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THE MANIFOLD OF THE LAGRANGEAN SUBSPACES OF A SYMPLECTIC VECTOR SPACE

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1. Introduction

Let (E^{2n}, α) be a real $2n$-dimensional symplectic vector space with symplectic form α, i.e., α is a nondegenerate skew-symmetric bilinear form on E. Then an n-dimensional subspace λ of E will be called a Lagrangean subspace if $\alpha_{|\lambda} \equiv 0$ holds. The set $\Lambda(E)$ of all Lagrangean subspaces of (E^{2n}, α) has a structure of $n(n+1)$-dimensional compact connected regular algebraic variety. If we put $\Lambda^k(\lambda) := \{\mu \in \Lambda(E) \mid \dim(\lambda \cap \mu) = k\}$ for $\lambda \in \Lambda(E)$, then $\Lambda^0(\lambda)$ is a cell (i.e., diffeomorphic to $\boldsymbol{R}^{n(n+1)/2}$) for any $\lambda \in \Lambda(E)$. Moreover $\sum(\lambda) := \bigcup_{k\geq 1} \Lambda^k(\lambda)$ is an algebraic subvariety of $\Lambda(E)$, and defines an oriented cycle of codimension one, whose Poincaré dual is a generator of $H^1(\Lambda(E), Z) \cong Z$ and defines the Maslov-Arnold index [1], [3], [4]. This index plays an important role in the proof of Morse index theorem in the calculus of variations [4]. In the present note, we shall give a differential geometric characterization of $\sum(\lambda)$, i.e., by introducing an appropriate riemannian metric on $\Lambda(E)$ we shall show that $\sum(\lambda)$ is the cut locus of some $\mu \in \Lambda(E)$ and $\Lambda^0(\lambda)$ is the interior set of μ. In fact, take a basis $\{e_i, f_j\}$ $(1 \leq i, j \leq n)$ of E such that $\alpha(e_i, e_j) = \alpha(f_i, f_j) = 0$ and $\alpha(e_i, f_j) = -\delta_{ij}$. Then with respect to this basis (E, α) may be identified with $(\boldsymbol{R}^{2n}, \alpha_0)$, where $\boldsymbol{R}^{2n} = \{(p, q) \mid p, q \in \boldsymbol{R}^n\}$ is $2n$-dimensional euclidean space with the canonical inner product $\langle\ ,\ \rangle$, and $\alpha_0((p, q), (p', q')) := \langle q, p'\rangle - \langle p, q'\rangle$. We put $\lambda_0 := \{(p, 0) \mid p \in \boldsymbol{R}^n\}$ and $\mu_0 := \{(0, q) \mid q \in \boldsymbol{R}^n\}$ which are of course Lagrangean subspaces. Then the (real representation of) unitary group $U(n)$ naturally acts on $\Lambda(n) := \Lambda(\boldsymbol{R}^{2n})$ transitively, and its isotropic subgroup at λ_0 is given by $O(n)$. Thus $\Lambda(n)$ is diffeomorphic to $U(n)/O(n)$. Now $M = U(n)/O(n)$ has a structure of a compact symmetric space whose riemannian structure comes from the Killing form of the Lie algebra of $U(n)$. In the present note we shall determine the cut locus and the first conjugate locus of a point of M, from which we may prove the assertion mentioned above. For compact simply connected symmetric spaces, it is known that the cut locus and the first conjugate locus of any point coincide with each other (see [2]). Note that $\pi_1(M) \cong Z$ for our manifold $M = U(n)/O(n)$. Finally we shall determine all closed geodesics of M and calculate their intersection number with the oriented cycle $\bigcup_{k\geq 1} \Lambda^k(n)$.

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2. Preliminaries

2.1. Let $\mathfrak{G}$ (resp. $\mathfrak{H}$) be the Lie algebra of $U(n)$ (resp. $O(n)$). We put

$$B_{ij} := E_{ij} - E_{ji}\ , \quad C_{ij} := \sqrt{-1}(E_{ij} + E_{ji})\ , \quad A_i := 1/\sqrt{2}\,C_{ii}\ ,$$

