J. DIFFERENTIAL GEOMETRY
16 (1981) 739–744

MINIMAL SETS OF FAMILIES OF VECTOR FIELDS ON COMPACT SURFACES

JORGE HOUNIE

1. Introduction

Let M be a compact connected smooth manifold of dimension two, and consider a subgroup G of the group of diffeomorphisms of M. A set $\Omega \subset M$ is G-invariant if $g\Omega \subset \Omega$ for all g in G. A set is said to be G-minimal if it is closed G-invariant nonempty, and contains no such proper subset. Let D be a set of smooth vector fields on M, and consider the group G_D generated by the one-parameter group whose infinitesimal generators are the elements of D. When D contains exactly one vector field, a well-known theorem of Schwartz [5] shows that a G_D-minimal set is either a point, a homeomorph of S^1 or all of M (in the last case M must be homeomorphic to a torus T^2). The purpose of this paper is to extend this result to arbitrary families of vector fields.

Theorem 1. *Let M be a compact connected two-dimensional smooth manifold. Let D be a set of smooth vector fields on M, and consider a G_D-minimal set $\Omega \subset M$. Then Ω must be one of the following*:

(a) *a point which is a common zero of the vector fields of D*;

(b) *a G_D-orbit homeomorphic to S^1*;

(c) *all of M.*

Proof. Let $m \in \Omega$, and denote by $\gamma(m)$ the G_D-orbit of m, i.e., the set of points of the form $g(m)$, $g \in G_D$. By a theorem of Sussmann [7], $\gamma(m)$ is a smooth connected paracompact submanifold of M (with a natural differentiable structure) of dimension k, $0 \leqslant k \leqslant 2$. All vector fields in D are tangent to $\gamma(m)$. If $k = 0$, $\gamma(m)$ is a point and we have (a). If $k = 2$, $\gamma(m)$ is open in M. Then $\overline{\gamma(m)} \setminus \gamma(m)$ is a closed invariant proper subset of Ω, so $\overline{\gamma(m)} = \gamma(m) = \Omega = M$. This gives (c). If $k = 1$, $\gamma(m)$ is homeomorphic to S^1 or $\mathbf{R}$. In the first case we get (b). Assume that $\gamma(m)$ is homeomorphic to $\mathbf{R}$, and consider $\overline{\gamma(m)} = \Omega$. If the interior of Ω is nonempty, we conclude as before that $\Omega = M$. The theorem will be proved if we show that Ω cannot be nowhere dense when $\gamma(m)$ is homeomorphic to $\mathbf{R}$. Let us reason by contradiction, and assume that Ω is nowhere dense.

Communicated by R. Bott, April 10, 1981. The author was partially supported by CNP_q, Brazil.

Consider a vector field X which belong to D and does not vanish at m, and consider an imbedding i: $[-1, 1] \to M$ such that

(a) X is transversal to $i((-1, 1)) = I$,
(b) $i(-1)$ and $i(1)$ are not in Ω,
(c) $i(0) = m$.

Given a point p in $I \cap \Omega$, $\gamma(p)$ is homeomorphic to $\mathbf{R}$. In fact, we may choose a diffeomorphism $j: \mathbf{R} \to \gamma(p)$ so that $j(0) = p$, $j'(0) = \lambda X(p)$, $\lambda > 0$. Since $\bigcap_n \overline{j[n, \infty)}$ is closed, invariant and nonempty, and is thus equal to Ω, it follows that there is a least positive s_0 such that $j(s_0) \in I$: "the first return to I of the G_D-orbit through p in the direction of X". It is easy to see that the vector $j'(s)$, $0 \leqslant s \leqslant s_0$, can be extended to a vector field Y in M, which is a finite linear combination with smooth coefficients of vector fields of D, that is, Y belongs to the $C^\infty(M)$-module D' generated by D. So in a neighborhood of p in I, the first return to I of the G_D-orbit of a point in $\Omega \cap I$ is also the first return to I through the orbit of Y. Since $\Omega \cap I$ is compact and nowhere dense in I, we may cover $\Omega \cap I$ with a finite number of disjoint open subsets of I, so that in each one of them the "first return" is performed through the orbit of a vector field of D'. Thus the "first return function" can be extended to a smooth function $\tilde{f}$ in a neighborhood of $\Omega \cap I$ in I. The latter induces a smooth function $f = i^{-1}\tilde{f}i$, in a neighborhood V of $i^{-1}(\Omega \cap I) = G$, $f: V \to (-1, 1)$.

