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THE CURVATURE OF HOMOGENEOUS SIEGEL DOMAINS

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1. Introduction

Our principal goal is to give fairly general sufficient conditions to guarantee that a homogeneous Siegel domain with the Bergman metric has some positive sectional curvatures. Examples have been found by K. H. Look and Hsu-I-Chau [4] and by S. Vagi (private communication) but our method, using j-algebras, appears to be different. We would like to thank J. Dorfmeister, R. Goodman, H. Rossi, S. Vagi, and E. Wilson for various helpful communications.

Throughout this paper, all vector spaces, algebras, etc., are finite dimensional. In this section, we give definitions and quote some of the fundamental results of Gindikin, Pjateckii-Sapiro, and Vinberg [2], [6], [9]. Additional references are [3], [5], [7], and [8].

By a Siegel domain, we mean a set $D = D(\Omega, F) = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m$: $\operatorname{Im} z - F(w, w) \in \Omega\}$ where Ω is an open convex cone (containing no lines) in $\mathbf{R}^n$ and F: $\mathbf{C}^m \times \mathbf{C}^m \to \mathbf{C}^n$ is Ω-Hermitian. We allow the possibility $m = 0$ so as to formally include the Siegel domains of type I. We consider D as a Riemannian manifold with respect to the Bergman metric which we denote by $\langle\, ,\, \rangle$. Let G be the group of all biholomorphic transformations of D onto D (every such transformation is also an isometry), and G_a the subgroup of complex affine automorphisms of $\mathbf{C}^{m+n}$ which take D onto D. Then G is transitive if and only if G_a is transitive, and in that case D is said to be homogeneous. Each homogeneous Siegel domain is biholomorphically equivalent (hence isometric) to a homogeneous bounded domain, and conversely. Suppose $S \subset G$ is a simply transitive Lie subgroup. Fixing a point $p \in D$, we can identify S with D by $g \to g(p)$. The pull back of the Bergman metric then becomes a left invariant Riemanian metric, also denoted $\langle\, ,\, \rangle$, on S. The almost complex structure of D at p then pulls back to a vector space endomorphism j on the Lie algebra $\mathfrak{s}$.

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Now start with a real Lie algebra $\mathfrak{s}$, an endomorphism j of the vector space $\mathfrak{s}$ and a linear form ω on $\mathfrak{s}$.

Definition 1. The triple $(\mathfrak{s}, j, \omega)$ is called a normal j-algebra if for all X, $Y \in \mathfrak{s}$

(1.1) j is an almost complex structure, i.e., $j^2X = -X$,

(1.2) $[X, Y] + j[jX, Y] + j[X, jY] = [jX, jY]$,

(1.3) $\omega[jX, jY] = \omega[X, Y]$,

(1.4) $\omega[jX, X] > 0$ for $X \neq 0$,

(1.5) $\mathfrak{s}$ is solvable, and the adjoint representation has only real eigenvalues.

The relation between normal j-algebras and homogeneous Siegel domains is given in the following.

Theorem 1. *Let D be a homogeneous Siegel domain. Then simply transitive split-solvable Lie subgroups $S \subset G_a$ exist, and for each such S and point $p \in D$ the Lie algebra $\mathfrak{s}$ of S has the structure of a normal j-algebra $(\mathfrak{s}, j, \omega)$, where j is the pull back of the almost complex structure of D at p, and the pull back of the Bergman metric satisfies $\omega[jX, Y] = \langle X, Y\rangle$. Further, up to isomorphism, all normal j-algebras arise in this way. Note however that two j-algebras are considered isomorphic if they are connected by an algebra isomorphism which preserves the almost complex structure but need not preserve the form (see Theorem* 4).

2. Structure of normal j-algebras

Fix a normal j-algebra $(\mathfrak{s}, j, \omega)$ with positive definite j-invariant inner product $\langle X, Y\rangle = \omega[jX, Y]$. The following is part of the basic structure theorem of Pjateckii-Sapiro [6], although (2.5) was stated most explicitly by Rossi and Vergne [7], [8].

