J. DIFFERENTIAL GEOMETRY
15 (1980) 217–223

GLOBAL UNIQUENESS IN THE DISC LIFTING PROBLEM

C. DENSON HILL & GERALDINE TAIANI

1. Introduction

In a previous paper [1] the authors have considered the problem of characterizing families of analytic discs in $\mathbf{C}^n$ whose boundaries lie on a prescribed *CR* submanifold. We obtained rather precise results which describe each such disc in the ambient space as the lift of a corresponding parameter disc in the tangent space to the manifold. The construction of each lifted disc involves solving a certain system of nonlinear singular integral equations in which the parameter discs occur as parameters. For a more complete discussion of these and related matters see [1].

In the work mentioned above we were concerned only with the local problem: we showed that there exists a unique local lifted disc, corresponding to each parameter disc, when the parameters occurring in the nonlinear integral equations are sufficiently small. This local problem has an interesting global analogue: does there exist a unique lifted disc associated with each parameter disc of arbitrary size? In fact, as the parameters in the system of nonlinear singular integral equations become larger and larger, one might well expect some kind of bifurcation phenomenon to take place.

In this paper we are concerned with the global uniqueness question. In the case where the prescribed *CR* submanifold has real codimension one, we show that no such bifurcation occurs; i.e., global solutions are unique. In §3 we prove such a uniqueness theorem under rather weak assumptions on the boundary values of the disc. In §4 we give a simpler proof which requires all the discs to be continuous on $\bar{D}$. In §5 we give a counterexample to global uniqueness for a submanifold having codimension two.

The authors would like to express their appreciation to Harold Widom for his enlightening conversations and especially for his help in the construction of the counterexample in §5.

Communicated by A. Nijenhuis, October 2, 1978. Research supported by National Science Foundation Grants.

2. Formulation of the problem

Let M be a real $(2n-1)$-dimensional hypersurface embedded in $\mathbf{C}^n$. In what follows we will merely need to assume that M satisfies a Lipschitz condition (so for example M could be a piecewise linear manifold). But our main assumption will be that M can be globally represented as a graph over some real linear affine hyperplane H in $\mathbf{C}^n$ (for example H might be the real tangent space to M at some point). Without loss of generality we may assume that $H = \mathbf{R} \times \mathbf{C}^m$, with $m = n - 1$, and $\mathbf{C}^n = \mathbf{C} \times \mathbf{C}^m$ with holomorphic coordinates $(z, w) = (z, w_1, \ldots, w_m)$, $z = x + iy$, such that $H = \{y = 0\}$. Thus our assumptions imply that M is globally defined by an equation of the form

$$M : y = h(x, w), \tag{1}$$

for all $(x, w) \in \mathbf{R} \times \mathbf{C}'$ where $h : \mathbf{R} \times \mathbf{C}^m \to \mathbf{R}$ is Lipschitz continuous.

Let D be the open unit disc in the complex ζ-plane with closure $\bar{D}$ and boundary S^1. We define an *analytic disc in* $\mathbf{C}^n$ to be a map $g : \bar{D} \to \mathbf{C}^n$ which is holomorphic in D. To be consistent an analytic disc in $\mathbf{R}$ should be thought of as a real constant map $c : \bar{D} \to \mathbf{R}$. Therefore an analytic disc in $H = \mathbf{R} \times \mathbf{C}^m$, which will be called a *parameter disc*, is actually a pair $(c, w) = (c, w(\zeta))$ where c is a real constant and $w : \bar{D} \to \mathbf{C}^m$ is an analytic disc in $\mathbf{C}^m$. Here we are concerned with analytic discs g in $\mathbf{C}^m$ whose boundaries $g(S^1)$ lie on M. Any such disc is of the form

$$g(\zeta) = (f(\zeta 0, w(\zeta))), \quad \zeta \in \bar{D}$$

with $f = u + iv$ an analytic disc in $\mathbf{C}$, w an analytic disc in $\mathbf{C}^m$, and where

$$v(e^{i\theta}) = h(u(e^{i\theta}), w(e^{i\theta})), \quad e^{i\theta} \in S^1. \tag{2}$$

