KATO'S INEQUALITY AND THE SPECTRAL DISTRIBUTION OF LAPLACIANS ON COMPACT RIEMANNIAN MANIFOLDS

H. HESS, R. SCHRADER & D. A. UHLENBROCK

1. Introduction

In this note we study Bochner Laplacians D_∇ given by connections ∇ on a smooth hermitian vector bundle V over a compact Riemannian manifold M (without boundary for simplicity). We will compare the spectrum of D_∇ with that of the Laplace-Beltrami operator Δ_g on M.

Our analysis is based on Kato's inequality [11] which we prove for the present situation, and the ensuing domination of the semigroup $\exp t\, D_\nabla$ by $\exp t\Delta_g$, [8] (see also [18], [19] for the special case of the complex line bundle over $\mathbf{R}^n$). This domination leads to the comparison of the spectra in the form

$$(1.1) \qquad \operatorname{Tr} \exp t\, D_\nabla \leqslant n \operatorname{Tr} \exp t\Delta_g \quad (t \geqslant 0),$$

where n is the rank of V. Estimate (1.1) of course yields inequalities for the corresponding Riemann zeta functions.

We extend this result by considering second order (linear) differential operators on V which differ from D_∇ by a zero order differential operator, i.e., a strict vector bundle endomorphism.

As an application we will consider Laplace-de Rham operators Δ and spinor Laplacians $\not\nabla^2$. They differ from the appropriate D_∇ by a strict vector bundle endomorphism involving the curvature of the connection employed. To cover the most general case, both exterior forms and spinors are allowed to have coefficients in an arbitrary hermitian vector bundle with connection over M. We then compute the differences $\Delta - D_\nabla$ and $\not\nabla^2 - D_\nabla$, obtaining the corresponding Weitzenböck formulas.

This note resulted from our study of the behavior of Yang-Mills potentials, which are the Christoffel symbols of connections (over $\mathbf{R}^n$). Over arbitrary

Riemannian manifolds they generalize to connections in hermitian vector bundles, including the electromagnetic potentials as the special case where the vector bundles are line bundles. In this context, estimate (1.1) reflects a diamagnetic effect of Yang-Mills potentials. Its various aspects (over $\mathbf{R}^n$) have been exploited by several authors [4], [6], [17].

2. Kato's inequality for the Bochner Laplacian

Let ∇ be a connection on V which is always understood to be linear and with respect to the hermitian structure. Furthermore, let $_g\nabla$ be the Levi-Cività connection on TM. These two connections induce a connection on $T^*M \otimes V$, the tensor product, which we denote by ∇^1 in the present section. For fixed M we follow the convention to abbreviate TM, T^*M by T, T^* respectively. The metric tensor g will variably be considered as a linear map from $C^\infty(T^* \otimes T^*)$ or as a bilinear map from $C^\infty(T^*) \times C^\infty(T^*)$ to $C^\infty(M \times \mathbf{R})$.

The Bochner (or reduced) Laplacian D_∇ is then defined by the diagram

$$D_\nabla : C^\infty(V) \xrightarrow{\nabla} C^\infty(T^* \otimes V) \xrightarrow{\nabla^1} C^\infty(T^* \otimes T^* \otimes V) \xrightarrow{g \otimes 1} C^\infty(V).$$

(Note that we use the physicist's sign convention.) The hermitian structure

$$\langle \cdot , \cdot \rangle : C^\infty(V) \times C^\infty(V) \to C^\infty(M \times \mathbf{C})$$

and the volume density $|d \operatorname{vol}|$ on M make $C^\infty(V)$ into a pre-Hilbert space whose completion will be denoted by $L^2(V)$, the Hilbert space of square integrable sections in V. We will write the scalar product as

(2.1) $$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle |d \operatorname{vol}|$$

for $\alpha, \beta \in L^2(V)$. A routine computation shows that D_∇ is symmetric on $C^\infty(V)$.

If V is the trivial bundle $M \times \mathbf{C}$ and ∇ the standard connection d, then D_∇ is the Laplace-Beltrami operator Δ_g.