Theorem 2. *Let $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{s}$. Then each of the following statements is true.*

(2.1) *$\mathfrak{a}$ is a commutative subalgebra, and $\mathfrak{n}$ can be represented as the direct sum of the root spaces $\mathfrak{n}_\alpha = \{X \in \mathfrak{n}\colon [H, X] = \alpha(H)X, H \in \mathfrak{a}\}$ of the adjoint action of $\mathfrak{a}$ on $\mathfrak{n}$.*

(2.2) *Let $\varepsilon_1, \cdots, \varepsilon_R$ be the roots whose root spaces are mapped into $\mathfrak{a}$ by j. Then $R = \dim \mathfrak{a}$ and, with proper labelling, all roots are of the form $\frac{1}{2}\varepsilon_k$, ε_k, $1 \leqslant k \leqslant R$; $\frac{1}{2}(\varepsilon_m \pm \varepsilon_n)$, $1 \leqslant m < n \leqslant R$ (although not all these need be roots)*

(2.3) *If $X \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$, $Y \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$ are nonzero, then $[X, Y] \neq 0$.*

(2.4) *If $X \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$, $Y \in \mathfrak{n}_{\varepsilon_n}$ are nonzero, then $[X, Y] \neq 0$.*

(2.5) $j\mathfrak{n}_{\frac{1}{2}\varepsilon_k} = \mathfrak{n}_{\frac{1}{2}\varepsilon_k}, j\mathfrak{n}_{\frac{1}{2}(\varepsilon_m + \varepsilon_n)} = \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$.

We will need to derive some minor corollaries from this theorem (some of which can in fact be found imbedded in its proof). First, for each linear functional α on $\mathfrak{a}$, let $H_\alpha \in \mathfrak{a}$ be defined by $\langle H_\alpha, H\rangle = \alpha(H)$, $H \in \mathfrak{a}$. Also observe $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \subset \mathfrak{n}_{\alpha+\beta}$, and (2.2) therefore forces many brackets to vanish. For example, $\Sigma\, \mathfrak{n}_{\varepsilon_k}$ is abelian for this reason. Next, each $\mathfrak{n}_{\varepsilon_k}$ is one-dimensional, and we will arbitrarily fix nonzero elements $X_k \in \mathfrak{n}_{\varepsilon_k}$ which then gives a basis $\{jX_k\colon 1 \leqslant k \leqslant R\}$ of $\mathfrak{a}$. If $k \neq m$, we have, by (2.1) and (1.2),

$$0 = [jX_k, jX_m] = [X_k, X_m] + j[jX_k, X_m] + j[X_k, jX_m]$$
$$= j\{\varepsilon_m(jX_k)X_m - \varepsilon_k(jX_m)X_k\}.$$

Since X_k and X_m are linearly independent, this gives

(1) $$\varepsilon_k(j\mathfrak{n}_{\varepsilon_m}) = 0 \text{ if } k \neq m.$$

However,

$$0 \neq \langle X_k, X_k\rangle = \omega[jX_k, X_k] = \varepsilon_k(jX_k)\omega(X_k),$$

so

(2) $$\varepsilon_k(j\mathfrak{n}_{\varepsilon_k}) \neq 0.$$

In particular, the root $\varepsilon_1, \cdots, \varepsilon_R$ are linearly independent. Now from (1), we have $\langle X_k, X_m\rangle = \omega[jX_k, X_m] = 0$ if $k \neq m$, so the root spaces $\mathfrak{n}_{\varepsilon_1}, \cdots, \mathfrak{n}_{\varepsilon_R}$ are pairwise orthogonal. Now using the orthogonality of the basis $\{jX_k\colon 1 \leqslant k \leqslant R\}$, we get

(3) $$\langle X_k, X_k\rangle H_{\varepsilon_k} = \varepsilon_k(jX_k)jX_k,$$

(4) $$\langle H_{\varepsilon_k}, H_{\varepsilon_m}\rangle = 0 \text{ if } k \neq m.$$

Now fix $X \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$. Using (2.2) and (1.2) gives

$$[X_n, X] = -j[jX_n, X] - j[X_n, jX] + [jX_n, jX]$$
$$= -\left(\tfrac{1}{2}(\varepsilon_m - \varepsilon_n)(jX_n)\right)jX + \left(\tfrac{1}{2}(\varepsilon_m + \varepsilon_n)(jX_n)\right)jX$$
$$= \varepsilon_n(jX_n)jX.$$