In the same way, we obtain a smooth function $g: V \to (-1, 1)$ induced by "the first return to I of the G_D-oribt of p in the direction of $-X$". Letting W be open in $(-1, 1)$ such that $G \subset W \subset \overline{W} \subseteq V$, we summarize the properties of f and g:

(1) $G = (-1, 1) \setminus \bigcup_{i=1}^{\infty}(a_i, b_i)$, G is perfect,
(2) $H = \{a_i, b_i, i = 1, 2, \cdots\}$, $f(H) \subseteq H$, $g(H) \subseteq H$,
(3) $(a_i, b_i) \subset W$ implies $f((a_i, b_i)) = (a_j, b_j)$, $g((a_i, b_i)) = (a_k, b_k)$ for some j, k,
(4) $f(G) \subset G$, $g(G) \subset G$.
(5) $0 < L \leqslant |f'(w)| \leqslant F$, $0 < L \leqslant |g'(w)| \leqslant F$, for all $w \in W$, $0 < L < 1 < F$,
(6) $|f''(w)| \leqslant M$, $|g''(w)| \leqslant M$, for all $w \in W$.

Consider the semigroup S generated by f and g, i.e., the functions $h: G \to G$ of the form $h = f^{n_1} \circ g^{m_1} \circ \cdots \circ f^{n_j} \circ g^{m_j}$, n_i, $m_i \in \mathbf{Z}^+$, where f^n indicates composition n-times. We shall denote the S-orbit of x by $[x]$, $x \in G$. Then

(7) $i[x] = \gamma(i(x)) \cap \Omega$, $x \in G$,
(8) If $h \in S$, $a \in G$ and $h(a) = a$, there is a neighborhood U of a such that $h(b) = b$ for all b in $U \cap G$.

The last property is proved by observing that locally f and g are induced by the first return functions of certain vector fields Y_j, transversal to I. Therefore h induces a piecewise differentiable path α made up to arcs of integral curves of the Y_j's. Since $h(a) = a$ and $\alpha \subseteq \gamma(i(a))$, each arc is traversed the same number of times in each direction. If $b \in G$ is sufficiently close to a, the path β induced by h starting at $i(b)$ will follow the arcs of integral curves of the same Y_j's used by α and in the same order. In particular, each arc will be traversed the same number of times in each direction. This implies that $h(b) = b$. Note that $U \cap G$ does not reduce to a point since G is perfect.

To prove the theorem we need only show that properties (1) to (8) lead to a contradiction.

To each sequence of positive integers $(n_1, m_1, n_2, m_2, \cdots)$ we associate a sequence F_j of functions of S so that

$$\begin{aligned} &F_0 = \text{identity}, \\ (9) \qquad &F_j = f^{j-M_k} \circ F_{M_k}, \quad M_k < j \leqslant N_{k+1}, \quad k = 0, 1, 2, \cdots, \\ &F_j = g^{j-M_k} \circ F_{N_k}, \quad N_k < j \leqslant M_k, \quad k = 1, 2, \cdots, \end{aligned}$$

where $N_k = n_1 + m_1 + \cdots + n_k$, $M_k = N_k + m_k$, $M_0 = 0$.

Lemma 1. *There exist a complementary interval* (a, b), $a, b \in G$, *and a sequence of positive integers* $n_1, m_1, n_2, m_2, \cdots$ *so that* F_j *defined by* (9) *satisfies* $F_j(a, b) \subset W$, $j = 1, 2, \cdots$, *and* $\{F_j(a),\ j = 1, 2, \cdots\}$ *is dense in* G.