where E_{ij} denotes the $n \times n$-matrix whose r-th row and s-th column are given by $\delta_{ir}\delta_{js}$. Then $\mathfrak{G}$ may be considered as a real Lie algebra with basis $\{B_{ij}(1 \leq i < j \leq n), C_{ij}(1 \leq i < j \leq n), A_i(1 \leq i \leq n)\}$, and we have the vector space direct sum $\mathfrak{G} = \mathfrak{M} + \mathfrak{H}$, where we put $\mathfrak{M} := \{A_i(1 \leq i \leq n), C_{ij}(i < j)\}$ and $\mathfrak{H} := \{B_{ij}(i < j)\}$. Now we define an inner product Q on $\mathfrak{G}$ by $Q(X, Y) := -\frac{1}{2}\operatorname{trace} XY$. Then $\{A_i, B_{ij}(i < j), C_{ij}(i < j)\}$ forms an orthonormal basis of $\mathfrak{G}$ with respect to Q. We shall give the Lie multiplication table.

$$\begin{aligned}
&[A_i, A_j] = 0\ ,\\
&[A_i, B_{jk}] = \sqrt{2}\{\delta_{ij}C_{ik} - \delta_{ik}C_{ij}\}\ ,\\
&[A_i, C_{jk}] = -\sqrt{2}\{\delta_{ij}B_{ik} + \delta_{ik}B_{ij}\}\ ,\\
&[B_{ij}, B_{kl}] = -\delta_{ik}B_{jl} + \delta_{il}B_{jk} + \delta_{jk}B_{il} - \delta_{jl}B_{ik}\ ,\\
&[B_{ij}, C_{kl}] = -\delta_{ik}C_{jl} - \delta_{il}C_{jk} + \delta_{jk}C_{il} + \delta_{jl}C_{ik}\ ,\\
&[C_{ij}, C_{kl}] = -\delta_{ik}B_{jl} - \delta_{il}B_{jk} - \delta_{jk}B_{il} - \delta_{jl}B_{ik}\ ,
\end{aligned} \tag{2.1}$$

If we define an involutive automorphism $s: U(n) \to U(n)$ by $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ denotes the real representation of an element of $U(n)$, then the fixed point set of s is $O(n)$ and $ds_{e|\mathfrak{H}} = \mathrm{id}_{\mathfrak{H}}$, $ds_{e|\mathfrak{M}} = -id_{\mathfrak{M}}$ does hold. Since $\mathfrak{H}$ is simple, it contains no nonzero ideal of $\mathfrak{G}$. Finally Q is a ds-invariant, ad $(\mathfrak{H})$-invariant positive definite bilinear form on $\mathfrak{G}$. We may define a riemannian structure g on $U(n)/O(n)$ by restricting Q to $\mathfrak{M} \times \mathfrak{M}$ and then translating with $U(n)$. Thus $(M = U(n)/O(n), g)$ is a riemannian symmetric space with an oila (orthogonal involutive Lie algebra) $(\mathfrak{G}, ds, Q)$. Note that $\mathfrak{A} := \{A_i(1 \leq i \leq n)\}$ forms a Cartan subalgebra of the oila $(\mathfrak{G}, ds, Q)$ (i.e., maximal abelian subalgebra in $\mathfrak{M}$), and the center of $\mathfrak{G}$ is generated by $c := (A_1 + \cdots + A_n)/\sqrt{n}$. Now let $\pi: U(n) \to U(n)/O(n)$ be the canonical projection and put $o = \pi(e)$ which may be identified with λ_0. As usual we shall identify $\mathfrak{M}$ and the tangent space T_oM via the map $d\pi_e$. We denote by τ_g the left translation on $U(n)/O(n)$ by an element $g \in U(n)$.

2.2. Lemma. *Let ∇ denote the covariant differentiation of Levi-Civita connection of M with respect to g, and let $R(X, Y)Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ be the curvature tensor. Then the curvature tensor at o is given as follows:*

$$\begin{aligned}
&R(A_i, A_j)A_k = R(A_i, A_j)C_{kl} = 0\ ,\\
&R(A_i, C_{jk})A_l = 2\{\delta_{ij}\delta_{il}C_{kl} + \delta_{ik}\delta_{il}C_{jl} - (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})C_{il}\}\ ,
\end{aligned}$$

