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REMARKS ON UNIFORMLY EXPANDING HOROCYCLE PARAMETERIZATIONS

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Dedicated to the memory of Rufus Bowen

1. The analysis of Anosov flows on manifolds depends on the analysis of the stable and unstable foliations associated with them, and these foliations have interesting dynamical properties in their own right. In case these foliations are one-dimensional, they give rise to flows, generalizing the horocycle flows on surfaces of negative curvature. B. Marcus[1] has shown that these horocycle flows admit reparameterizations with especially nice properties, which he exploited in proving unique ergodicity. However, he proved that the resulting systems were smooth only rarely, forming closed, nowhere dense sets in appropriate classes of flows. In this note we show that these sets consist essentially of single points. Namely, we prove

Theorem A. *Let $\{f_t\}$ be a C^2 Anosov flow on a compact connected three-dimensional manifold M with stable and unstable orientable foliations W^s and W^u respectively. If W^s and W^u admit C^2 uniformly expanding and contracting parameterizations, then M supports the structure of a homogeneous space of a Lie group with the flow and foliations induced by one-parameter subgroups.*

The three-dimensional G-induced flows which are Anosov are known: they occur in the classification in [1, Chapter III], and are either constant time suspensions of hyperbolic toral automorphisms or are generalized geodesic flows (i.e., flows finitely covered by geodesic flows on tangent bundles of surfaces of constant negative curvature.)

If the flow on M was actually the geodesic flow in the unit tangent bundle of a surface S of negative curvature, Theorem A fails to say much directly about S itself, even though M turns out to be very special. However, it develops that in this case we need only assume one of the horocycle flows to be uniformly reparameterizable:

Theorem B. *Let S be a compact C^∞ surface with negative Gaussian curvature K, and H be the vector field (in the unit tangent bundle) of its unit speed expanding horocycle flow. If H admits a uniformly expanding C^2 reparameterization, then K is constant.*

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[1] [6], [7]; see also Bowen and Marcus [2], and the references cited in these papers to work of Margulis and Lifshitz.

The author is indebted to Brian Marcus for various suggestions; in particular, for the generality of the formulation of Theorem A. The author is especially grateful to the referee for supplying a more general proof of Theorem A and pointing out a serious error in the author's original Theorem B (see the remarks at the end of § 3.)

2. In this section we make precise the notion of uniformly expanding parameterization, and prove Theorem A. The exposition and notation follows Marcus [7] fairly closely. Proofs or references to proofs of the following facts may be found in that paper.

$W^u(x)$, the unstable manifold through x to the Anosov flow $\{f_t\}$, is characterized by the relation

$$W^u(x) = \left\{ y \in M \colon \lim_{t \to -\infty} \rho(f_t x, f_t y) = 0 \right\} ,$$

where ρ is some smooth metric on the manifold M. The stable manifold $W^s(x)$ through x is defined similarly, but for t approaching $+\infty$. Under the assumption that M is three-dimensional, each $W^u(x)$ and $W^s(x)$ is one-dimensional. By requiring these foliations to be orientable, we ensure that there is a continuous one-parameter group of homeomorphisms $\{\varphi_s\}$ whose orbits are the unstable manifolds. A choice of such a one-parameter group is called a W^u *flow* or W^u *parameterization.* The differentiability class of the function

$$\varphi \colon (s, x) \to \varphi_s(x)$$

is the class of the parameterization.

Bcause $f_t(W^u(x)) = W^u(f_t(x))$, there is a function

$$s^* \colon \boldsymbol{R} \times \boldsymbol{R} \times M \to \boldsymbol{R}$$

such that

$$f_t \circ \varphi_s(x) = \varphi_{s^*(t,s,x)} \circ f_t(x) . \tag{2.1}$$

Definition. *A uniformly expanding* W^u parameterization is one for which

$$s^*(t, s, x) = \lambda^t s$$

for some constant $\lambda > 1$. λ is called the expansion coefficient.

If the W^u parameterization φ is merely differentiable in s, it is possible to define the vector field of the flow E^u by the equation

$$E_x^u(g) = \frac{d}{ds} g(\varphi_s(x))|_{s=0} ,$$

for smooth functions g. If φ is C^2, $x \to E_x^u$ is C^1. X will always designate the vector field of $\{f_t\}$.