Proof. Let $\mu = \text{dist}(G, (-1, 1) \setminus W)$, $A = \{i \mid b_i - a_i \geqslant \mu\}$ and $B = \{a_i, b_i, i \in A\}$. The sets A and B are finite. By (7) we may identify $[a_1]$ with the integers $\mathbf{Z}$, where $k \in \mathbf{Z}$ corresponds to the $|k|$-th return to I in the direction of X or $-X$ according to the sign of k. Denote by $\bar{f}$, $\bar{g}$, $\bar{F}_j$ the functions induced by f, g, F_j in this identification. Then $\bar{f}(k) = k \pm 1$, $\bar{g}(k) = k \pm 1$ and $|\bar{f}(k) - \bar{g}(k)| = 2$. Hence there is a sequence of positive integers $(n_1, m_1, \cdots)$ such that either $\bar{F}_j(0) = j$ or $\bar{F}_j(0) = -j$, $j = 1, 2, \cdots$, (according to the sign of $\bar{f}(0)$). It follows from (2) and the construction of F_j that there exists N such that $F_k(a_1) \notin B$ for $k \geqslant N$ and $F_N(a_1) = a_i$ or b_i for some $i \notin A$. Hence $(a_i, b_i) \subset W$, and it follows from (3) and the choice of N that $|F_j(a_i) - F_j(b_i) < \mu$ for all $j = 1, 2, \cdots$. Then setting $(a, b) = (a_i, b_i)$, $F_j((a, b)) \subseteq W$ for all $j = 1, 2, \cdots$. The density of $\{F_j(a)\}$ follows from $\Omega = \bigcap_n \overline{j[n, \infty)} = \bigcap_n \overline{j(-\infty, n]}$, where $j\colon \mathbf{R} \to \gamma(i(a))$ is a diffeomorphism.

Using Lemma 1, the mean value theorem and estimates (5) and (6) we may find, adapting the reasonings of [5, p. 456], a positive $\nu < \mu$ so that $|F_j(x) - F_j(a)| < \mu$ for $|x - a| < \nu$, $j = 1, 2, \cdots$, and $F_j'(x) \to 0$ uniformly for $|x - a| \leqslant \nu$, $j \to \infty$, where a is the left endpoint of the interval of Lemma 1.

Select j such that

$$|F_j'(x)| \leq \tfrac{1}{2} \quad \text{if } |x - a| \leq \nu, \tag{11}$$
$$|F_j(a) - a| \leq \nu/2.$$

It follows that $F_j: [a - \nu, a + \nu] \to [a - \nu, a + \nu]$ has a *unique* fixed point p in $[a - \nu, a + \nu]$. Obtaining the fixed point by successive approximations starting at a we see that $p \in G$. This contradicts (8).

Remarks. (1) Each one of the alternatives of Theorem 1 for a minimal set Ω actually occurs for suitable D. For instance, (c) is obtained if D is such that to every point p of M there corresponds a pair of vectors of D which are linearly independent at p. On the other hand, if $M = \Omega = \overline{\gamma(m)}$ but $\dim \gamma(m) = 1$, M must be homeomorphic to a torus T^2, since in this case any two vectors of D are linearly dependent at every point of M, and D defines a line field without singularities (see next section).

(2) When D contains exactly one vector field, the functions f and g appearing in the proof of Theorem 1, satisfy $f = g^{-1}$, and the semigroup S is a group, so proofs become simpler (see [5]).

(3) It is clear that "smooth" may be replaced by C^2 everywhere. A well-known example of Denjoy [1], showed that the theorem is false in the C^1 case.

