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ABSTRACT PRESTRATIFIED SETS ARE (*b*)-REGULAR

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Introduction

The concept of abstract prestratified sets is central in the proof that topologically stable maps are dense. It allows the construction of homeomorphisms by integrating certain discontinuous vector fields.

One very important fact is that (b)-regularly stratified closed subsets of manifolds are abstract prestratified sets. In this paper we give a proof of the converse: every abstract prestratified set can be imbedded in some $\mathbf{R}^n$ as a closed (b)-regularly stratified subset. This result was independently proved and used by M. Goresky in [2]. Our proof is different from his and less complicated, and is based on ideas of A. Verona [14]. We use a double induction on the dimension and the height of the stratification. See also the paper [6] by Natsume, who published another proof of this result. Precisely we want to prove

Theorem 1. *For every abstract prestratified set $(A, \mathcal{S}, \mathcal{T})$ of finite dimension there exist $n \in \mathbf{N}$, a closed set $B \subset \mathbf{R}^n$, and a homeomorphism $\phi\colon A \to B$ such that*

(i) *for all strata $X \in \mathcal{S}$ the map $\phi \mid X$ is a C^∞-imbedding,*

(ii) *B is (b)-regularly stratified by $\{\phi(X) \mid X \in \mathcal{S}\}$,*

(iii) *for all controlled maps $f\colon A \to \mathbf{R}$ the composition $f \circ \phi^{-1}$ is a C^∞-map.*

1. Preliminaries

We first repeat the main definitions and some properties of abstract prestratified sets. For details see [4] and [13].

An *abstract prestratified set* (a.p.s.) is a triple $(A, \mathcal{S}, \mathcal{T})$, where A is a locally compact Hausdorff-space with countable basis, $\mathcal{S}$ is a locally finite partition of A into locally closed subsets, called the *strata*, which are C^∞-manifolds without

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boundary, and $\mathfrak{T} = \{(T_X, \pi_X, \rho_X) \mid X \in \mathbb{S}\}$ is a system of tubular neighborhoods: π_X is a retraction of the open set T_X onto X, ρ_X is a distance function defined on T_X with $\rho_X^{-1}(0) = X$. Furthermore, there must hold the well known commutation relations $\pi_X \circ \pi_Y = \pi_X$ and $\rho_X \circ \pi_Y = \rho_X$, whenever these are defined, and some differentiability conditions on π_X, ρ_X. Making the T_X smaller and redefining ρ_X we may assume the following:

(i) For all $X, Y \in \mathbb{S}$, $X \cap \overline{Y} \neq \emptyset$ iff $T_X \cap Y \neq \emptyset$.

(ii) For all $X, Y \in \mathbb{S}$ with $T_X \cap T_Y \neq \emptyset$ we have $X \subset \overline{Y}$ or $Y \subset \overline{X}$ or $X = Y$.

(iii) There is an $\varepsilon_A > 0$ such that for all $X \in \mathbb{S}$ the map (π_X, ρ_X): $T_X \to [0, \varepsilon_A)$ is proper.

(iv) The commutation relations $\pi_X \circ \pi_Y = \pi_X$ and $\rho_X \circ \pi_Y = \rho_X$ hold on the whole intersection $T_X \cap T_Y$ for $X \subset \overline{Y} \setminus Y$ (see [11, p. 76]).

So in the following we shall only consider a.p.s.'s satisfying these conditions too.

For $\varepsilon > 0$ we define

$T_X^\varepsilon := \{a \in T_X \mid \rho_X(a) \leqslant \varepsilon\}$,

$T_X^{(\varepsilon)} := \{a \in T_X \mid \rho_X(a) < \varepsilon\}$,

$S_X^\varepsilon := T_X^\varepsilon \setminus T_X^{(\varepsilon)}$.

The *dimension of* A is $d(A) := \sup\{\dim X \mid X \in \mathbb{S}\}$, and the *height* of the stratification is $\sigma(A) := \sup\{p \in \mathbf{N} \mid$ there exist strata $X_i \in \mathbb{S}$, $1 \leqslant i \leqslant p$, with $\dim X_1 < \cdots < \dim X_p\}$.