Thus

$$\varepsilon_n(jX_n)\omega(jX) = \omega[X_n, X] = \omega[jX_n, jX] = \left(\tfrac{1}{2}(\varepsilon_m + \varepsilon_n)(jX_n)\right)\omega(jX).$$

From (1) and (2), this gives

(5) $$\omega(\mathfrak{n}_{\frac{1}{2}(\varepsilon_m + \varepsilon_n)}) = 0 \text{ for } m < n.$$

Again, by (2.2),

$$0 = \omega[jX, X_n] = \langle X, X_n\rangle = \omega[jX_n, X] = \left(\tfrac{1}{2}(\varepsilon_m - \varepsilon_n)(jX_n)\right)\omega(X),$$

which proves

(6) $$\omega(\mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}) = 0 \text{ for } m < n.$$

Now for each root $\alpha \neq \varepsilon_k$, let $\bar{\alpha}$ be the root for which $j\mathfrak{n}_\alpha \subset \mathfrak{n}_{\bar{\alpha}}$. For such α and any root β, (2.2), (5), (6) show that

$$\langle \mathfrak{n}_\alpha, \mathfrak{n}_\beta \rangle = \omega[j\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \subset \omega(\mathfrak{n}_{\bar{\alpha}+\beta}) = 0,$$

unless $\bar{\alpha} + \beta$ is a root of the form ε_k or $\frac{1}{2}\varepsilon_k$. It is easy to check that the only distinct roots α, β (not both of the form ε_k) which are exceptions are $\alpha = \frac{1}{2}\varepsilon_n$, $\beta = \frac{1}{2}(\varepsilon_m - \varepsilon_n)$ but then we need only reverse the roles of α and β. Thus we have the following corollary:

(7) *The root spaces are all pairwise orthogonal.*

Finally we need some more technical results. Choose any elements

$$Y_m \in \mathfrak{n}_{\frac{1}{2}\varepsilon_m}, \quad Y_n \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}, \quad Z \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}.$$

Then (1.2) can be written as

$$[jY_m, Y_n] + [Y_m, jY_n] = j[Y_m, Y_n] - j[jY_m, jY_n],$$

where the left-hand side is in $\mathfrak{n}_{\frac{1}{2}(\varepsilon_m + \varepsilon_n)}$, while the right-hand side is in $\mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$. Thus each side vanishes, and we have $[jY_m, Y_n] = -[Y_m, jY_n]$ or equivalently,

(8) $$(\operatorname{ad} Y_n) \circ j | \mathfrak{n}_{\frac{1}{2}\varepsilon_m} = -(\operatorname{ad} jY_n) | \mathfrak{n}_{\frac{1}{2}\varepsilon_m}.$$

Similarly, (1.2) and (2.1) give

$$0 = [jY_n, jZ] - j[Y_n, jZ] = [Y_n, Z] + j[jY_n, Z],$$

or equivalently,

(9) $$(\operatorname{ad} jY_n) | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)} = j \circ (\operatorname{ad} Y_n) | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}.$$

Thus

(10) $$\begin{aligned}(\operatorname{ad} Y_n)(\operatorname{ad} jY_n) | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)} &= (\operatorname{ad} Y_n) \circ j \circ (\operatorname{ad} Y_n) | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)} \\ &= -(\operatorname{ad} jY_n)(\operatorname{ad} Y_n) | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)},\end{aligned}$$

and so

(11) $$(\operatorname{ad} Y_n)(\operatorname{ad} jY_n) | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)} = \tfrac{1}{2} \operatorname{ad}[Y_n, jY_n] | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}.$$

Now $\langle Y_n, Y_n \rangle = \omega[jY_n, Y_n]$ which together with (2.4) implies:

(12) If $Y_n \neq 0$, then $(\operatorname{ad} Y_n)(\operatorname{ad} jY_n) | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$ is nonsingular.