To each such disc g we will associate the parameter disc (c, w) where $c = \operatorname{Re} f(0)$, and call g the *lift* of (c, w). This situation can be summarized by the commuting diagram

$$\begin{tikzcd}
 & M \arrow[r, hook] & \mathbf{C}^n = \mathbf{C} \times \mathbf{C}^m \arrow[d, "(\operatorname{Re} f(0), \mathrm{id})"] \\
S^1 \arrow[ur, "g|_{S^1}"] \arrow[r, hook] & \bar{D} \arrow[ur, "g=(f,w)"] \arrow[r, "(c,w)"'] & H = \mathbf{R} \times \mathbf{C}^m .
\end{tikzcd}$$

Let T be the Hilbert transform on the circle; i.e., the bounded linear singular integral operator on $L^2(S^1)$ given by

$$Tu(e^{i\theta}) = \frac{1}{2\pi} \mathrm{p.v.} \int_{-\pi}^{\pi} u(e^{i(\theta - \varphi)}) \operatorname{Im}\left(\frac{1 + e^{i\varphi}}{1 - e^{i\varphi}} \right) d\varphi.$$

T is the operator which takes the boundary values u of a harmonic function U to the boundary values v of the harmonic conjugate V with the normalization $V(0) = 0$. We will restrict our attention to analytic discs g, f, w, etc., whose components are all in the Hardy space H^2. Within this context there is a one-to-one correspondence between the possible liftings g of a given parameter disc (c, w) and the possible $L^2 = L^2(S^1)$ solutions u of the nonlinear singular integral equation

$$u(e^{i\theta}) = c - T\left[h(u(e^{i\theta}), w(e^{i\theta}))\right], \quad e^{i\theta} \in S^1. \tag{3}$$

Thus the question of uniqueness in the disc lifting problem is equivalent to the question of uniqueness of solutions of (3).

3. Global uniqueness in codimenson one

We have the following theorem.

Theorem 1. *Let M be uniformly Lipschitz continuous on compact subsets of $\mathbf{R} \times \mathbf{C}^m$, and let (c, w) be any parameter disc with bounded measurable boundary values. Then*

(a) *bounded measurable solutions u of* (3) *are unique,*

(b) *liftings g of (c, w) with bounded measurable boundary values are unique.*

Remark. Our proof shows that the boundedness assumptions above can be relaxed as follows: assume that M is uniformly Lipschitz continuous on $\mathbf{R} \times \mathbf{C}^m$ with Lipschitz constant L, and consider a paramater disc whose boundary values are finite almost anywhere. Then solutions u of (3), or liftings g, which are in L^p for some $p > (1 - (2/\pi)\tan^{-1} L)^{-1}$, are unique.

Proof. From what we have said above it suffices to prove part (a): let u_1, u_2 be two bounded measurable solutions of (3) corresponding to the same parameter (c, w),

$$u_1 = c - T\left[h(u_1, w)\right], \quad u_2 = c - T\left[h(u_2, w)\right].$$

Then $u = u_1 - u_2$ satisfies the equation

$$(I + TK)u = 0, \tag{4}$$

where K is the operator on $L^2(S^1)$ of multiplication by the function

$$k(e^{i\theta}) = \begin{cases} \dfrac{h(u_1(e^{i\theta}), w(e^{i\theta})) - h(u_2(e^{i\theta}), w(e^{i\theta}))}{u_1(e^{i\theta}) - u_2(e^{i\theta})}, & \text{if } u_1(e^{i\theta}) \neq u_2(e^{i\theta}), \\ 0, & \text{otherwise.} \end{cases}$$

Note that k is a bounded measurable function, bounded by the uniform Lipschitz constant for h on an appropriate compact set in $\mathbf{R} \times \mathbf{C}^m$. In order to conclude that $u = 0$ almost anywhere, it will suffice to show that the bounded measurable function $\varphi = Ku$ vanishes almost anywhere on S^1, since $u = -T\varphi$ by (4).