$$
\begin{aligned}
R(A_i, C_{jk})C_{lm} &= \sqrt{2}\{\delta_{ij}\delta_{il}C_{km} + \delta_{ij}\delta_{im}C_{kl} + \delta_{ik}\delta_{il}C_{jm} + \delta_{ik}\delta_{im}C_{jl} \\
&\qquad - (\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm})C_{il} - (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})C_{im}\}\ , \\
R(C_{ij}, C_{kl})A_m &= \sqrt{2}\{-(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km})C_{im} - (\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km})C_{jm} \\
&\qquad + (\delta_{il}\delta_{jm} + \delta_{jl}\delta_{im})C_{km} + (\delta_{ik}\delta_{jm} + \delta_{jk}\delta_{im})C_{lm}\}\ , \\
R(C_{ij}, C_{kl})C_{pq} &= -(\delta_{jk}\delta_{lq} + \delta_{jl}\delta_{kq})C_{ip} - (\delta_{jk}\delta_{lp} + \delta_{jl}\delta_{kp})C_{iq} \\
&\quad - (\delta_{ik}\delta_{lq} + \delta_{il}\delta_{kq})C_{jp} - (\delta_{ik}\delta_{lp} + \delta_{il}\delta_{kp})C_{jq} \\
&\quad + (\delta_{il}\delta_{jq} + \delta_{jl}\delta_{iq})C_{kp} + (\delta_{il}\delta_{jp} + \delta_{jl}\delta_{ip})C_{kq} \\
&\quad + (\delta_{ik}\delta_{jq} + \delta_{jk}\delta_{iq})C_{lp} + (\delta_{ik}\delta_{jp} + \delta_{jk}\delta_{ip})C_{lq}\ .
\end{aligned} \tag{2.2}
$$

Proof. Direct calculation by the formula $R(X, Y)Z = [[X, Y], Z]$ for $X, Y, Z \in \mathfrak{M}$, [5].

2.3. Proposition. *We denote by $K(\sigma)$ the sectional curvature for the plane section σ. Then we have $0 \leq K(\sigma) \leq 4$ for all σ.*

Proof. Since M is homogeneous, we may restrict our attention to T_0M. Let $\{U, V\}$ be an orthonormal basis for σ, and let $\mathfrak{A}'$ be a Cartan subalgebra containing U. Then there exists an $h \in SO(n)$ such that $\mathrm{Ad}\, h(\mathfrak{A}) = \mathfrak{A}'$. We may assume $\sigma = \{\mathrm{Ad}\,(h)X, \mathrm{Ad}\,(h)Y\}$, where $X = \sum_i \alpha_i A_i$, $Y = \sum \beta_i A_i + \sum_{p<q} \gamma_{pq} C_{pq}$, $\sum \alpha_i^2 = 1$, $\sum \beta_i^2 + \sum_{p<q} \gamma_{pq}^2 = 1$, $\sum \alpha_i \beta_i = 0$. Since $\mathrm{Ad}\, h$ acts on $\mathfrak{M}$ as an isometry, we get by Lemma 2.2

$$
\begin{aligned}
K(\sigma) &= Q(R(U, V)U, V) = Q(R(X, Y)X, Y) \\
&= 2 \sum_{p<q} (\alpha_p - \alpha_q)^2 \gamma_{pq}^2\ 2 \operatorname*{Max}_{p<q} |\alpha_p - \alpha_q|^2 \leq 4\ ,
\end{aligned} \tag{2.3}
$$

where the equality holds if and only if $X = (A_p - A_q)/\sqrt{2}$ and $Y = C_{pq}$ for some $p < q$.