Proof of Theorem A. We let $\{\psi_s\}$ designate the uniformly reparameterized flow corresponding to the stable manifolds W^s defined analogously to the $\{\varphi_s\}$. Relation (2.1) may be restated for these flows by the equations

$$f_t \circ \varphi_s = \varphi_{\lambda^t s} \circ f_t\ , \qquad f_t \circ \psi_s = \psi_{\delta^t s} \circ f_t\ , \tag{2.2}$$

where the expansion coefficient $\lambda > 1$ and the contraction coefficient δ is in $(0, 1)$,. In terms of the vector fields E^u and E^s of these flows, (2.2) may be rewritten as

$$(f_t)_* E_x^u = \lambda^t E_{x_t}^u\ , \qquad (f_t)_* E_x^s = \delta^t E_{x_t}^s\ , \tag{2.3}$$

where $x_t = f_t(x)$. Since X, E^u, and E^s are independent at every point of M, there are continuous functions a, b, and c such that

$$[E^u, E^s] = aX + bE^u + cE^s\ .$$

Then

$$(f_t)_*[E^u, E^s] = (a \circ f_{-t})X + (b \circ f_{-t})\lambda^t E^u + (c \circ f_{-t})\delta^t E^s\ .$$

However, using (2.3) we compute

$$\begin{aligned}(f_t)_*[E^u, E^s] &= [(f_t)_* E^u, (f_t)_* E^s] = (\lambda\delta)^t [E^u, E^s] \\ &= (\lambda\delta)^t aX + (\lambda\delta)^t bE^u + (\lambda\delta)^t E^s\ .\end{aligned}$$

Equating coefficients, we obtain

$$a \circ f_{-t} = (\lambda\delta)^t a\ , \quad b \circ f_{-t} = \delta^t b\ , \quad c \circ f_{-t} = \lambda^t c\ . \tag{2.4}$$

But since M is compact, the continuous functions a, b, and c cannot satisfy these equations for all t and remain bounded unless b and c are identically zero, $\lambda\delta = 1$, and a is constant on f_t orbits.

To show that a is constant everywhere we apply a so-called "Mautner lemma" argument: In the Banach space of continuous functions on M with the supremum norm, the flows $\{f_t\}$ and $\{\varphi_s\}$ define groups of isometries $\{U_t\}$, $\{V_s\}$ respectively. ($(U_t a)(x) = a(f_t x)$, *etc.*).

The first equation of (2.2) may be written $U_t V_s = V_{\lambda^{-t}s} U_t$. Then, because the function a is U_t-invariant, and U_t is an isometry,

$$\|a - V_s a\| = \|U_t a - U_t V_s a\| = \|U_t a - V_{\lambda^{-t}s} U_t a\| = \|a - V_{\lambda^{-t}s} a\|\ .$$

But, for each fixed s, as $t \to \infty$, $V_{\lambda^{-t}s} a \to a$ in norm. Hence a is also invariant under the flow $\{\varphi_s\}$. Similarly, a is invariant under $\{\psi_s\}$.

At any point x of M the map $(t, u, s) \to (f_t \circ \varphi_u \circ \psi_s)(x)$ is nonsingular at $(0, 0, 0)$. Hence its image contains an open neighborhood of x, throughout which, by the conclusions of the previous paragraph, a is constant. By the connectedness of M, a is constant on M.

We conclude that the fields X, E^u, and E^s form a three-dimensional real Lie algebra g with multiplication table (setting $\mu = \log \lambda$)

$$[X, E^u] = -\mu E^u \, , \quad [X, E^s] = \mu E^s \, , \quad [E^u, E^s] = aX \, . \tag{2.5}$$

If a is zero, the corresponding algebra is solvable; otherwise $\mathfrak{g}$ is isomorphic to $sl(2, \boldsymbol{R})$. The fact that M becomes a homogeneous space of the corresponding group follows from Palais' theorem ([9], see also Loos [6, p. 34]), and the proof is complete.