2. Line fields

A smooth line field with singularities Λ on a manifold M is a smooth one-dimensional distribution defined on an open subset V of M. The points of $M \setminus V$, where the distribution is not defined, are the singularities of Λ; if $V = M$ we say that Λ is without singularities. By Frobenius theorem, the maximal integral curves of Λ constitute a regular one-dimensional foliation of V. Thus we may consider an equivalence relation on M, whose equivalence classes are (i) the leaves of this foliation, and (ii) single points of $M \setminus V$. A subset of M is Λ-invariant if it is a union of equivalence classes. A Λ-minimal set is a closed nonempty invariant set which contains no such proper subset. Two line fields with singularities Λ_1, Λ_2 defined on manifolds M_1 and M_2 respectively are equivalent if there exists a homeomorphism of M_1 onto M_2 which preserve the equivalence relations induced by Λ_1 and Λ_2. In particular, if Λ_1 and Λ_2 are equivalent, M_1 and M_2 are homeomorphic.

A line field induced on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ by a straight line with irrational angular coefficient will be referred to as "irrational line field on T^2".

Theorem 2. *Let M be a compact connected two-dimensional smooth manifold, and let Λ be a smooth line field with singularities on M. Then a Λ-minimal set Ω must be one of the following*:

(a) *a singularity of* Λ;

(b) *a closed integral curve of* Λ, *homeomorphic to* S^1;

(c) *all of* M. *In this case* Λ *is equivalent to an irrational line field on* T^2.

Proof. Let V be the open subset of M where Λ is not singular, and consider a family of vector fields D which vanish on $M \setminus V$ such that to every point p of V, there are a neighborhood U of P and a vector field X of D which spans Λ over U. It follows that Ω is G_D-minimal so (a), (b) or (c) or Theorem 1 must hold. If (c) holds, Λ has no singularities. This implies (see for instance [3, p. 275]) that the Euler characteristic of M is zero, so M is homeomorphic to a torus T^2 or a Klein bottle K^2. In the latter case, every regular one-dimensional foliation of M has a closed leaf (Kneser [2, p. 153]), so Ω cannot be all of M. Then M must be homeomorphic to T^2. Consider a smooth closed curve Γ everywhere transversal to Λ, and consider a vector $X \neq 0$ on Γ which spans Λ over Γ. Let $f(x)$ be the first return to Γ of the leaf through x in the direction of X. Suppose that for a certain $x \in \Gamma$ the arc of integral curve of Λ which joins x to $f(x)$ enters Γ in the direction of $-X$. Then the same will happen for all $x \in \Gamma$ since the set of those points is open and closed in Γ. This implies that f reverses the orientation of Γ and has a fixed point, which is impossible. Thus the arcs leaving Γ in the direction of X, also enter Γ in the direction of X. This induces a coherent orientation on the leaves of Λ, and Λ may be spanned by a single vector field X_1 which extends X. The "first return to Γ" function induced by X_1 must have an irrational rotation number. Therefore Λ is equivalent to an irrational line field on T^2 [6, Chap. III].

Remark. Related results concerning line fields spanned by a single vector field were studied in [4, p. 210]. When the set V where Λ is regular is simply connected, Λ is spanned by a single vector field. However, this is not true in general, as simple examples show.

References

[1] A. Denjoy, *Sur les courbes définies par des équations differentielles à la surface du tore*, J. Math. Pures Appl. **11** (1932) 333–375.

[2] H. Kneser, *Regulare Kurvenscharen auf den Ringflächen*, Math. Ann. **91** (1924) 135–154.

[3] J. Little, *On singularities of submanifolds of higher dimensional Euclidean spaces*, Ann. Mat. Pura Appl. **83** (1969) 261–335.

[4] R. Sacksteder & A. Schwartz, *Limit sets of foliations*, Ann. Inst. Fourier (Grenoble) **15** (1965) 201–213.

[5] A. Schwartz, *A generalization of the Poincaré-Bendixon theorem to closed two-dimensional manifolds*, Amer. J. Math. **85** (1963) 453–458.

[6] S. Sternberg, *Celestial mechanics*, *Part* II, Benjamin, New York, 1969.

[7] H. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973) 171–188.

Universidade Federal de Pernambuco
Recife, Brasil