Lemma 1. *Suppose $\frac{1}{2}\varepsilon_m$, $\frac{1}{2}\varepsilon_n$, $\frac{1}{2}(\varepsilon_m - \varepsilon_n)$ are roots with* $\dim \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)} = 1$. *Choose any nonzero $Y \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$, and let $\phi = \operatorname{ad} Y | \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$, $\psi = \operatorname{ad} Y | \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$ so we have*

$$\mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)} \xrightarrow{\phi} \mathfrak{n}_{\frac{1}{2}\varepsilon_m} \xrightarrow{\psi} \mathfrak{n}_{\frac{1}{2}(\varepsilon_m + \varepsilon_n)}.$$

Then $\langle \operatorname{Im} \phi, \operatorname{Ker} \psi \rangle \neq 0$.

Proof. Let $\bar{\phi} = \operatorname{ad} jY|\mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$ and suppose $\langle \operatorname{Im} \phi, \operatorname{Ker} \psi \rangle = 0$. Now ϕ is injective by (2.3) so $\dim \operatorname{Im} \phi = 1$ which in particular shows $\psi \neq 0$. Then $\dim \mathfrak{n}_{\frac{1}{2}(\varepsilon_m \pm \varepsilon_n)} = 1$ implies $1 = \dim \operatorname{Im} \psi = \dim \mathfrak{n}_{\frac{1}{2}\varepsilon_m} - \dim \ker \psi$ so $\mathfrak{n}_{\frac{1}{2}\varepsilon_m} = \operatorname{Im} \phi + \operatorname{Ker} \psi$ (orthogonal direct sum). Thus $j \operatorname{Im} \phi \subset \operatorname{Ker} \psi$. But (9) shows that $j \operatorname{Im} \phi = \operatorname{Im} \bar{\phi}$ so $0 = \psi\bar{\phi}$. This contradicts (12).

3. Results on curvature

S will be a connected Lie group whose Lie algebra $\mathfrak{s}$ is a normal j-algebra. Thus $\mathfrak{s}$ has the canonical inner product $\langle X, Y \rangle = \omega[jX, Y]$, and S has the induced left invariant metric. The Levi-Civita connection ∇ is computed by

$$(13)\quad 2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle, \qquad X, Y, Z \in \mathfrak{s}.$$

Note that ∇_X becomes a skew symmetric linear transformation on $\mathfrak{s}$, and that $\nabla_X Y - \nabla_Y X = [X, Y]$. We will constantly use the results of §2, especially (2.2), (4), and (7). An easy application of these gives

$$(14)\qquad \nabla_H = 0, H \in \mathfrak{a},$$

$$(15)\qquad \nabla_Y Y = \langle Y, Y \rangle H_\alpha \text{ for } Y \in \mathfrak{n}_\alpha.$$

In computing sectional curvatures, we use

$$(16)\quad \langle R(Y, Z)Z, Y \rangle = -\langle \nabla_Y Y, \nabla_Z Z \rangle + \langle \nabla_Y Z, \nabla_Z Y \rangle - \langle \nabla_{[Y,Z]} Z, Y \rangle.$$

For $H \in \mathfrak{a}$, $Z \in \mathfrak{n}_\alpha$, we find

$$\langle R(H, Z)Z, H \rangle = -\langle \nabla_{[H,Z]} Z, H \rangle = -\alpha(H)\langle \nabla_Z Z, H \rangle = -\alpha(H)^2 \langle Z, Z \rangle,$$

which shows there are always plane sections with negative curvature. Now in contrast, we can state our main result.

Theorem 3. *Suppose $\mathfrak{s}$ is a normal j-algebra, and there exist roots $\frac{1}{2}\varepsilon_m$, $\frac{1}{2}\varepsilon_n$, $\frac{1}{2}(\varepsilon_m - \varepsilon_n)$ with* $\dim \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)} = 1$. *Then the corresponding left invariant metric on S has plane sections with positive curvature.*

Proof. By Lemma 1, there exist nonzero elements $Y \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$, $Z \in \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$, $V \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$ such that

$$[Y, Z] = 0, \langle [Y, V], Z \rangle \neq 0.$$

Then we compute

$$\langle \nabla_Y Y, \nabla_Z Z \rangle = \langle Y, Y \rangle \langle Z, Z \rangle \langle H_{\frac{1}{2}\varepsilon_n}, H_{\frac{1}{2}\varepsilon_m} \rangle = 0,$$

$$\langle \nabla_Y Z, \nabla_Z Y \rangle = \langle \nabla_Y Z, \nabla_Y Z \rangle = |\nabla_Y Z|^2,$$

$$\langle \nabla_Y Z, V \rangle = \tfrac{1}{2}\langle [V, Y], Z \rangle \neq 0.$$

But then $\langle R(Y, Z)Z, Y \rangle > 0$.