Let $\varepsilon \leqslant \varepsilon_A$ be given. A continuous map f: $A \to M$ into some manifold is said to be *ε-controlled*, if it is of class C^∞ on each stratum, and there holds $f \circ \pi_X = f$ on $T_X^{(\varepsilon)}$ for all $X \in \mathbb{S}$. An *ε-isomorphism* between a.p.s.'s is a homeomorphism f: $A \to A'$, mapping strata X of A diffeomorphically onto strata X' of A' and $T_X^{(\varepsilon)}$ into $T'_{X'}$, such that

$$(\pi'_{X'}, \rho'_{X'}) \circ f = (f \circ \pi_X, \rho_X) \quad \text{on } T_X^{(\varepsilon)}.$$

An important example of an a.p.s. is the *mapping cylinder* example: Let $(A, \mathbb{S}, \mathfrak{T})$ be an a.p.s. and p: $A \to M$ a proper ε-controlled submersion onto a connected C^∞-manifold. Then the (open) mapping cylinder

$$Z^{(\varepsilon)}(p) := A \times [0, \varepsilon) + M / (a, 0) \sim p(a)$$

is in a canonical way an a.p.s. with strata $X \times (0, \varepsilon)$ and M and with naturally defined tubular data.

In [13, Proposition (2.1)] it was shown that for all $0 < \delta < \varepsilon < \varepsilon_A$ and all $X \in \mathbb{S}$ there exists an ε-isomorphism

$$T_X^{(\varepsilon)} \to Z^{(\varepsilon)}\left(\pi_X \mid S_X^\delta\right)$$

which we can get by integrating an arbitrary controlled vector field on $T_X^{(\varepsilon)} \setminus X$ which projects by (π_X, ρ_X) to the vector field $(0, d/dt)$ on $X \times \mathbf{R}$. If we choose these vector fields compatible with one another, we can get what in [3] is called a *family of lines* (for details see [11, p. 79–84]).

2. (b)-regular imbeddings

For details on Whitney's conditions (a), (b) and (b′) see for example [12].

A topological imbedding $\phi: A \to \mathbf{R}^n$ of an a.p.s. is called a *(b)-regular imbedding*, if $\phi \mid X$ is C^∞-imbedding for all strata X, and $\phi(A)$ is (b)-regularly stratified by $\{\phi(X) \mid X \in \mathbb{S}\}$. So, if ϕ is a "stratified" imbedding of A into some $\mathbf{R}^n$, for ϕ to be a (b)-regular imbedding we have only to test whether $\phi(Y)$ is (b)-regular over $\phi(X)$ for all $X \subset \overline{Y}$. For this point the following lemma is very useful in our proof of Theorem 1.

Lemma 1. *Suppose $X, Y \subset \mathbf{R}^n$ are C^1-submanifolds, $x \in X \subset \overline{Y} \setminus Y$, and Y is (a)- or (b)- or (b')-regular over X at x. Let $A = X \cup Y$ and let $\phi: A \to B \subset \mathbf{R}^m$ be a C^1-diffeomorphism of reduced C^1-spaces. Then the triple $(\phi(Y), \phi(X), \phi(x))$ also fulfils condition (a) or (b) or (b') respectively.*

In fact we do not yet need all of our assumptions for these conclusions. The proof of our lemma needs the most elementary parts of the theory of differentiable spaces (see [8], [9], and [10] for some more details): A map $f: A \to \mathbf{R}^m$ of an arbitrary subset $A \subset \mathbf{R}^n$ is said to be *differentiable*, if it is locally the restriction of a differentiable map. Subsets with this structure are called *reduced differentiable spaces*. A *diffeomorphism* is a bijective differentiable map whose inverse is also differentiable. Every subset $A \subset \mathbf{R}^n$ has a *tangent space $Tg_x A^N$* at $x \in A$:

$$Tg_x A^N := \{v \in \mathbf{R}^n \mid df(x)v = 0 \text{ for all } f \text{ of class } C^N \text{ defined in some neighborhood of } x \text{ such that } f \mid A \equiv 0\}.$$

With these definitions the proof of Lemma 1 is just the same as the usual proof of the invariance of Whitney's conditions under diffeomorphisms of open subsets of $\mathbf{R}^n$.