We need some further notation and conventions. First g and $\langle \cdot , \cdot \rangle$ induce a hermitian structure on $T^* \otimes V$ which will be denoted by $\langle \cdot , \cdot \rangle_g$. In addition, $\langle \cdot , \cdot \rangle$ induces sesquilinear maps

$$C^\infty(T^* \otimes V) \times C^\infty(V) \to C^\infty(T^* \otimes \mathbf{C}),$$
$$C^\infty(V) \times C^\infty(T^* \otimes V) \to C^\infty(T^* \otimes \mathbf{C}),$$
$$C^\infty(T^* \otimes V) \times C^\infty(T^* \otimes V) \to C^\infty(T^* \otimes T^* \otimes \mathbf{C}),$$

which we also denote by $\langle \cdot , \cdot \rangle$. Furthermore, the continuous nonnegative function $|\alpha|$ on M will mean $\langle \alpha, \alpha \rangle^{1/2}$ for $\alpha \in C^\infty(V)$, $\langle \alpha, \alpha \rangle_g^{1/2}$ for $\alpha \in C^\infty(T^* \otimes V)$, or $g(\alpha, \alpha)^{1/2}$ for $\alpha \in C^\infty(T^*)$. Finally, write $|\alpha|_\varepsilon := (|\alpha|^2 + \varepsilon^2)^{1/2}$ for $\varepsilon > 0$. Then $|\alpha|_\varepsilon$ is C^∞ on M and $|\alpha|_\varepsilon \geqslant \varepsilon$.

Now let $\alpha \in C^\infty(V)$. We will compute $\Delta_g |\alpha|_\varepsilon$ in two ways. First we have

$$\Delta_g |\alpha|_\varepsilon^2 = g \circ {}_g\nabla \circ d|\alpha|_\varepsilon^2 = 2g \circ {}_g\nabla(|\alpha|_\varepsilon d|\alpha|_\varepsilon)$$

(2.2)
$$= 2|\alpha|_\varepsilon \Delta_g |\alpha|_\varepsilon + 2g(d|\alpha|_\varepsilon, d|\alpha|_\varepsilon)$$

$$= 2|\alpha|_\varepsilon \Delta_g |\alpha|_\varepsilon + 2|d|\alpha|_\varepsilon|^2.$$

On the other hand

$$\Delta_g |\alpha|_\varepsilon^2 = \Delta_g(\langle \alpha, \alpha \rangle + \varepsilon^2) = g \circ {}_g\nabla \circ d\langle \alpha, \alpha \rangle$$

(2.3)
$$= g \circ {}_g\nabla \langle \nabla\alpha, \alpha \rangle + g \circ {}_g\nabla\langle \alpha, \nabla\alpha \rangle$$

$$= g\langle \nabla^1 \circ \nabla\alpha, \alpha \rangle + 2g\langle \nabla\alpha, \nabla\alpha \rangle + g\langle \alpha, \nabla^1 \circ \nabla\alpha \rangle$$

$$= 2\,\mathrm{Re}\langle D_\nabla \alpha, \alpha \rangle + 2\langle \nabla\alpha, \nabla\alpha \rangle_g = 2\,\mathrm{Re}\langle D_\nabla \alpha, \alpha \rangle + 2|\nabla\alpha|^2.$$

Next we claim
(2.4)
$$|d|\alpha|_\varepsilon|^2 \leqslant |\nabla\alpha|^2.$$

Indeed we have $d|\alpha|_\varepsilon^2 = 2|\alpha|_\varepsilon d|\alpha|_\varepsilon$ and also $d|\alpha|_\varepsilon^2 = 2\,\mathrm{Re}\langle \nabla\alpha, \alpha \rangle \in C^\infty(T^*)$. Thus

$$|\alpha|_\varepsilon^2 |d|\alpha|_\varepsilon|^2 = g\,(\mathrm{Re}\langle \nabla\alpha, \alpha \rangle, \mathrm{Re}\langle \nabla\alpha, \alpha \rangle) \leqslant |\langle \nabla\alpha, \alpha \rangle|^2.$$

We now claim the following inequality
(2.5)
$$|\langle \nabla\alpha, \alpha \rangle| \leqslant |\nabla\alpha| \cdot |\alpha| \leqslant |\nabla\alpha| \cdot |\alpha|_\varepsilon,$$

from which relation (2.4) follows. Indeed (2.5) is a statement on each fibre. Therefore it is an immediate consequence of the following easy lemma in linear algebra.