We conclude this section with some results pertaining to the freedom of choice for the form ω in a j-algebra. First, note that for $H \in \mathfrak{a}$, $X \in \mathfrak{n}_\alpha$, (14) implies that

$$R(X, H)H = -\nabla_{[X,H]}H = \alpha(H)\nabla_X H = \alpha(H)\{\nabla_H X + [X, H]\} = -\alpha(H)^2 X,$$

while

$$R(H', H)H = 0 \text{ for } H' \in \mathfrak{a}.$$

Thus we can compute the Ricci tensor by

$$\operatorname{Ric}(H, H) = -\sum_\alpha \alpha(H)^2 \dim \mathfrak{n}_\alpha.$$

In particular

$$\operatorname{Ric}(H_{\varepsilon_k}, H_{\varepsilon_k}) = -\langle H_{\varepsilon_k}, H_{\varepsilon_k}\rangle^2 \Big\{1 + \tfrac{1}{4} \dim \mathfrak{n}_{\frac{1}{2}\varepsilon_k} + \tfrac{1}{2} \sum_{m \neq k} \dim \mathfrak{n}_{\frac{1}{2}(\varepsilon_k + \varepsilon_m)}\Big\}. \tag{17}$$

However by (3) and the formula above (2),

$$\langle H_{\varepsilon_k}, H_{\varepsilon_k}\rangle = \frac{\varepsilon_k(jX_k)}{\omega(X_k)}, \tag{18}$$

which is independent of choice of X_k but does depend on ω. Remembering that the Bergman metric on a homogeneous domain is always Einstein and in fact satisfies Ric $= -\langle\ ,\ \rangle$, we have

Theorem 4. *Suppose $\mathfrak{s}$ is a normal j-algebra. If the left invariant metric on S corresponding to ω is Einstein, then*

$$\frac{\varepsilon_k(jX_k)}{\omega(X_k)} \Big\{1 + \tfrac{1}{4} \dim \mathfrak{n}_{\frac{1}{2}\varepsilon_k} + \tfrac{1}{2} \sum_{m \neq k} \dim \mathfrak{n}_{\frac{1}{2}(\varepsilon_k + \varepsilon_m)}\Big\}$$

is independent of $k = 1, \cdots, R$. Moreover, if the metric is the Bergman metric, the above expression is 1.

Example 1. In order to present concrete Siegel domains which fall under Theorem 3, we give an example, the details of whose structure are given by Murakami [5, pp. 76–95]. Here m, q are positive integers, $n = \frac{1}{2}(m + q)(m + q + 1)$, $\mathbf{R}^n$ is identified with the space of symmetric $(m + q)$-square real matrices, $\mathbf{C}^n$ is the complexification of $\mathbf{R}^n$ identified with the corresponding space of complex matrices, Ω is the cone of positive definite matrices in $\mathbf{R}^n$, $\mathbf{C}^m$ is identified with the space of m by 1 column matrices, and F: $\mathbf{C}^m \times \mathbf{C}^m \to \mathbf{C}^n$ is the Ω-Hermitian map defined by

$$F(w, w') = \begin{pmatrix} \frac{1}{2}(w'{}^t\overline{w}' + \overline{w}'{}^t w) & 0 \\ 0 & 0 \end{pmatrix}.$$

This gives a homogeneous Siegel domain $D \subset \mathbf{C}^{m+n}$. As usual, the Lie algebra $\mathfrak{g}_a$ of the affine group of D has the gradation (Murakami's indexing)

$$\mathfrak{g}_a = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0,$$

where $\mathfrak{g}_0$ is the Lie algebra of the group of linear transformations preserving D. After some identifications, we have [5, p. 81]

$$\mathfrak{g}_0 = \Big\{(A, B): A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, B = a + i\theta I_m \text{ where } a \text{ is } m \times m \text{ real},$$
$$b \text{ is } m \times q \text{ real}, c \text{ is } q \times q \text{ real}, \theta \in \mathbf{R}\Big\}.$$