The following two lemmas give us the standard representatives of our (b)-regular imbeddings. The proof of the first one is elementary, and we omit it:

Lemma 2. *Suppose $(A, \mathbb{S}, \mathfrak{T})$ is an a.p.s. with $A \subset \mathbf{R}^n$, $M \subset \mathbf{R}^l$ is a connected C^∞-manifold, and $p: A \to M$ is a proper ε-controlled submersion. Then the map $\phi: Z^{(\varepsilon)}(p) \to \mathbf{R}^{n+l+1}$ given by $\phi([(a, t)]) := (t \cdot a, p(a), t)$ is a stratified imbedding, i.e., a homeomorphism onto some subset, whose restrictions to strata are C^∞-imbeddings.*

Lemma 3. *Suppose $A \subset \mathbf{R}^n$ is (b)-regularly stratified by $\mathfrak{S}_A$ (i.e., the inclusion is a (b)-regular imbedding), $M \subset \mathbf{R}^l$ is a connected C^∞-manifold, and $p\colon A \to M$ is a proper C^∞-map of reduced C^∞-spaces, whose restriction to each stratum is a surjective submersion. Let $\varepsilon > 0$. Then $B := \{(t \cdot x, p(x), t) \mid x \in A, 0 \leqslant t < \varepsilon\}$ is (b)-regularly stratified by $\mathfrak{S}_B = \{M, \{(t \cdot x, p(x), t) \mid x \in X, 0 < t < \varepsilon\}_{X \in \mathfrak{S}_A}\}$.*

Proof. Let $f\colon U \to \mathbf{R}^l$ be a C^∞-representative of p on a neighborhood U of A (which one can construct with a partition of unity), and let $g\colon U \times \mathbf{R} \to \mathbf{R}^{n+l+1}$ be defined by $g(x, t) = (t \cdot x, f(x), t)$. If $X, Y \in \mathfrak{S}_A$ with $X \subset \overline{Y} \setminus Y$ and $(x_0, t_0) \in X \times (0, \varepsilon)$, then for the triple $(g(Y \times (0, \varepsilon)), g(X \times (0, \varepsilon)), g(x_0, t_0))$ condition (b) holds, because the triple is diffeomorphic to $(Y \times (0, \varepsilon), X \times (0, \varepsilon), (x_0, t_0))$. So we must only prove condition (b) in the case where one stratum is M. By Lemma 1 we can assume $M = \mathbf{R}^l$. Let $X \in \mathfrak{S}_A$, let

$$(0, x_0, 0) \in 0 \times \mathbf{R}^l \times 0 \subset \overline{g(X \times (0, \varepsilon))},$$

and let $x_\nu \in X$, $t_\nu \in (0, \varepsilon)$ be such that $y_\nu = g(x_\nu, t_\nu)$ converges to $(0, x_0, 0)$ and $Tg_{y_\nu} g(X \times (0, \varepsilon)) = dg(x_\nu, t_\nu)(Tg_{x_\nu} X \times \mathbf{R})$ converges to τ (in the Grassmannian $G_{n+l+1,m+1}$, $m = \dim X$). We first want to show $0 \times \mathbf{R}^l \times 0 \subset \tau$.

We have (t_ν) converges to 0, and (because p is proper) (x_ν) converges without loss of generality to some $y_0 \in A \cap \overline{X}$. Then $y_0 \in Y \in \mathfrak{S}_A$, and we may suppose that $(Tg_{x_\nu} X)$ converges to some subspace $\tau^* \subset \mathbf{R}^n$. By our assumption we have $Tg_{y_0} Y \subset \tau^*$, and because $p \mid Y$ is a submersion we get $0 \times \mathbf{R}^l \times 0 \subset dg(y_0, 0)(Tg_{y_0} Y \times \mathbf{R})$, and the assertion is proved.

Let now $r\colon \mathbf{R}^{n+l+1} \to 0 \times \mathbf{R}^l \times 0$ be the standard projection. Then the normal directions $y_\nu - r(y_\nu)$ are given by $(x_\nu, 0, 1)$. If they converge in $G_{n+l+1,1}$ to $v \in \mathbf{R}^{n+l+1}$ with $|v| = 1$, we may assume

$$\lim_{\nu \to \infty} \frac{(x_\nu, 0, 1)}{\left(1 + |x_\nu|^2\right)^{\frac{1}{2}}} = v \quad (\text{in } \mathbf{R}^{n+l+1}),$$

and we get

$$v = \frac{(y_0, 0, 1)}{\left(1 + |y_0|^2\right)^{\frac{1}{2}}} = \frac{1}{\left(1 + |y_0|^2\right)^{\frac{1}{2}}} dg(y_0, 0)(0, 1) \in \tau.$$

Thus we have conditions (a) and (b′), which together are equivalent to (b).