Lemma 2.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two (finite-dimensional) Hilbert spaces with scalar products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Denote by $\langle \cdot, \cdot \rangle$ the induced scalar product in $\mathcal{H}_1 \otimes \mathcal{H}_2$, and also by $\langle \cdot, \cdot \rangle_2$ the induced map $(\mathcal{H}_1 \otimes \mathcal{H}_2) \times \mathcal{H}_2 \to \mathcal{H}_1$ given by*

$$\left(h_1 \otimes h_2, \tilde{h}_2\right) \mapsto h_1 \quad \langle h_2, \tilde{h}_2 \rangle_2, \quad \left(h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2, \tilde{h}_2 \in \mathcal{H}_2\right).$$

Then for any $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and $y \in \mathcal{H}_2$ the following generalized Schwarz inequality holds:

$$\langle \langle x, y \rangle_2, \langle x, y \rangle_2 \rangle_1 \leqslant \langle x, x \rangle \cdot \langle y, y \rangle_2.$$

Proof. Write

$$x = h_1 \otimes y + \sum_i h_1^i \otimes h_2^i = h_1 \otimes y + \tilde{x},$$

with $h_1, h_1^i \in \mathcal{H}_1, h_2^i \in \mathcal{H}_2$ and $\langle y, h_2^i \rangle_2 = 0$. Then

$$\langle x, x \rangle = \langle h_1, h_1 \rangle_1 \cdot \langle y, y \rangle_2 + \langle \tilde{x}, \tilde{x} \rangle$$

$$\geqslant \langle h_1, h_1 \rangle_1 \cdot \langle y, y \rangle_2,$$

$$\langle x, y \rangle_2 = h_1 \langle y, y \rangle_2,$$

such that

$$\langle\langle x, y \rangle_2, \langle x, y \rangle_2 \rangle_1 = \langle h_1, h_1 \rangle_1 \cdot (\langle y, y \rangle_2)^2$$
$$\leqslant \langle x, x \rangle \cdot \langle y, y \rangle_2. \qquad \text{q.e.d.}$$

We now compare (2.2) with (2.3) and use (2.4) to obtain the first part of

Proposition 2.2. *The Bochner Laplacian D_∇ satisfies Kato's inequality*

$$(2.6) \qquad \text{Re}\langle D_\nabla \alpha, \alpha \rangle \leqslant |\alpha|_\varepsilon \Delta_g |\alpha|_\varepsilon, \quad (\alpha \in C^\infty(V)).$$

In the limit $\varepsilon \searrow 0$ the following inequality between distributions in $\mathcal{D}'(M \times \mathbf{R})$ holds:

$$(2.7) \qquad \text{Re}\langle D_\nabla \alpha, \text{sign}_\xi \alpha \rangle \leqslant \Delta_g |\alpha|, \quad (\alpha \in C^\infty(V)),$$

where

$$\text{sign}_\xi \, \alpha = \begin{cases} \dfrac{\alpha}{|\alpha|} & \text{on } \mathbf{supp} \, \alpha, \\[2mm] \xi & \text{otherwise}, \end{cases}$$

ξ *being an arbitrary measurable section in the sphere bundle of V.*

The second part of the proposition is an easy consequence of the first part (see, e.g., [8]).

Remark 2.3. It is not difficult to show the existence of measurable sections in the sphere bundle of V. The above definition makes $\text{sign}_\xi \, \alpha$ also a measurable section in the sphere bundle of V. The proof of the proposition is the base-free formulation of the original proof of Kato's inequality. Note that no use has been made of the compactness, so it holds in the noncompact case as well. However, compactness is used in the following corollary, although there are cases where it is possible to do without, as for example when $M = \mathbf{R}^n$ (see, e.g., [8]).

Corollary 2.4. *D_∇ is essentially self-adjoint on $C_c^\infty(V)$.*

Proof. Since $-D_\nabla$ is symmetric and nonnegative on $C_c^\infty(V)$, it is sufficient to show that $\mathbf{Ker}(-D_\nabla + \lambda)^* = 0$ for some $\lambda > 0$ by a standard criterion for essential self-adjointness, (see, e.g., [10]). Thus let $\alpha_1 \in L^2(V)$ be such that

$$\langle\langle \alpha_1, (-D_\nabla + 1)\alpha \rangle\rangle = 0$$

for all $\alpha \in C_c^\infty(V)$. But then as a relation in $\mathcal{D}'(V)$, the dual of $C_c^\infty(V)$, $(-D_\nabla + 1)\alpha_1 = 0$, and therefore by Kato's inequality we have

$$(-\Delta_g + 1)|\alpha_1| \leqslant |(-D_\nabla + 1)\alpha_1| = 0.$$

Now the resolvent $(-\Delta_g + 1)^{-1}$ maps $C_c^\infty(M \times \mathbf{R})$ and hence its dual into itself. Furthermore it is positivity preserving since its kernel is positive. This again follows from the fact that $\exp - t(-\Delta_g + 1)$ has positive kernel [13].