Let p be the point $(iI_{m+q}, 0)$ in D. Then the isotropy subalgebra of $\mathfrak{g}_a$ at p is

$$\mathfrak{k}_0 = \{(A, B) \in \mathfrak{g}_0: a, c \text{ are skew symmetric}, b = 0\}.$$

Define the subalgebra $\mathfrak{s}$ of $\mathfrak{g}_a$ by $\mathfrak{s} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{s}_0$ where

$$\mathfrak{s}_0 = \{(A, B) \in \mathfrak{g}_0: A \text{ is upper triangular}, \theta = 0\}.$$

Let S be the analytic subgroup of G_a corresponding to $\mathfrak{s}$. Then S is solvable and simply transitive on D, so $\mathfrak{s}$ is a normal j-algebra (Theorem 1). An abelian complement to $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, of dimension $m + q$, is

$$\mathfrak{a} = \{(A, B) \in \mathfrak{s}_0: A \text{ is diagonal}\}.$$

For $(A, B) \in \mathfrak{a}$, $A = \operatorname{diag}(d_1, \cdots, d_{m+q})$, let $\varepsilon_k(A, B) = 2d_k$. Then the roots of the adjoint action of $\mathfrak{a}$ on $\mathfrak{n}$ are precisely ε_k, $1 \leqslant k \leqslant m + q$; $\frac{1}{2}\varepsilon_k$, $1 \leqslant k \leqslant m$; $\frac{1}{2}(\varepsilon_j \pm \varepsilon_k)$, $1 \leqslant j < k \leqslant m + q$ and

$$\dim \mathfrak{n}_\alpha = \begin{cases} 2 & \text{if } \alpha = \frac{1}{2}\varepsilon_k, \\ 1 & \text{if } \alpha \neq \frac{1}{2}\varepsilon_k. \end{cases}$$

To specify the root spaces precisely, introduce the following notation:

E_{jk} is the $m + q$ square matrix with $(E_{jk})_{rs} = \delta_{jr}\delta_{ks}$.

X_{jk}, $1 \leqslant j \leqslant k \leqslant m + q$, is the element of $\mathfrak{g}_a$ tangent to the one-parameter group $(z, w) \to (z + t(E_{jk} + E_{kj}), w)$, $z \in \mathbf{C}^n$, $w \in \mathbf{C}^m$.

Y_j, $Y_{\bar{j}}$, $1 \leqslant j \leqslant m$, are independent elements of the plane in $\mathfrak{g}_a$ tangent to all the one-parameter groups $(z, w) \to (z + 2iF(w, tb) + iF(tb, tb), w + tb)$, where $b \in \mathbf{C}^m$ is 0 in all except the jth entry.

Z_{jk}, $1 \leqslant j < k \leqslant m + q$, is the element (A, B) of $\mathfrak{s}_0$ with $A = E_{jk}$.

Then we find

$$\mathfrak{n}_{\frac{1}{2}(\varepsilon_j+\varepsilon_k)} = \mathbf{R}X_{jk}, \quad 1 \leqslant j \leqslant k \leqslant m+q,$$
$$\mathfrak{n}_{\frac{1}{2}\varepsilon_k} = \mathbf{R}Y_k + \mathbf{R}Y_{\bar{k}}, \quad 1 \leqslant k \leqslant m,$$
$$\mathfrak{n}_{\frac{1}{2}(\varepsilon_j-\varepsilon_k)} = \mathbf{R}Z_{jk}, \quad 1 \leqslant j < k \leqslant m+q,$$

and the root spaces $\mathfrak{n}_{\varepsilon_k}$, $1 \leqslant k \leqslant m+q$, are precisely those mapped into $\mathfrak{a}$ by the almost complex structure map.

Finally, one observes by [1, Prop. 6.6 (see also the following Remark)], that the root space structure with respect to the *orthogonal* complement of $\mathfrak{n}$ in $\mathfrak{s}$ agrees with this one up to an automorphism of $\mathfrak{s}$. Alternatively, one can show $\mathfrak{a}$ is actually the orthogonal complement by explicitly computing the Bergman metric at p by [5, Lemma 8.2] and by explicitly realizing the map from $\mathfrak{s}$ to the tangent space of D at p.

