. Studies, No. 102, Princeton University Press, Princeton, 1982, 669–706.

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