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COMPACT REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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Introduction

Let M be an n-dimensional real hypersurface of a complex projective space $CP^{(n+1)/2}$ of complex dimension $(n+1)/2$, and H the Weingarten map of the immersion $i: M \to CP^{(n+1)/2}$. It is known [1] that if a compact minimal hypersurface M of $CP^{(n+1)/2}$ satisfies trace $H^2 \leq n-1$, then trace $H^2 = n-1$, and up to isometries of $CP^{(n+1)/2}$, M is a certain distinguished minimal hypersurface $M^c_{p,q}$ for some p and q.

The purpose of the present paper is to generalize the above result in such a way that we have an integral inequality which is still valid even if the immersion i is not necessarily minimal. Two main tools for this purpose are Lemma 1.1, to be stated in § 1, and the following integral formula established by Yano [3], [4]:

$$(0.1) \qquad \int_M \{\mathrm{Ric}\,(X,X) + \tfrac{1}{2}|L(X)g|^2 - |\nabla X|^2 - (\mathrm{div}\, X)^2\} {*1} = 0\,,$$

where X is an arbitrary tangent vector field on M, $*1$ is the volume element of M, and $|Y|$ denotes the length with respect to the Riemannian metric of a vector field Y on M.

In § 1 we explain the model space $M^c_{p,q}$, and in § 2 we present some formulas to be used in § 3. Finally in § 3 we prove our main result.

1. Submersion, immersion and the model $M^c_{p,q}$

Let S^{n+2} be an odd-dimensional sphere of radius 1 in a Euclidean $(n+3)$-space E^{n+3}, $CP^{(n+1)/2}$ the complex projective space, and $\tilde{\pi}$ the Riemannian submersion with totally geodesic fibres, which is defined by the Hopf fibration $S^{n+2} \to CP^{(n+1)/2}$. The almost complex structure J of $CP^{(n+1)/2}$ is nothing but the fundamental tensor of the submersion $\tilde{\pi}$, and the Riemannian metric G of $CP^{(n+1)/2}$ is induced naturally from that of S^{n+2}. With respect to (J, G), $CP^{(n+1)/2}$ is a Kaehlerian manifold of constant holomorphic sectional curvature 4. The curvature tensor $\bar{R}$ of $CP^{(n+1)/2}$ is given by

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$$\bar{R}(\bar{X},\bar{Y})\bar{Z} = G(\bar{Y},\bar{Z})\bar{X} - G(\bar{X},\bar{Z})\bar{Y} + G(J\bar{Y},\bar{Z})J\bar{X} - G(J\bar{X},\bar{Z})J\bar{Y} - 2G(J\bar{X},\bar{Y})J\bar{Z}\,, \tag{1.1}$$

where $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ are tangent vector fields on $CP^{(n+1)/2}$.

For a real hypersurface M of $CP^{(n+1)/2}$ and the circle bundle $\bar{M}$ over M we can construct a Riemannian submersion π compatible with the Hopf fibration $\tilde{\pi}$ in such a way that $\bar{M}$ is a hypersurface of S^{n+2} and that for $\pi\colon \bar{M} \to M$, the following diagram commutes:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\ \tilde{i}\ } & S^{n+2} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow{\ i\ } & CP^{(n+1)/2}\,. \end{array}$$

In this case $\tilde{i}$ is an isometry on the fibres. We take the family of generalized Clifford surfaces $M_{r,s} = S^r \times S^s$, where $r + s = n + 1$. Regarding E^{n+3} as a complex $\frac{1}{2}(n + 3)$-space, we choose the spheres to lie in complex subspaces. Then we get fibrations $S^1 \to M_{2p+1,2q+1} \to M^c_{p,q}$ compatible with the Hopf fibration, where $2(p + q) = n - 1$. $M^c_{p,q}$ thus obtained are remarkable classes of real hypersurfaces of $CP^{(n+1)/2}$.

Remark. In [1], $M_{r,s}$ always means $S^r \times S^s$ which is immersed in S^{n+2} minimally. But in this paper we do not assume that $M_{r,s}$ is minimal.

A fundamental relation between M and $\bar{M}$ is the following [2].

Lemma. 1.1. *In order that the Weingarten map $\bar{H}$ of $\bar{M}$ is covariant constant, it is necessary and sufficient that the Weingarten map H of M commutes with the fundamental tensor F of π.*

From this lemma we know that if the Weingarten map H commutes with the fundamental tensor F of π, $\bar{M}$ must be $M_{r,s}$ and consequently M must be $M^c_{p,q}$ for some p, q.