Since $|\alpha_1| \in \mathcal{D}'(M \times \mathbf{R})$ we obtain

$$|\alpha_1| = (-\Delta_g + 1)^{-1}|\alpha_1| \leqslant 0,$$

implying $\alpha_1 = 0$ as desired.

To extend Kato's inequality to some second order differential operators, which are not necessarily of the form D_∇, let L be a strict C^∞-vector bundle endomorphism of V, whose restriction to each fibre is hermitian, i.e.,

$$\langle L_x\alpha, \beta \rangle = \langle \alpha, L_x\beta \rangle, \quad (\alpha, \beta \in V_x, x \in M).$$

Since M is compact, L induces a bounded, selfadjoint operator on $L^2(V)$, which also will be denoted by L. Therefore $D_\nabla + L$ is essentially selfadjoint on $C_c^\infty(V)$ and semibounded from above.

Furthermore, for $x \in M$ let

$$l_x := \sup \operatorname{spec} L_x.$$

By pointwise multiplication, $(l_x)_{x \in M}$ defines a C^0-vector bundle endomorphism of $M \times \mathbf{R}$ (see, e.g., [10]). Since M is compact, this endomorphism extends to a bounded selfadjoint operator on $L^2(M \times \mathbf{R})$ which we denote by l satisfying

$$\tilde{l} := \sup_{x \in M} l_x = \sup \operatorname{spec} l = \sup \operatorname{spec} L.$$

This gives

$$\operatorname{Re}\langle L\alpha, \alpha \rangle \leqslant |\alpha| l |\alpha| \leqslant \tilde{l}|\alpha|^2 \text{ for } \alpha \in C^\infty(V),$$

and thus relations (2.6) and (2.7) extend, respectively, to

$$(2.6')\quad \operatorname{Re}\langle (D_\nabla + L)\alpha, \alpha \rangle \leqslant |\alpha|_\varepsilon(\Delta_g + l)|\alpha|_\varepsilon \leqslant |\alpha|_\varepsilon(\Delta_g + \tilde{l})|\alpha|_\varepsilon,$$

$$(2.7')\quad \operatorname{Re}\langle (D_\nabla + L)\alpha, \operatorname{sign}_\xi \alpha \rangle \leqslant (\Delta_g + l)|\alpha| \leqslant (\Delta_g + \tilde{l})|\alpha|.$$

3.   The main result

The general framework in [8] allows to discuss the relation between the semigroups $\exp t(D_\nabla + L)$ and $\exp t(\Delta_g + l)$. First we note that the map

$$|\cdot| : L^2(V) \to L^2(M \times \mathbf{R})$$

is an absolute map which is absolutely pairing in the sense of [8]. With the help of sign_ξ as given above, this indeed follows as in [8]. Also, $\exp t\Delta_g$ is positivity preserving [13], therefore by Trotter's product formula [20], [5], $\exp t(\Delta_g + l)$ is positivity preserving. It follows that $(-\Delta_g - l + c)^{-1}$ is positivity preserving for each $c > l$.

Hence by Kato's inequality (2.7') and theorem 2.15 in [8], $\exp t(D_\nabla + L)$ is dominated by $\exp t(\Delta_g + l)$ which again is dominated by $e^{t\tilde{l}} \exp t\Delta_g$ for all

$t \geqslant 0$. More explicitly, this reads as follows:

Let $\exp t(D_{\nabla} + L)(x, y) \in \mathrm{Hom}(V_y, V_x)$, $(x, y \in M)$, be the kernel of $\exp t(D_{\nabla} + L)$. We take the operator norm fibrewise and obtain

$$\|\exp t(D_{\nabla} + L)(x, y)\| \leqslant \exp t(\Delta_g + l)(x, y)$$

$$\leqslant e^{t\tilde{l}}(\exp t\Delta_g)(x, y).$$

Setting $x = y$ and integrating out, we obtain the following a priori estimate.