3. Stratified imbeddings of cylindrical type

Let $(A, \mathfrak{S}, \mathfrak{T})$ be a finite-dimensional a.p.s., and fix a family of lines on A. In particular, for any given $\varepsilon > 0$ with $3\varepsilon < \varepsilon_A$ and each $X \in \mathfrak{S}$ with $\dim X \leqslant \dim A - 1$, we have 2ε-controlled retractions $\phi_x\colon T_x^{(2\varepsilon)} \setminus X \to \mathfrak{S}_x^\varepsilon$ such that

$\pi_x \circ \phi_x = \pi_x$, and such that the map $\psi_x\colon T_x^{(2\varepsilon)} \to Z_x^{(2\varepsilon)} := Z^{(2\varepsilon)}(\pi_x \mid S_x^\varepsilon)$ given by

$$\psi_x(a) = \begin{cases} [(\phi_x(a), \rho_x(a))] & \text{for } a \notin X, \\ [a] & \text{for } a \in X \end{cases}$$

is a 2ε-isomorphism. For $X, Y \in \mathbb{S}$ with $X \subset \overline{Y} \setminus Y$, $\phi_y \circ \phi_x$ and $\phi_x \circ \phi_y$ are defined on $T_x^{(2\varepsilon)} \cap T_y^{(2\varepsilon)} \setminus Y$, and $\phi_x \circ \phi_y = \phi_y \circ \phi_x$. Furthermore, $(\phi_x, \rho_x)\colon T_x^{(2\varepsilon)} \setminus X \to S_x^\varepsilon \times (0, 2\varepsilon)$ is a 2ε-isomorphism; in particular,

$$\begin{aligned} (\pi_y, \rho_y) \circ \phi_x &= (\phi_x \circ \pi_y, \rho_y) && \text{on } T_x^{(2\varepsilon)} \cap T_y^{(2\varepsilon)}, \\ (\pi_x, \rho_x) \circ \phi_y &= (\pi_x, \rho_x) && \text{on } T_x^{(2\varepsilon)} \cap_y^{(2\varepsilon)} \setminus Y. \end{aligned}$$

The following definition gives us a control condition describing how stratified imbeddings $\phi\colon A \to \mathbf{R}^n$ of a certain type glue together: ϕ is said to be of *δ-cylindrical type* (for some given $\delta > 0$ with $\delta \leqslant 2\varepsilon$) *relative to* $\{\phi_x\}$ if for all X the map

$$g\colon \phi(T_x^{(\delta)}) \to Z_x^{(2\varepsilon)}(\phi)(T_x^{(\delta)}) \subset \mathbf{R}^{2n+1},$$

defined by $g(\phi(a)) = (\rho_x(a) \cdot \phi(\phi_x(a)), \phi(\pi_x(a)), \rho_x(a)) =: Z_x^{(2\varepsilon)}(\phi)(a)$ for all $a \in T_x^{(\delta)}$, is a C^∞-diffeomorphism of reduced spaces. Then for all $X, (\pi_x, \rho_x) \circ \phi^{-1}\colon \phi(T_x^{(\delta)}) \to X \times \mathbf{R}$ is a C^∞-map of reduced spaces, and each controlled map is differentiable.

We can now formulate

Theorem 2. *Suppose $(A, \mathbb{S}, \mathfrak{T})$ is an a.p.s. of finite dimension, and $\{\phi_x\}$ is chosen as above. Let $\varepsilon < \delta \leqslant 2\varepsilon$, and let D be open and B be closed in A such that $B \subset D$ and $M \cap T_x^{(\delta)} = \pi_x^{-1}(M \cap X) \cap T_x^{(\delta)}$ for $M = B$, D and all $X \in \mathbb{S}$. Let $\psi_0\colon D \to \mathbf{R}^n$ be a (b)-regular imbedding such that $\psi_0(C)$ is closed in $\mathbf{R}^n$ for all closed $C \subset A$ such that $C \subset D$. Suppose further that ψ_0 is of δ-cylindrical type relative to $\{\phi_x \mid D\}$. Then for each $\varepsilon < \delta^* < \delta$ there exist an integer $N > n$ and a closed (b)-regular imbedding $\psi\colon A \to \mathbf{R}^N$ with the following properties:*

$\psi = \psi_0$ (which means "$= (\psi_0, 0)$") on B;

ψ is of δ^*-cylindrical type relative to $\{\phi_x\}$;

for all $a \in A$ with $\pi_{N-n}(\psi(a)) = 0$ we have $a \in D$ and $\psi(a) = \psi_0(a)$, where for $l < k$, $\pi_{k-l}\colon \mathbf{R}^k \to \mathbf{R}^{k-l}$ is the natural projection onto the last coordinates.