The referee has pointed out that the compactness of M may be replaced by a much weaker hypothesis. Namely, call an open set 0 *divergent* in the *positive* (*negative*) sense if for every compact set $K \subseteq M$ there exists a $T > 0$ such that $f_t(0) \cap K$ is empty for all $t \geq T$ (all $t \leq -T$).

Theorem A*. *The conclusion of Theorem A holds if the hypothesis that M is compact is replaced by the condition that no open set of M be divergent in either the positive or negative sense.*

This hypothesis is satisfied in the following situations (M is assumed separable):

(1) M is compact.

(2) Every point of M is nonwandering. (In particular, if the flow is topologically transitive.)

(3) M admits a finite invariant measure, positive on open sets.

3. In Theorem B, the manifold in which the geodesic flow takes place is T_1S, the unit tangent bundle of the surface S. The assumption on the curvature K has as consequences that the geodesic flow, whose vector field we designate by X, is an Ansov flow, and that the unit speed expanding horocycle flow has a C^1 vector field H. Rename H as H^+ to distinguish it from the corresponding contracting field H^-. These fields satisfy the relations

$$[X, H^+] = -uH^+ \, , \qquad [X, H^-] = -\bar{u}H^- \, . \tag{3.1}$$

Here u, $\bar{u}$ are C^1 functions bounded away from zero—u positive, $\bar{u}$ negative—and both satisfy the Riccati equation

$$X(v) + v^2 + K \circ \pi = 0 \, . \tag{3.2}$$

where π is the projection from T_1S to S. The construction of u and proof that it is smooth are due to E. Hopf [5]; the subsequent definition of H^+ is reviewed in [3]. The hypothesis on the reparameterization of H^+ means that there exists a nowhere zero C^1 function g such that, if $E^u = gH^+$, then E^u satisfies the commutation relation $[X, E^u] = -\mu E^u$, with μ a positive constant. If we compare this with the commutator computed using (3.1), we find that

$$X(g) = (u - \mu)g \ . \tag{3.3}$$

Lemma 3.1. *H^- admits a C^2 uniformly contacting parameterization with contraction coefficient $e^{-\mu}$.*

Proof. Because the C^1 functions $g, u, \bar{u}$ are all bounded away from zero and u is positive, $\bar{u}$ negative, the function h defined by the equation

$$h = [g(u - \bar{u})]^{-1}$$

is C^1 and everywhere positive. Using (3.2) and (3.3), we compute that

$$\begin{aligned} X(\log h) &= -X(\log g) - X(\log(u - \bar{u})) \\ &= -(u - \mu) - (u - \bar{u})^{-1}(-u^2 - K \circ \pi + \bar{u}^2 + K \circ \pi) \\ &= -u + \mu - [u - \bar{u}]^{-1}(\bar{u}^2 - u^2) \\ &= -u + \mu + [\bar{u} - u]^{-1}[\bar{u} - u][\bar{u} + u] = \bar{u} + \mu \ . \end{aligned}$$

Now set $E^s = hH^-$. Inserting this in the second equation of (3.1) we find that

$$[X, E^s] = X(h)H^- + h[X, H^-] = (\bar{u} + \mu)hH^- - h\bar{u}H^- = \mu E^s \ .$$

This result allows us to apply Theorem A to the flow.

Lemma 3.2. $\qquad [E^u, E^s] = X.$

A consequence of this lemma is that the Lie algebra involved is $sl(2, \boldsymbol{R})$. That we must be in the semi-simple case would also follow from consideration of the fundamental group, but we need the stronger result that the constant a which arose in Theorem A is actually 1.