Theorem 3.1. *With the notation as above, the estimate*

$$\mathrm{Tr}\, \exp t(D_{\nabla} + L) \leqslant n\, \mathrm{Tr}\, \exp t(\Delta_g + l) \leqslant n\, e^{t\tilde{l}}\, \mathrm{Tr}\, \exp t\Delta_g$$

holds

We note that this a priori result may be extended to noncompact cases as well. Then however, l_x has to tend sufficiently strongly to $-\infty$ as $x \to \infty$ to make $\exp t(\Delta_g + l)$ of trace class giving in particular $\Delta_g + l$ a discrete spectrum. As an example, consider the case $M = \mathbf{R}^m$ where Δ_g is the ordinary Laplacian Δ. In addition, for simplicity, let L be such that l is bounded above. Then as in, e.g., [6] we have the following additional result:

$$\mathrm{Tr}\, \exp t(D_{\nabla} + L) \leqslant n\, \mathrm{Tr}\, \exp t(\Delta + l)$$

$$\leqslant n(2\pi)^{-m} \int \exp -t(k^2 + l_x)\, d^m k d^m x.$$

Since the last estimate is given in terms of an integral, involving the symbol of $\Delta + l$, over phase space (the cotangent bundle), it is called a classical bound by physicists.

4. Laplace-de Rham operators

Consider first the vector bundle $V := \Lambda^p T^* \otimes \mathbf{C}$ for some $p \in \mathbf{N}_0$, $p \leqslant \dim M$. By means of the Gram determinant, the metric g makes V a hermitian vector bundle. The Levi-Cività connection $^g\nabla$ on T induces a connection on $\Lambda^p T^*$ and hence on V, which is denoted by $^g\nabla^p$ and is with respect to the hermitian structure.

The Laplace-de Rham operator $\Delta^p = -(d\delta + \delta d)^p$ and the Bochner Laplacian $D_{g_{\nabla p}}$ on V are related by the Weitzenböck formula involving the curvature tensor R, [21], [15], [16],

$$(4.1) \qquad\qquad \Delta^p = D_{g_{\nabla p}} + D^p R,$$

where $D^p R$ is a strict vector bundle endomorphism of V which is selfadjoint on each fibre. Hence the discussion of the previous section may be applied.

In two special cases it is easily computed: $D^0 R = 0$, and $D^1 R$ is minus the

Ricci tensor (with the sign convention $R_{ij} = R^k{}_{ikj}$ by which ordinary spheres have positive curvature). Note that $\sup_{x \in M} \sup \operatorname{spec}(D^p R)_x \leqslant 0$ if the curvature is nonnegative constant [12]. (The same reference discusses conditions for $D^p R$ to be negative definite on each fibre with its consequences.)

This result may be generalized in the following way: Let F be an arbitrary hermitian vector bundle over M with hermitian connection $_F\nabla$. Consider now the hermitian vector bundle $V := \Lambda^p T^* \otimes F$, and denote by $\hat{\nabla}^p$ the exterior covariant derivative on V induced by $_F\nabla$

$$\hat{\nabla}^p : C^\infty(\Lambda^p T^* \otimes F) \to C^\infty(\Lambda^{p+1} T^* \otimes F).$$

(For normalized conventions we refer to [7].) We will compare $\hat{\nabla}^p$ with the tensor product of the connections $^g\nabla^p$ and $_F\nabla$, which will be denoted by ∇^p:

$$\nabla^p : C^\infty(\Lambda^p T^* \otimes F) \to C^\infty(T^* \otimes \Lambda^p T^* \otimes F).$$

Therefore let

$$\hat{A}^p : T^* \otimes \Lambda^p T^* \to \Lambda^{p+1} T^*$$

be the strict vector bundle morphism induced by the wedge product via

$$\hat{A}^p(\eta \otimes \alpha) = \eta \wedge \alpha, \quad (\eta \in T^*, \alpha \in \Lambda^p T^*),$$

implying [15] that

$$\hat{\nabla}^p = (\hat{A}^p \otimes 1_F) \circ \nabla^p.$$

Consider the strict vector bundle morphism

$$\hat{g}^p : T^* \times \Lambda^{p+1} T^* \to \Lambda^p T^*$$

given in terms of the canonical isomorphism $T^* \to T, \eta \mapsto \eta^\#$ (induced by the metric) via

$$\hat{g}^p(\eta \otimes \alpha) = \eta^\# \lrcorner \alpha \ (\eta \in T^*, \alpha \in \Lambda^{p+1} T^*).$$