The function φ satisfies the equation

$$(I + KT)\varphi = 0 \tag{5}$$

obtained by applying the operator K to (4). Let

$$\Phi^+(z) = \begin{cases} a_0 + 2\sum_{n=1}^{\infty} a_n z^n, & z \in D, \\ (\varphi + iT\varphi)(e^{i\theta}), & z = e^{i\theta}, \end{cases}$$

$$\Phi^-(z) = \begin{cases} a_0 + 2\sum_{n=1}^{\infty} a_{-n} z^{-n}, & z \in E, \\ (\varphi - iT\varphi)(e^{i\theta}), & z = e^{i\theta}, \end{cases}$$

with E equal to the complement of $\overline{D}$ in the Riemann sphere, and where

$$\varphi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

is the Fourier series for φ. Then Φ^+ and Φ^- are holomorphic in D and E, respectively, with boundary values in L^p for all $1 \leqslant p < \infty$, and $\Phi^+(0) = \Phi^-(\infty) = a_0$. Moreover

$$\varphi(e^{i\theta}) = \tfrac{1}{2}\left[\Phi^+(z) + \Phi^-(z)\right], \quad z = e^{i\theta},$$

and

$$i(T\varphi)(e^{i\theta}) = \tfrac{1}{2}\left[\Phi^+(z) - \Phi^-(z)\right], \quad z = e^{i\theta},$$

are valid in $L^p(S^1)$. Inserting these relations in (5) we obtain

$$\Phi^+(e^{i\theta}) + G(e^{i\theta})\Phi^-(e^{i\theta}) = 0, \tag{6}$$

where the function

$$G(e^{i\theta}) = \frac{1 + ik(e^{i\theta})}{1 - ik(e^{i\theta})}$$

has its range contained in $S^1 - \{-1\}$, since k is real and bounded. Then

$$\psi(e^{i\theta}) = \log G(e^{i\theta}) = i \operatorname{Arg} G(e^{i\theta})$$

is a well defined bounded measurable function with

$$-\pi < -\gamma_0 \leqslant \operatorname{Arg} G(e^{i\theta}) \leqslant \gamma_0 < \pi,$$

for some constant γ_0.

In exactly the same way that the functions Φ^+ and Φ^- were constructed from φ, we construct the functions $2\Psi^+$ and $-2\Psi^-$ from ψ. Then Ψ^+ and Ψ^- are holomorphic in D and E, respectively, with the boundary values in L^p for all $1 \leqslant p < \infty$:

$$\Psi^+(z) = \tfrac{1}{2}(\psi + iT\psi)(e^{i\theta}), \quad z = e^{i\theta},$$

$$\Psi^-(z) = -\tfrac{1}{2}(\psi - iT\psi)(e^{i\theta}), \quad z = e^{i\theta}.$$

Moreover, $\Psi^+(0) = -\Psi^-(\infty) = \frac{1}{2}b_0$ and

$$\psi(e^{i\theta}) = \Psi^+(z) - \Psi^-(z), \quad z = e^{i\theta}.$$

Next we show that the functions $e^{-\Psi^+}$ and $e^{-\Psi^-}$ have boundary values in L^p for $1 \leqslant p \leqslant 1 + \delta$, where $\delta > 0$ is chosen small enough so that $\gamma \equiv (1 + \delta)\gamma_0 < \pi$. In fact

$$\int_{S^1} |e^{-\Psi^\pm}|^{1+\delta} = \int_{S^1} |e^{(\mp\psi - iT\psi)/2}|^{1+\delta}$$

$$\leqslant \int_{S^1} e^{(1+\delta)/2}|T\psi| = \int_{S^1} e^{\gamma|T(-i\psi/\gamma_0)|/2}.$$