2. Local formulas for a real hypersurface

Let X be a vector field over a real hypersurface M of $CP^{(n+1)/2}$, and N the unit normal local field to M. Then the transforms JX and JN of X and N respectively by the almost complex structure J of $CP^{(n+1)/2}$ can be expressed by

$$JX = FX + u(X)N\,, \qquad JN = -U\,, \tag{2.1}$$

where F is the fundamental tensor of the submersion $\pi\colon \bar{M} \to M$, [2]. F, u and U thus obtained define, respectively, antisymmetric linear transformation of the tangent bundle $T(M)$, a 1-form and a vector field on M. In terms of the induced Riemannian metric g we have

$$g(U, X) = u(X) \ . \tag{2.2}$$

Iterating J to X and N we can easily see that

$$F^2X = -X + g(U, X)U \ , \tag{2.3}$$

$$FU = 0 \ , \tag{2.4}$$

$$g(U, U) = 1 \ . \tag{2.5}$$

The second fundamental form h and the corresponding Weingarten map H of $T(M)$ are defined and related to covariant differentiation $\bar{\nabla}$ and ∇ in $\bar{M}$ and M respectively by the following formulas:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \ , \tag{2.6}$$

$$\bar{\nabla}_X N = -HX \ , \tag{2.7}$$

and $h(X, Y) = g(HX, Y)N = g(X, HY)N$.

Since the Riemannian connection of $CP^{(n+1)/2}$ leaves the almost complex structure J invariant, (2.1), (2.6) and (2.7) imply that

$$(\nabla_Y F)Z = g(U, Z)HY - g(HY, Z)U \ , \tag{2.8}$$

$$\nabla_Y U = FHY \ . \tag{2.9}$$

Lemma 2.1. *In order that the Weingarten map H of M commutes with the fundamental tensor F of π, it is necessary and sufficient that the vector field U is an infinitesimal isometry.*

Proof. We compute the Lie derivative $L(U)g$ of the Riemannian metric g with respect to U, and obtain

$$\begin{aligned}(L(U)g)(X, Y) &= g(\nabla_X U, Y) + g(\nabla_Y U, X) \\ &= g(FHX, Y) + g(FHY, X) = g((FH - HF)X, Y) \ ,\end{aligned}$$

because of the fact that H is symmetric and F is antisymmetric. Thus we have proved Lemma 2.1.

Let R and Ric be respectively the curvature tensor and the Ricci tensor of M. Then from (1.1) we have

$$\begin{aligned}R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY \\ &\quad - 2g(FX, Y)FZ + g(HY, Z)HX - g(HX, Z)HY \ ,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\mathrm{Ric}\,(X, Y) &= (n + 2)g(X, Y) - 3g(U, X)g(U, Y) \\ &\quad + (\mathrm{trace}\, H)g(HX, Y) - g(H^2X, Y) \ .\end{aligned} \tag{2.11}$$

3. A generalization of Lawson's theorem

Here we prove a theorem which is a generalization of Lawson's theorem stated in the beginning of the introduction. First we apply (0.1) to the vector field U. Since F is antisymmetric and H is symmetric, (2.9) implies that div U = trace $FH = 0$ and consequently (0.1) becomes

$$\int_M \{\mathrm{Ric}\,(U, U) - |\nabla U|^2\}*1 = -\tfrac{1}{2}\int_M |L(U)g|^2*1 \leq 0 \,, \tag{3.1}$$

where equality holds if ond only if U is an infinitesimal isometry. On the other hand, (2.3) (2.5) (2.9) and (2.11) imply that

$$\mathrm{Ric}\,(U, U) = n - 1 + (\mathrm{trace}\, H)g(HU, U) - g(H^2U, U) \,, \tag{3.2}$$

$$|\nabla U|^2 = \mathrm{trace}\, FH^t(FH) = -\mathrm{trace}\, F^2H^2 = \mathrm{trace}\, H^2 - g(H^2U, U) \,. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$\int_M \{n - 1 + (\mathrm{trace}\, H)g(HU, U) - \mathrm{trace}\, H^2\}*1 \leq 0 \,, \tag{3.4}$$

where equality holds if and only if U is an infinitesimal isometry. Thus combinning Lemma 1.1 with Lemma 2.1 gives

Theorem. *Let M be a compact orientable real hypersurface of $CP^{(n+1)/2}$ over which the following inequality*

$$\int_M \{n - 1 + (\mathrm{trace}\, H)g(HU, U) - \mathrm{trace}\, H^2\}*1 \geq 0 \tag{3.5}$$

holds. Then, up to isometries of $CP^{(n+1)/2}$, M is $M^c_{p,q}$ for some p and q.

Corollary 1. *Let M be a compact orientable real hypersurface of $CP^{(n+1)/2}$. If the Weingarten map H of M satisfies*

$$\mathrm{trace}\, H^2 \leq n - 1 + (\mathrm{trace}\, H)g(HU, U) \,, \tag{3.6}$$

then, up to isometries of $CP^{(n+1)/2}$, M is $M^c_{p,q}$ for some p and q.

Corollary 2, [1]. *Let M be a compact orientable minimal hypersurface of $CP^{(n+1)/2}$ over which* trace $H^2 \leq n - 1$ *holds. Then, up to isometries of $CP^{(n+1)/2}$, M is $M^c_{p,q}$ for some p, q.*

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