The exterior covariant coderivative then is defined to be the composition

$$\hat{\nabla}^{*p} : C^\infty(\Lambda^{p+1} T^* \otimes F) \overset{\nabla^{p+1}}{\to} C^\infty(T^* \otimes \Lambda^{p+1} T^* \otimes F) \overset{-\hat{g}^p \otimes 1_F}{\to} C^\infty(\Lambda^p T^* \otimes F).$$

Thus $\hat{\nabla}^p$ and $\hat{\nabla}^{*p}$ are first order linear differential operators which are formal adjoints of each other, generalizing d and δ respectively. We define

(4.2) $$\Delta^p := -\left(\hat{\nabla}^{p-1} \circ \hat{\nabla}^{*p-1} + \hat{\nabla}^{*p} \circ \hat{\nabla}^p\right)$$

to be the generalized Laplace-de Rham operator on F-valued p-forms.

Denoting the tensor product of the connections $^g\nabla^1$ and ∇^p by $^g\nabla^1 \otimes \nabla^p$, we may rewrite (4.2) as

$$\Delta^2 = \left(\beta^{2,p} \otimes 1_F\right) \circ \left(^g\nabla^1 \otimes \nabla^p\right) \circ \nabla^p$$

with

$$\beta^{2,p} := \hat{A}^{p-1} \circ \left(1_{T^*} \otimes \hat{g}^{p-1}\right) + \hat{g}^p \circ \left(1_{T^*} \otimes \hat{A}^p\right).$$

An easy calculation shows that the symmetric part of $\beta^{2,p}$ is equal to $g \otimes 1_{\Lambda^p T^*}$. Therefore the difference of the Laplacians $\Delta^p - D_{\nabla^p}$ is a differential operator of zero order, i.e., a strict vector bundle morphism, which can be expressed in terms of the curvature of ∇^p. The latter in turn is determined by the Riemann curvature tensor R, or more precisely its dual

$R^* \in C^\infty(T^* \wedge T^* \otimes \mathbf{End}(T^*))$, and by $\mathbf{curv}_F \nabla \in C^\infty(T^* \wedge T^* \otimes \mathbf{End}(F))$.

In more detail, let

$$\lambda^p : \mathbf{End}(T^*) \to \mathbf{End}(\Lambda^p T^*)$$

and

$$\tilde{\circ} : \mathbf{End}(\Lambda^p T^*) \otimes \mathbf{End}(\Lambda^p T^*) \to \mathbf{End}(\Lambda^p T^*)$$

denote the derivation extension of endomorphisms and the linear map induced by composition $\circ$ of endomorphisms, respectively. Then the difference of the Laplacians is given by the generalized Weitzenböck formula (extending (4.1)):

$$\Delta^p - D_{\nabla^p} = -\left(\tilde{\circ}(\lambda^p \otimes \lambda^p)\left((1_{T^*} \otimes \cdot^\# \otimes 1_{\mathbf{End}(T^*)})R^*\right)\right) \otimes 1_F$$

(4.3)

$$+ (\lambda^p \otimes 1_{\mathbf{End}(F)})\left((1_{T^*} \otimes \cdot^\# \otimes 1_{\mathbf{End}(F)})\mathbf{curv}_F \nabla\right).$$

Comparison with [16] shows that the first term on the right-hand side is just $D^p R \otimes 1_F$.