Proof of Lemma 3.2. Recall the definitions of H^+ and H^- in terms of the basic and fundamental vector fields of $T_1(S)$ (see [3], [4] for their geometric meanings and explication of their brackets):

$$H^+ = Y + uA \ , \qquad H^- = -Y - \bar{u}A \ .$$

Here Y is the basic field complementary to X, A is the fundamental field corresponding to rotations of the fiber, and the orientation is such that

$$[Y, A] = X \ .$$

In the following computation, only terms ultimately involving X will be explicitly retained, since Theorem A says the others will vanish — i.e., "$\equiv$" will denote "congruence modulo $\{Y, A\}$".

$$\begin{aligned} aX = [E^u, E^s] &= [gH^+, hH^-] = gH^+(h)H^- - hH^-(g)H^+ + gh[H^+, H^-] \\ &\equiv gh[Y + uA, -Y - \bar{u}A] \equiv -gh\{u[A, Y] + \bar{u}[Y, A]\} \\ &= gh(u - \bar{u})X = X . \end{aligned}$$

The last equation is a result of the definition of h in Lemma 3.1.

Abusing the notation, we let X, E^u, E^s represent both the fields on $T_1(S)$ and the corresponding left invariant fields on G, the universal covering group of $SL(2, \boldsymbol{R})$. There is a diffeomorphism

$$\eta: T_1(S) \to G/\Gamma ,$$

where Γ is a discrete subgroup of G isomorphic to the fundamental group of $T_1(S)$. The one-parameter subgroup of G defined by $g_t = \exp(tX)$ satisfies

$$\eta \circ f_t = g_t \cdot \eta , \tag{3.4}$$

where the operation on the right is multiplication. Let ν be the measure induced in G/Γ by Haar measure, normalized so $\nu(G/\Gamma) = 1$. If Ω is the measure in $T_1(S)$ induced by the Riemannian structure of S (the kinematic density), we also normalize to it give total measure 1. Now $\eta_*\Omega$ is a smooth measure on G/Γ, invariant with respect to the flow $\{g_t\}$, since Ω was invariant with respect to $\{f_t\}$. Hence $\eta_*\Omega = c\nu$, for the latter is a smooth ergodic measure, and by the above normalization, $c = 1$.

Lemma 3.3. $$\int_{T_1(S)} u\Omega = \mu .$$

Proof. Integrating the relation $X(\log g) = u - \mu$ along an $\{f_t\}$ trajectory we find that

$$(\log g)(x_1) - (\log g)(x_0) = \int_0^1 u(f_t(x_0))dt - \mu ,$$

where $x_1 = f_1(x_0)$. Now integrate both sides of this equation with respect to Ω, using the fact that Ω is invariant with respect to the flow.

$$\begin{aligned} 0 &= \int_{T_1(S)} \int_0^1 u(f_t(x_0))dt\,\Omega - \int_{T_1(S)} \mu\Omega \\ &= \int_0^1 \int_{T_1(S)} u(f_t(x_0))\Omega dt - \mu = \int_{T_1(S)} u\Omega - \mu . \end{aligned}$$

(This is an old trick of E. Hopf's; compare [5, p. 608].)

This lemma just rederives in this case the known relation between the local coefficient of expansion and the entropy (Sinai [11, Theorem 7.1]). For μ is the entropy of $\{f_t\}$ with respect to Ω. Because of the isomorphism η, it is also the entropy of $\{g_t\}$ on G/Γ with respect to ν. But we can compute the entropy of this latter flow another way. Because Γ is the fundamental group of T_1S, its algebraic structure is completely determined by the Euler characteristic of S [1, p. 24, Theorem 4.5]. Hence in G there is a one-parameter subgroup R containing the center of G such that $R\Gamma$ is closed. Then, if $S' = R\backslash G/\Gamma$, S' is a surface homeomorphic to S. But S' has a natural Riemannian metric of constant negative curvature K' induced by the hyperbolic metric on $R\backslash G$. The geodesic flow in T_1S' is the flow $\{g_t\}$ in G/Γ, and it leaves the measure ν invariant, since, by an argument similar to that preceding Lemma 3.3, that is the kinematic density of the Riemannian structure of S'.

Lemma 3.4. $\mu^2 = -4\pi^2\chi(S)$, *where* $\chi(S)$ *is the Euler characteristic of* S.

Proof. In view of the topological equivalence of S and S', it is sufficient to prove the lemma with S' in place of S. But for constant curvature this has been done by Sinai [10, Theorem 3]. One could also proceed directly by noticing that the constant function $v = \mu$, being the geodesic curvature of the horocycles in this metric, satisfies the Riccati equation (3.2) with K' in place of K. But the Gauss-Bonnet theorem says that

$$\int_{G/\Gamma} (K' \circ \pi)\nu = 2\pi \int_{S'} K'\omega' = 4\pi^2\chi(S') ,$$

if we designate the area form of S' by ω'.