Remark. Using the results of §2, it is easy to see that, in the terminology of Azencott-Wilson [1], every j-algebra is an *NC* algebra, and the canonical inner product is admissible. This fact is already known, for example by Azencott and Wilson, but does not seem to be noted in the literature. On the one hand, this says that the Azencott-Wilson theory is no help in deciding which homogeneous Siegel domains have nonpositive sectional curvatures in the Bergman metric. On the other hand, it says that if S is a connected simply transitive Lie group of holomorphic transformations of the Siegel domain D, then D has S-invariant Riemannian metrics with nonpositive sectional curvatures. These metrics can be described quite explicitly using the structure of the j-algebra $\mathfrak{s}$ and the construction of [1, pp. 355–357]. One forms a new inner product $(\ ,\ )_n$ on $\mathfrak{s}$ such that $(\mathfrak{a}, \mathfrak{n})_n = 0$, $(\mathfrak{n}, \mathfrak{n})_n = n^2\langle \mathfrak{n}, \mathfrak{n}\rangle$ and, after canonical transference to $\mathfrak{a}^*$, $(\alpha, \beta)_n = (\alpha, \beta)_1 > 0$ for all roots α, β (possible because all roots are positive on the element $H_0 = \Sigma(k\varepsilon_k(jX_k))^{-1}jX_k)$. This gives a left invariant Riemannian metric on S (hence an S-invariant metric on D after suitable identification) which has nonpositive sectional curvature for sufficiently large n. However such metrics are unlikely to be invariant under the full group of holomorphic, or even affine transformations of D (computing the full isometry group would be an interesting problem).

Example 2. We present another example, which is actually the original 8 (real)-dimensional example of Pjateckii-Sapiro. Take a real vector space $\mathfrak{s}$ with an almost complex structure j. In $\mathfrak{s}$, fix a 2 (real)-dimensional j-invariant subspace Z and a basis $jr_1, jr_2, r_1, r_2, x, jx$ of a vector space complement of Z. Define the bracket product so that the only nonzero terms involving basis elements or elements of Z satisfy (see [6, pp. 63–64])

$$[jr_1, r_1] = r_1, \ [jr_1, x] = \tfrac{1}{2}x, \ [jr_1, jx] = \tfrac{1}{2}jx,$$
$$[jr_1, z] = \tfrac{1}{2}z \ \text{ for } z \in Z,$$
$$[jr_2, x] = -\tfrac{1}{2}x, \ [jr_2, jx] = \tfrac{1}{2}jx, \ [jr_2, r_2] = r_2,$$
$$[r_2, x] = jx, \ [x, jx] = -r_1, \ [z, z'] \in \mathbf{R}r_1 \ \text{ for } z, z' \in Z.$$

We want to define ω to make $(\mathfrak{s}, j, \omega)$ a normal j-algebra. From the bracket relations and (1.3), we see ω must vanish on x, jx, and Z, and the values of ω on jr_1, jr_2 are irrelevant for computing $\langle X, Y\rangle = \omega[jX, Y]$. Multiplying ω by a constant just does the same for the induced left invariant metric on the corresponding Lie group S. We can then define ω so that $\omega(r_1) = 1$, $\omega(r_2) = t > 0$, and ω vanishes on $Z + \operatorname{span}\{jr_1, jr_2, x, jx\}$. Then we find that $jr_1, jr_2, r_1, r_2, x, jx$ are pairwise orthogonal and orthogonal to Z, $\langle\ ,\ \rangle$ is j invariant, and

$$\langle r_1, r_1\rangle = \langle x, x\rangle = 1, \ \langle r_2, r_2\rangle = t,$$
$$[z, z'] = \langle z, jz'\rangle r_1 \ \text{ for } z, z' \in Z,$$