2.4. Now we shall review the notion of cut locus and conjugate locus of a point of a riemannian manifold. Let (M, g) be a compact riemannian manifold, and let Exp denote the exponential mapping. Let X be a unit tangent vector at $x \in M$. Then $t \to \mathrm{Exp}\, tX$ is a geodesic emanating from x with the initial direction X and parametrized by the arc length. t_0X (resp. $\mathrm{Exp}\, t_0X$) is called a tangential conjugate point (resp. conjugate point) of x along the geodesic $t \to \mathrm{Exp}\, tX$, if there exists a nonzero Jacobi field $J(t)$ along $t \to \mathrm{Exp}\, tX$ such that $J(0) = J(t_0) = 0$. Next, t_0X (resp. $\mathrm{Exp}\, t_0X$) will be called a tangential cut point (resp. cut point) of x along $t \to \mathrm{Exp}\, tX$, if the geodesic segment $t \to \mathrm{Exp}\, tX$ $(0 \leq t \leq s)$ is a minimal geodesic for any $s \leq t_0$, but $t \to \mathrm{Exp}\, tX$ $(0 \leq t \leq s)$ is not a minimal geodesic for any $s > t_0$. Then it is easy to see the following: Assume that $\mathrm{Exp}\, t_0X$ is a cut point of x along a geodesic $t \to \mathrm{Exp}\, tX$, which is not a conjugate point. Then there exists a unit vector $Y \in T_xM$, $Y \neq X$, such that $\mathrm{Exp}\, t_0X = \mathrm{Exp}\, t_0Y$. (Tangential) conjugate locus (resp. (tangential) cut locus) of x is defined as the set of (tangential) conjugate points (resp. (tangential) cut points) of x along all the geodesics emanating from x. Finally the interior set $\mathrm{Int}(x)$ of x is defined as $M \backslash$cut locus of x. Let $\alpha(X)$ be a positive number such that $\alpha(X)X$ is the tangential cut point of x along $t \to \mathrm{Exp}\, tX$. Then the exponential mapping Exp maps $\{tX; X \in T_xM, g(X,X) = 1, 0 \leq t < \alpha(X)\}$ diffeomorphically onto $\mathrm{Int}(x)$. Thus $\mathrm{Int}(x)$ is a cell for any $x \in M$.

Now for our manifold $M = U(n)/O(n)$, note that every geodesic $t \to \mathrm{Exp}\, tX$ emanating from o with the initial direction $d\pi_e X (X \in \mathfrak{M}, g(X,X) = 1)$ may be expressed in the form $t \to \exp tX \cdot o$, where exp denotes the exponential mapping from the Lie algebra $\mathfrak{G}$ to $U(n)$, [5]. Then we get

2.5. Proposition. *Tangential conjugate locus (resp. tangential cut locus) of o is a hypersurface of the revolution about the line generated by $c = (A_1 + \cdots + A_n)/\sqrt{n}$ and may be obtained by rotating the tangential conjugate points (resp. tangential cut points) in $\mathfrak{A}$ about the above line by the action of* $\mathrm{Ad}\,(SO(n))$.

Proof. We put $\mathfrak{C} := \{C_{jk}(1 \leq j < k \leq n)\}$. Since $d\pi \circ \mathrm{Ad}\, hX = d\tau_h \circ d\pi(X)$ for $h \in SO(n)$ and $X \in \mathfrak{M}$, $\mathrm{Ad}\,(SO(n))$ acts on $\mathfrak{M}$ as an isometry group and transfers tangential conjugate (resp. cut) points into tangential conjugate (resp. cut) points. By (2.1) we have $\mathrm{ad}\,(\mathfrak{H})\mathfrak{A} = \mathfrak{C}$ and consequently $\mathrm{Ad}\,(SO(n))\mathfrak{A} = \mathfrak{M}$. Moreover since $c = (A_1 + \cdots + A_n)/\sqrt{n}$ belongs to the center of $\mathfrak{G}$, $\mathrm{Ad}\,(SO(n))$ leaves c invariant and maps the orthogonal complement of c in $\mathfrak{A}$ onto the orthogonal complement of c in $\mathfrak{M}$.

3. Conjugate locus

In this section we shall determine the tangential conjugate locus of o. By Proposition 2.5, it suffices to consider the tangential conjugate locus along a geodesic $t \to \mathrm{Exp}\, t(\sum_i \alpha_i A_i)$, $\sum_i \alpha_i^2 = 1$.

3.1. Proposition. *Let $X = \sum_i \alpha_i A_i$ be a unit vector in $\mathfrak{A}$. Then the symmetric linear transformation of $\mathfrak{M}$ which is defined by $V \to R(X,V)X$ has the following eigenvalues: 0 with the eigenspace $\mathfrak{A}$ and $2(\alpha_j - \alpha_k)^2$ with eigenvector $C_{jk}(1 \leq j < k \leq n)$. The first tangential conjugate point of o along a geodesic $t \to \mathrm{Exp}\, t(\sum_i \alpha_i A_i)$ is given by* $(\mathrm{Min}_{j<k}\, \pi/(\sqrt{2}\, |\alpha_j - \alpha_k|)) \sum_i \alpha_i A_i$.