We are now ready to complete the proof of Theorem B. Integrating (3.2) on T_1S, first along the flow and then with respect to Ω, just as in Lemma 3.3, we find that

$$\begin{aligned} 0 &= \int_{T_1S} [u(x_1) - u(x_0)]\Omega = \int_{T_1S} \int_0^1 X(u)(f_t x_0)dt\Omega \\ &= -\int_{T_1S} \int_0^1 u^2(f_t x_0)dt\Omega - \int_{T_1S} \int_0^1 (K \circ \pi)(f_t x_0)dt\Omega \\ &= -\int_{T_1S} u^2\Omega - \int_{T_1S} (K \circ \pi)\Omega = -\int_{T_1S} u^2\Omega - 2\pi \int_S K\omega . \end{aligned}$$

Hence using the Gauss-Bonnet theorem again on the original metric of S, and substituting the value of the Euler characteristic obtained in Lemma 3.4, we find that

$$\int_{T_1S} u^2\Omega = \mu^2 .$$

Interpreted in $L^2(T_1S, \Omega)$, this says that the norm of u is μ. Lemma 3.3 claims, however, that in the same Hilbert space, $(u, 1) = \mu$. The Schwarz inequality then implies that u is proportional to 1 almost everywhere. Hence the continuous function u is constant and (3.2) leads immediately to the conclusion that K is constant.

Remarks. 1. Just as in § 2, there is a Theorem B′ which covers the case when S is a complete surface of finite area. In order to ensure the existence and smoothness of the functions u and $\bar{u}$, it is convenient to assume that the curvature lie between negative bounds. This also guarantees that an extended Gauss-Bonnet formula holds, and the rest of the proof goes through unchanged.

2. In an earlier version of this paper the conclusion of Theorem B was asserted even if S was simply-connected. That such a result is false may be seen from the following construction: at a given point in the simply-connected, complete surface S construct all the expanding horocycles through that point. The unit vectors normal in the expanding direction to all these curves form a surface F in T_1S which is a global cross-section to the geodesic flow. Then one may prescribe arbitrary smooth values on F to the function $\log g$ and integrate (3.3). The resulting function provides a smooth reparameterization of the expanding horocycle flow.

4. Further remarks and conjectures. W. Perrizo has pointed out that the techniques of Theorem A, applied to the Riccati equation (3.2), yield a proof of the fact that, if the function u is constant on fibers (of the unit tangent bundle), then the curvature of the surface is necessarily constant. For, using the fields introduced in the proof of Lemma 3.2, and the relation

$$[A, X] = Y ,$$

we find that, applying A to (3.2),

$$Y(u) = [A, X]u = -A(u^2 + K \circ \pi) - XA(u) = 0 ,$$

under the assumption that $A(u) = 0$. Similarly, $X(u) = [Y, A]u = 0$, so u is constant. (3.2) then implies that K is constant. This observation may be given a differential-geometric formulation as follows:

Theorem C. *Let S be a complete surface with negative curvature bounded away from zero. If at every point the geodesic curvatures of the horocycles passing through that point coincide, S is a surface of constant curvature.*

The assumption on the curvature is to ensure the existence of the horocycles and the differentiability of the function u. Generalizations to higher dimensions leap to mind; the following conjecture seems nontrivial.

Conjecture. *If V is a simply-connected manifold of bounded negative sectional curvatures such that the mean curvatures of the horospheres through each point depend only on the point, then V is a symmetric space of rank one.*

Finally, we remark that the assumption $A(u) = 0$ has another interpretation, as pointed out in [3]. Namely, the natural measure in the tangent bundle is preserved by the horocycle flow if and only if $A(u) = 0$. Thus Theorem C is in agreement with (but certainly does not provide an independent proof of) that application of Proposition (6.5) in Marcus [7] which states that the unique horocycle invariant measure is smooth if and only if the parameterization was uniform.

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