5. Spinor Laplacians

The rôle which exterior covariant (co-)derivatives played in the previous section will now be taken over by Dirac operators. Assume that M is equipped with a spin structure [1], [2], [3], [14]. Thus suppose M to be oriented and let $P(M, g)$ be the orthonormal oriented $\mathbf{SO}(m)$-frame bundle of T ($m = \dim M$). Denote by

$$\rho : \mathbf{Spin}(m) \to \mathbf{SO}(m)$$

the canonical epimorphism. The spin structure is a principal $\mathbf{Spin}(m)$-bundle $\tilde{P}(M, g)$ being a two-fold covering of $P(M, g)$ in a ρ-equivariant way. It follows that we can identify T with the associated vector bundle $\tilde{P}(M, g) \times_\rho \mathbf{R}^m$. Let $\mathbf{Cl}(\mathbf{R}^m)$ be the Clifford algebra of $\mathbf{R}^m$ relative to the standard (positive) bilinear form which we also denote by g, and S be a hermitian Clifford module via

$$\sigma : \mathbf{Cl}(\mathbf{R}^m) \to \mathbf{End}(S)$$

such that σ restricts to a unitary representation of $\mathbf{Spin}(m)$. Note that we make no use of any irreducibility assumption on S.

For each $k \in \mathbf{N}_0$, σ combined with the canonical map $\otimes^k \mathbf{R}^m \to \mathbf{Cl}(\mathbf{R}^m)$

yields a map

$$\hat{\sigma}^k : (\otimes^k \mathbf{R}^m) \otimes S \to S,$$

which is a morphism of **Spin**(m)-modules. We define the spinor bundle to be the associated (hermitian) vector bundle over M

$$E := \tilde{P}(M, g) \times_\sigma S.$$

$\hat{\sigma}^k$ then defines an associated strict vector bundle morphism which together with the canonical isomorphism $\cdot^\# : T^* \to T$ yields a strict vector bundle morphism

$$\gamma^k : (\otimes^k T^*) \otimes E \to E$$

satisfying

(5.1) $$\gamma^k \circ \big((\otimes^k 1_{T^*}) \otimes \gamma^l\big) = \gamma^{k+l}, \quad (k, l \in \mathbf{N}_0).$$

γ^1 may be viewed as the generalization of Dirac matrices.

Next the Levi-Cività connection induces an **so**(m)-valued connection form ω on $P(M, g)$ which lifts uniquely to a connection form $\tilde{\omega}$ on $\tilde{P}(M, g)$. Association to $\tilde{\omega}$ with σ then yields a hermitian connection on E which will be denoted by $_E\nabla$. By construction of $\tilde{\omega}$, $_g\nabla$ is associated to $\tilde{\omega}$ via ρ. (For details of these associated procedures, we refer to [7].)

Furthermore, let F be an arbitrary hermitian vector bundle over M with a hermitian connection $_F\nabla$, and denote the resulting connection on $E \otimes F$ by ∇.

The Dirac operator is defined to be the composition

$$\not{V} : C^\infty(E \otimes F) \xrightarrow{\nabla} C^\infty(T^* \otimes E \otimes F) \xrightarrow{\gamma^1 \otimes 1_F} C^\infty(E \otimes F).$$

Its square $\not{V}^2$ is called the spinor Laplacian. Denoting the tensor product of the connections $^g\nabla^1$ and ∇ by $^g\nabla^1 \otimes \nabla$, in virtue of (5.1) we may write $\not{V}$ as

(5.2) $$\not{V}^2 = \big(\gamma^2 \otimes 1_F\big) \circ (^g\nabla^1 \otimes \nabla) \circ \nabla.$$

From the defining property of Clifford algebras it easily follows that the symmetric part of γ^2 is equal to $g \otimes 1_E$. Therefore the difference of the Laplacians $\not{V}^2 - D_\nabla$ is a strict vector bundle morphism which can be expressed in terms of the curvature of ∇. The latter, in turn is determined by the Riemann curvature tensor R and $\mathbf{curv}_F\nabla$. In more detail, note that the bundle **End**(T, g) of fibrewise antiselfadjoint endomorphisms of T can be identified with the associated vector bundle $\tilde{P}(M, g) \times_{\mathbf{Ad} \circ \rho} \mathbf{so}(m)$, and **End**$(E)$ can be identified with $\tilde{P}(M, g) \times_{\mathbf{Ad} \circ \sigma} \mathbf{End}(S)$. The infinitesimal operation

$$L\sigma : \mathbf{so}(m) \to \mathbf{End}(S)$$

now defines an associated strict vector bundle morphism which will be

denoted by

$$\mu : \mathbf{End}(T, g) \to \mathbf{End}(E).$$

Again, let

$$\circ^{\sim} : \mathbf{End}(E) \otimes \mathbf{End}(E) \to \mathbf{End}(E)$$

be the linear map induced by composition $\circ$. Finally $\cdot^{b} = (\cdot^{\#})^{-1} : T \to T^{*}$ denotes the isomorphism induced by the metric.