However since $-i\psi/\gamma_0$ is real valued, with absolute value bounded by one, and $0 < \gamma/2 < \pi/2$, a well known theorem about conjugate functions [2, Theorem 1.9, p. 70] implies that the last integral above is bounded by

$$\frac{4\pi}{\cos\gamma/2}.$$

Since $e^{-\Psi^+}$ and $e^{-\Psi^-}$ are holomorphic in D and E, respectively, it follows, as is well known, that their boundary values are assumed in an L^p sense for $1 \leqslant p \leqslant 1 + \delta$.

Multiplying (6) by $e^{-\Psi^+}$ and using the relation

$$G = e^{\Psi^+} \cdot e^{-\Psi^-}$$

on S^1, we obtain the equation

$$F^+(e^{i\theta}) + F^-(e^{i\theta}) = 0, \quad e^{i\theta} \in S^1, \tag{7}$$

where

$$F^+(z) = e^{-\Psi^+(z)}\Phi^+(z), \quad z \in \bar{D}, \tag{8}$$

$$F^-(z) = e^{-\Psi^-(z)}\Phi^-(z), \quad z \in \bar{E},$$

are holomorphic in D and E, respectively. Moreover F^+ and F^- have L^1 boundary values, assumed in an L^1 sense, because in particular Φ^+ and Φ^- have boundary values in $L^{(1+\delta)/\delta}$. Next consider the function F defined by

$$F(z) = \begin{cases} F^+(z), & z \in \bar{D}, \\ -F^-(z), & z \in E. \end{cases}$$

It now follows from (7) that F satisfies the Cauchy-Riemann equations across S^1 in the sense of distributions and therefore is an entire function. However F

is bounded, as

$$F(\infty) = -e^{b_0/2}a_0 \equiv c;$$

hence $F(z) \equiv c$ by Liouville's theorem.

By the definition of b_0 we have that b_0 is purely imaginary and $|b_0| < \pi$ by the mean value property; therefore $e^{\pm b_0/2}$ cannot be purely imaginary. From the relations $\Phi^+(0) = \Phi^-(\infty) = a_0$, $\Psi^+(0) = -\Psi^-(\infty) = b_0/2$, and the definition of F, we obtain $c = a_0e^{-b_0/2} = -a_0e^{b_0/2}$. Thus $c^2 = -a_0^2$, and a_0 is real. Therefore $c = 0$ since c cannot be purely imaginary. It follows from (8) that $\Phi^+ \equiv \Phi^- \equiv 0$, and therefore $\varphi = 0$ almost anywhere on S^1. This completes the proof of the theorem.

4. A simple proof for the case of continuous boundary values

In this section we make no explicit assumptions about the smoothness of M; we merely assume that it is expressible as above by a real valued function h which is defined on all of $\mathbf{R} \times \mathbf{C}^m$, $m = n - 1$. Instead, we will assume that all analytic discs under consideration have components which are holomorphic in D and continuous on $\overline{D}$. Let (c, w) be such a parameter disc. Suppose that $g_1 = (f_1, w)$ and $g_2 = (f_2, w)$ are two such liftings of (c, w). We will show that g_1 and g_2 must coincide.

Consider the complex valued function $f \in \mathcal{O}(D) \cap C(\overline{D})$ defined by $f = f_1 - f_2$. We will denote the boundary values of f_1, f_2 and f by $u_1 + iv_1$, $u_2 + iv_2$ and $u + iv$, respectively. Consider the mapping f from $\overline{D}$ to the complex $u + iv$ plane: let K be the compact set on the real line formed by the intersection of $f(\overline{D})$ with the pure imaginary axis. We will assume that $f \not\equiv 0$ and derive a contradiction.