Proof. We can assume that $d(A) = k + 1$ and that the assertion is true for all a.p.s.'s with dimension $\leqslant k$. If $\sigma(A) = 1$, then A itself is a C^∞-manifold, and we have only to show the existence of an extension of ψ_0 to a closed imbedding; it is easily done.

Now suppose $\sigma(A) = l + 1$, and let $B_1, B_2 \subset_{\text{closed}} A$ with $B \subset \mathring{B}_1$, $B_1 \subset \mathring{B}_2$ and $B_2 \subset D$. Making δ smaller if necessary we may suppose that for all $Y \in \mathbb{S}$

$$B_j \cap T_Y^{(\delta)} = \pi_Y^{-1}(B_j \cap Y) \cap T_Y^{(\delta)}, \quad j = 1, 2.$$

Without loss of generality we can further assume that there is only one stratum X of smallest dimension. Let $\delta^* < \eta < \delta$.

Assertion. There are an integer $M > n$ and a (b)-regular imbedding $\psi_1\colon T_x^{(\eta)} \to \mathbf{R}^M$ with the following properties:

(1) ψ_1 is of η-cylindrical type relative to $\{\phi_y \mid T_x^{(\eta)}\}$.

(2) For all $C \subset T_x^{(\eta)}$ which are closed in A, $\psi_1(C)$ is closed in $\mathbf{R}^M$.

(3) If $D \cap X \neq \varnothing$, then $\psi_1 = \psi_0$ on $B_1 \cap T_x^{(\eta)}$, and $\pi_{M-n}(\psi_1(a)) = 0$ implies $a \in D \cap T_x^{(\eta)}$ and $\psi_1(a) = \psi_0(a)$. Furthermore: $\psi_0 \cup \psi_1\colon \mathring{B}_1 \cup T_x^{(\eta)} \to \mathbf{R}^M$ is a topological imbedding.

The proof of this assertion requires several steps. First of all S_x^ε is an a.p.s. in a natural way (compare for example [13]). Because $d(S_x^\varepsilon) \leqslant k$ we can apply our induction hypothesis to S_x^ε together with $\{\phi_Y \mid S_x^\varepsilon\}$ and $\psi_0 \mid S_x^\varepsilon$, and get for some $m > n$ a closed (b)-regular imbedding $\phi_0\colon S_X^\varepsilon \to \mathbf{R}^m$ such that

(4) $\phi_0 = \psi_0$ on $S_X^\varepsilon \cap B_2$,

(5) ϕ_0 is of η-cylindrical type relative to $\{\phi_Y \mid S_X^\varepsilon\}$.

$\pi_x \mid S_x^\varepsilon$ is controlled, so $\pi_x \circ \phi_0^{-1}$ is a C^∞-map of reduced spaces. Now let $g\colon X \to \mathbf{R}^s$ be a closed imbedding with $g \mid X \cap B_2 = \psi_0 \mid X \cap B_2$. By Lemmas 2 and 3 we get

(6) $\phi\colon T_x^{(\eta)} \to \mathbf{R}^{m+s+1}$, $\phi(a) := (\rho_x(a) \cdot \phi_0(\phi_x(a)), g(\pi_x(a)), \rho_x(a))$, is a (b)-regular imbedding,

A direct computation shows that ϕ itself is of η-cylindrical type relative to $\{\phi_Y \mid T_x^{(\eta)}\}$. For this point one needs again the differentiability of $\pi_x \circ \phi_0^{-1}$.