Then the difference of the Laplacians is given by the Weitzenböck type formula (in close analogy to (4.3)):

$$(5.3) \qquad \overset{\vee}{\nabla}{}^{2} - D_{\nabla} = -2\big(\circ^{\sim}(\mu \otimes \mu)\big((\cdot^{\#} \otimes 1_{T^{*}} \otimes 1_{\mathbf{End}(T)})R)\big) \otimes 1_{F}$$

$$-2(\mu \otimes 1_{\mathbf{End}(F)})\big((\cdot^{\#} \otimes 1_{T^{*}} \otimes 1_{\mathbf{End}(F)})\mathbf{curv}_{F}\nabla).$$

Furthermore, the first term on the right-hand side can also be expressed as

$$\tfrac{1}{8} \gamma^{4}\big((1_{T^{*}} \otimes 1_{T^{*}} \otimes \cdot^{b} \otimes 1_{T^{*}})R\big) \otimes 1_{F} = \tfrac{1}{4} R^{sc} 1_{E} \otimes 1_{F}$$

(cf. [9]), where R^{sc} is the scalar curvature. Thus (5.3) becomes

$$\overset{\vee}{\nabla}{}^{2} - D_{\nabla} = \tfrac{1}{4} R^{sc} 1_{E} \otimes 1_{F} - 2(\mu \otimes 1_{\mathbf{End}(F)})\big((\cdot^{\#} \otimes 1_{T^{*}} \otimes 1_{\mathbf{End}(F)})\mathbf{curv}_{F}\nabla).$$

References

[1] M. F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) **4** (1971) 47–62.

[2] M. F. Atiyah, R. Bott & V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973) 279–330.

[3] M. F. Atiyah & I. M. Singer, *The index of elliptic operators. III*, Ann. of Math. **87** (1968) 546–604.

[4] Y. Avron, I. Herbst & B. Simon, *Schrödinger operators with magnetic fields. I*, Duke Math. J. **45** (1978) 847–883.

[5] P. P. Chernoff, *Product formulas, nonlinear semigroups and addition of unbounded operators*, Mem. Amer. Math. Soc. No. 140, 1974.

[6] J. M. Combes, R. Schrader & R. Seiler, *Classical bounds and limits for energy distributions of Hamilton operators in electromagnetic fields*, Ann. Physics **111** (1978) 1–18.

[7] W. Greub, S. Halperin & R. Vanstone, *Connections, curvature and cohomology. II*, Academic Press, New York, 1973.

[8] H. Hess, R. Schrader & D. A. Uhlenbrock, *Domination of semigroups and generalization of Kato's inequality*, Duke Math. J. **44** (1977) 893–904.

[9] N. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974) 1–55.

[10] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1976.

[11] _____, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972) 135–148.

[12] A. Lichnérowicz, *Géométrie des groupes de transformations*, Dunod. Paris, 1958.

[13] S. Minakshisundaram, *Eigenfunctions on Riemannian manifolds*, J. Indian Math. Soc. **17** (1953) 158–165.

[14] J. Milnor, *Spin structures on manifolds*, Enseignement Math. **9** (1963) 190–203.

[15] E. Nelson, *Tensor analysis*, Math. Notes, Princeton University Press, Princeton, 1967.

[16] V. K. Patodi, *Curvature and the eigenforms of the Laplace operator*, J. Differential Geometry **5** (1971) 233–249.

[17] R. Schrader, *Towards a constructive approach of a gauge invariant, massive $\mathscr{P}(\phi)_2$ theory*, Comm. Math. Phys. **58** (1978) 299–312.

[18] B. Simon, *An abstract Kato's inequality for generators of positivity preserving semigroups*, Indiana Univ. Math. J. **26** (1977) 1067–1073.

[19] _____, *Kato's inequality and the comparison of semigroups*, J. Functional Analysis **32** (1979) 97–101.

[20] H. F. Trotter, *On the product of semigroups of operators*, Proc. Amer. Math. Soc. **10** (1959) 545–551.

[21] R. Weitzenböck, *Invariantentheorie*, P. Noordhoff, Groningen, 1923.

FREIE UNIVERSITÄT BERLIN
UNIVERSITY OF MICHIGAN