On the one hand K must have nonvoid interior. Since $\operatorname{Re} f(0) = 0$, u has mean value zero; but $u \not\equiv 0$, therefore u assumes both positive and negative values. By continuity there exist a point in the open right half plane and a point in the open left half plane which belong to $f(D)$. However $f(D)$ is connected, hence $K \cap f(D) \neq \varnothing$; but $f(D)$ is open, so K must have nonvoid interior.

On the other hand K must reduce to a point, namely the origin. The continuous coordinate function v in the $u + iv$ plane attains its maximum value v^* and its minimum value v_* on the compact set K. The point $v^* \in f(S^1)$ because $f(D)$ is open. But going back to the boundary values $u + iv$ of f, we observe that $v = 0$ at a point of S^1 if $u = 0$ at that same point, since $v = h(u_1, w) - h(u_2, w)$. Therefore $v^* = 0$; likewise $v_* = 0$. This completes the proof.

5. An example of bifurcation in codimension two

The uniqueness problem for the hypersurface M discussed above has an analogue for a submanifold $M = M^{2n-l}$ of codimension l: now set $m = n - l$ and introduce holomorphic coordinates

$$(z, w) = (z_1, \cdots, z_l, w_1, \cdots, w_m), \quad z = x + iy, \quad \text{on } \mathbf{C}^n = \mathbf{C}^l \times \mathbf{C}^m.$$

Assume that the l-codimensional submanifold M is globally defined by the vector equation

$$M : y = h(x, w), \tag{9}$$

for all $(x, w) \in \mathbf{R}^l \times \mathbf{C}^m$, where $h : \mathbf{R}^l \times \mathbf{C}^m \to \mathbf{R}^l$. The formulation of the disc lifting problem for codimension l is exactly the same as in §2, except for these minor changes: $c = (c_1, \cdots, c_l) \in \mathbf{R}^l$ is a constant vector, $f = (f_1, \cdots, f_l) : \bar{D} \to \mathbf{R}^l$ denotes the first l components of a lifted disc g, and (2) becomes a vector equation. Likewise (3) becomes an $l \times l$ system of nonlinear singular integral equations. As before the question of uniqueness in the disc lifting problem is equivalent to the question of uniqueness of solutions of (3).

Unfortunately neither the method of proof of §3 nor the method of proof of §4 extends in a straightforward manner to the case of codimension greater than one; in fact in codimension 2 we have the following counterexample.

Consider the real 4 dimensional manifold M^4 embedded in $\mathbf{C}^3$ which has the defining equations

$$M^4 : \begin{cases} y_1 = h_1(x, w) = -(x_1^2 + x_2^2)x_2, \\ y_2 = h_2(x, w) = (x_1^2 + x_2^2)x_1. \end{cases}$$

Note that M^4 is a graph over the 2-codimensional real linear subspace $\mathbf{R}^2 \times \mathbf{C}$. The results of [1] apply to this M^4 in a sufficiently small neighborhood of the origin; namely, corresponding to any parameter disc $(c, w) \in \mathbf{R}^2 \times \mathbf{C}$, with sufficiently small C^α norm there is a unique local lifting to an analytic disc g in $\mathbf{C}^3$ with boundary on M. However, corresponding to any parameter disc $(0, w) \in \mathbf{R}^2 \times \mathbf{C}$, even those with arbitrarily small C^α norm, we have, for this example, two distinct global liftings:

$$g_1(\zeta) = (0, 0, w(\zeta)), \quad \zeta \in \bar{D},$$

$$g_2(\zeta) = (\zeta, i\zeta, w(\zeta)), \quad \zeta \in \bar{D}.$$

References

[1] C. D. Hill & G. Taiani, *Families of analytic discs in $\mathbf{C}^n$ with boundaries on a prescribed CR submanifold*, Ann. Scuola Norm. Sup. Pisa **2** (1978) 327–380.

[2] Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968.

STATE UNIVERSITY OF NEW YORK, STONY BROOK
INSTITUTE FOR ADVANCED STUDY