In order to obtain (3) we have to redefine ϕ. Take any C^∞-function $f\colon X \to \mathbf{R}$ such that $0 \leqslant f \leqslant 1$, $f \mid B_1 \cap X \equiv 1$, and support $(f) \subset \mathring{B}_2 \cap X$. Then the map

$$\psi_1\colon T_x^{(\eta)} \to \mathbf{R}^n \times \mathbf{R}^{m+s+1} \times \mathbf{R} = \mathbf{R}^M$$

defined by $\psi_1(a) := (f \circ \pi_x(a) \cdot \psi_0(a), (1 - f \circ \pi_x(a)) \cdot \phi(a), 1 - f \circ \pi_x(a))$ is a stratified imbedding. Because of (4) and the assumption on ψ_0 we have

(7) The map F such that

$$\begin{aligned} F \circ \psi_0(a) &= \big(\rho_x(a) \cdot \psi_0(\phi_X(a)), \psi_0(\pi_X(a)), \phi_X(a)\big) \\ &= \big(\rho_x(a) \cdot \phi_0(\phi_X(a)), g(\pi_X(a)), \rho_X(a)\big) \\ &= \phi(a) \end{aligned}$$

for all $a \in B_2 \cap T_x^{(\eta)}$ is a C^∞-diffeomorphism of reduced spaces. This crucial point gives us:

(i) ψ_1 is locally equivalent to ϕ or ψ_0 by a C^∞-diffeomorphism of reduced spaces and is thus a (b)-regular imbedding (Lemma 1), and furthermore

(ii) ψ_1 is of η-cylindrical type relative to $\{\phi_y \mid T_x^{(\eta)}\}$ (notice that this is a local question).

It is now easy to see that ψ_1 satisfies our assertion.

We will now use a second induction on the height of the stratification to get the theorem. Let $A^* = A \setminus X$, $\mathfrak{S}^* = \mathfrak{S} \setminus \{X\}$, and $\mathfrak{T}^* = \mathfrak{T} \setminus \{(T_x, \pi_x, \rho_x)\}$. Take $D^* = (\mathring{B}_1 \cup T_x^{(\eta)}) \setminus T_x^{\varepsilon/4}$, $B^* = (B \cup T_x^{\delta^*}) \setminus T_x^{(\varepsilon/2)}$, and $\psi^* = \psi_0 \cup \psi_1 \mid D^*$. Our assertion and the fact that $\sigma(A^*) = l$, $d(A^*) = k + 1$ allow us to apply Theorem 1, and thus we get an integer $N > M$ and a closed (b)-regular imbedding ψ_2: $A^* \to \mathbf{R}^N$ such that

(8) ψ_2 is of δ^*-cylindrical type relative to $\{\phi_Y \mid Y \in \mathfrak{S}^*\}$,

(9) $\psi_2 = \psi_0^*$ on B^*, and if $\pi_{N-M}(\psi_2(a)) = 0$ for $a \in A^*$, then $a \in D^*$ and $\psi_2(a) = \psi_0^*(a)$.

Then the map ψ: $A \to \mathbf{R}^N$ given by

$$\psi(a) := \begin{cases} \psi_1(a) & \text{for } a \in T_x^{(\delta^*)}, \\ \psi_2(a) & \text{for } a \in A \setminus T_x^{\varepsilon/2} \end{cases}$$

is a well-defined, continuous and (because of (9)) injective map.

We want to show that ψ is an open map onto its image. Suppose a, $a_\nu \in A$ with $\lim \psi(a_\nu) = \psi(a)$. If $\pi_{N-M}(\psi(a)) \neq 0$, we have $\lim \psi_2(a_\nu) = \psi_2(a)$ and thus $\lim a_\nu = a$.

Suppose now $\pi_{N-M}(\psi(a)) = 0$. By (9) we have $a \in D^*$ and $\psi(a) = \psi_0^*(a)$. First we consider only those a_ν which are in $T_x^{(\delta^*)}$. If these are infinitely many, we have $\psi(a_\nu) = \psi_1(a_\nu)$, and $\lim a_\nu = a$ by (3). For all the other a_ν we have $a_\nu \in A \setminus T_x^{\varepsilon/2}$ and $\psi(a_\nu) = \psi_2(a_\nu)$. We may suppose that these too are infinitely many. Because ψ_2 is closed, we get $\psi_0^*(a) = \psi_2(\tilde{a})$ for some $\tilde{a} \in A^*$ and $\lim a_\nu = \tilde{a} \notin T_x^{(\varepsilon/2)}$. Then by definition $\psi(\tilde{a}) = \psi_2(\tilde{a}) = \psi(a)$ and $\tilde{a} = a$. It is now easy to see that ψ satisfies all conditions of our theorem.

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