In this example, note that $\mathfrak{n} = Z + \operatorname{span}\{r_1, r_2, x, jx\}$, $\mathfrak{a} = \operatorname{span}\{jr_1, jr_2\}$, $\varepsilon_k(jr_m) = \delta_{km}$ $(k, m = 1, 2)$. Further, $\mathfrak{n}_{\varepsilon_1} = \mathbf{R}r_1$, $\mathfrak{n}_{\varepsilon_2} = \mathbf{R}r_2$, $\mathfrak{n}_{\frac{1}{2}\varepsilon_1} = Z$, $\mathfrak{n}_{\frac{1}{2}\varepsilon_2} = 0$, $\mathfrak{n}_{\frac{1}{2}(\varepsilon_1 - \varepsilon_2)} = \mathbf{R}x$, $\mathfrak{n}_{\frac{1}{2}(\varepsilon_1 + \varepsilon_2)} = \mathbf{R}jx$. In particular, the hypothesis of Theorem 3 does not hold, and Theorem 4 implies that we must take $t = 3/4$ to get an Einstein metric (which by Theorem 1 will be a multiple of the Bergman metric).

Now we compute the Leve-Civita connection by (13). Explictly for

$$U = u_1 jr_1 + u_2 jr_2 + u_3 r_1 + u_4 r_2 + u_5 x + u_6 jx + z, z \in Z,$$
$$V = v_1 jr_1 + v_2 jr_2 + v_3 r_1 + v_4 r_2 + v_5 x + v_6 jx + z', z' \in Z,$$

we find

$$\begin{aligned} 2\nabla_U V = {} & (2u_3v_3 + u_5v_5 + u_6v_6 + \langle z, z'\rangle)jr_1 \\ & + (2tu_4v_4 - u_5v_5 + u_6v_6)jr_2/t \\ & + (-2u_3v_1 + u_6v_5 - u_5v_6 + \langle z, jz'\rangle)r_1 \\ & + (-2tu_4v_2 + u_5v_6 + u_6v_5)r_2/t \\ & + (-u_5v_1 + u_5v_2 - u_3v_6 - u_6v_3 - u_6v_4 - u_4v_6)x \\ & + (-u_6v_1 - u_6v_2 + u_4v_5 - u_5v_4 + u_3v_5 + u_5v_3)jx \\ & -v_1 z + u_3 jz' + v_3 jz. \end{aligned}$$

For $t = 3/4$, $U = -jr_1 - jr_2 + jx$, $V = 3r_1 + 2r_2 + 6x$, we find

$$2\nabla_U V = 6r_1 + 8r_2 - 5x, \qquad 2\nabla_U U = jr_1 + \tfrac{4}{3}jr_2 + 2jx,$$
$$2\nabla_V U = 12r_2 - 5x, \qquad 2\nabla_V V = 54jr_1 - 40jr_2 + 36jx,$$
$$[U, V] = 3r_1 - 2r_2, \qquad 2\nabla_{[U,V]} V = 18jr_1 - 8jr_2 + 6jx,$$

and $4\langle R(U, V)V, U\rangle = 23$. Thus the sectional curvature through U and V is positive in the Bergman metric. However, a very nasty but elementary calculation shows that for $t = 1$ and arbitrary U, V

$$\begin{aligned} &4\langle R(U, V)V, U\rangle \\ &\quad= -(u_1v_6 - u_6v_1 + u_4v_5 - u_5v_4 + u_2v_6 - u_6v_2 + u_3v_5 - u_5v_3)^2 \\ &\qquad -(u_1v_5 - u_5v_1 + u_4v_6 - u_6v_4 - u_2v_5 + u_5v_2 - u_3v_6 + u_6v_3)^2 \\ &\qquad -(2u_2v_4 - 2u_4v_2 + u_6v_5 - u_5v_6)^2 \\ &\qquad -|u_1z' - v_1z - u_3jz' + v_3jz|^2 - |u_5z' - v_5z - u_6jz' + v_6jz|^2 \\ &\qquad -4\left\{\langle z, jz'\rangle + \tfrac{1}{2}(2u_1v_3 - 2u_3v_1 + u_6v_5 - u_5v_6)\right\}^2 \leqslant 0. \end{aligned}$$

Thus we have a deformation, through S-invariant *Kähler* metrics, from the Bergman metric to a metric with only nonpositive sectional curvatures. Again, it would be interesting to know the full isometry group of this new metric.

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