Proof. The first assertion is clear, because by Lemma 2.2 we have

$$(3.1) \qquad R(X, A_j)X = 0\,, \qquad R(X, C_{jk})X = 2(\alpha_j - \alpha_k)^2 C_{jk}\,.$$

Next take an orthonormal basis $\{A_i, C_{jk}\}$ of $T_oM \cong \mathfrak{M}$. By parallel translating $\{A_i, C_{jk}\}$ along $t \to \mathrm{Exp}\, tX$, we have an orthonormal frame field $\{A_i(t), C_{jk}(t)\}$ along the above geodesic. Let $J(t) := \sum_i a_i(t)A_i(t) + \sum_{j<k} b_{jk}(t)C_{jk}(t)$ be a Jacobi field with $J(0) = 0$. Then since M is a symmetric space, the Jacobi equation $\nabla_{\dot{c}(t)}\nabla_{\dot{c}(t)}J(t) + R(\dot{c}(t), J(t))\dot{c}(t) = 0$, where we put $c(t) = \mathrm{Exp}\, tX$, takes the form

$$(d^2/dt^2)a_i(t) = 0 \ , \qquad (d^2/dt^2)b_{jk}(t) + 2(\alpha_j - \alpha_k)^2 b_{jk}(t) = 0 \ ,$$

with $a_i(0) = b_{jk}(0) = 0$. So we have

$$J(t) = t \sum_i a_i A_i(t) + \sum_{j<k} b_{jk}(\sin \sqrt{2}\,|\alpha_j - \alpha_k|\,t) C_{jk}(t)$$

for some constants a_i, b_{jk}. Then $J(t_0) = 0$ holds for some $t_0 > 0$ if and only if $a_i = 0$ (i $= 1, \cdots, n$) and $\sin \sqrt{2}\,|\alpha_j - \alpha_k|\,t_0 = 0$ for some $j < k$.

3.2. Remark. Let the multiplicity of the first tangential conjugate point t_0X along $t \to \operatorname{Exp} tX$ be equal to a, i.e., $|\alpha_{j_1} - \alpha_{k_1}| = \cdots = |\alpha_{j_a} - \alpha_{k_a}| = \operatorname{Max}_{j<k} |\alpha_j - \alpha_k|$. Then by variational completeness, $\{\operatorname{Exp} t_0 \operatorname{Ad}(h_s)X \mid h_s = \exp sY\,;\ Y \in \{B_{j_1k_1}, \cdots, B_{j_ak_a}\} \subset \mathfrak{H}\}$ reduces to a point $\operatorname{Exp} t_0X$.

4. Cut locus

First we shall give the following lemma.

4.1. Lemma. *Let t_0X be the tangential cut point of o along a geodesic $t \to \operatorname{Exp} tX$, where $X = \sum \alpha_i A_i \in \mathfrak{A}$, $\sum \alpha_i^2 = 1$. Then either t_0X is a tangential conjugate point of o along $t \to \operatorname{Exp} tX$ or there exists a unit $Y = \sum \beta_i A_i \in \mathfrak{A}$, $Y \neq X$, such that* $\operatorname{Exp} t_0X = \operatorname{Exp} t_0Y$.

Proof. Suppose that t_0X is not a conjugate point. Then there exists a unit vector $Z \in \mathfrak{M}$ such that $\operatorname{Exp} t_0X = \operatorname{Exp} t_0Z$ and $Z \neq X$. We shall show $[X, Z] = 0$. In fact, suppose $[X, Z] \neq 0$. We may assume $Z \notin \mathfrak{A}$. Since M is a symmetric space, $\operatorname{Exp} t_0X = \pi \exp t_0X$ holds and we have $\exp t_0X = \exp(t_0Z)h$ for some $h \in O(n)$. Then $\exp(-t_0X)\exp(sZ)\exp(t_0X) = \operatorname{Ad} h^{-1} \exp(sZ)$, and consequently $\operatorname{Ad}(\exp(-t_0X))Z = \operatorname{Ad} h^{-1}Z$ holds. But $\operatorname{Ad}(\exp(-t_0X))C_{jk} = C_{jk} \cos a_{jk} + B_{jk} \sin a_{jk}$ with $a_{jk} = \sqrt{2}(\alpha_j - \alpha_k)t_0$. So if we put $Z = \sum z_i A_i + \sum_{j<k} z_{jk} C_{jk}$, then we have

$$\begin{aligned}
[X, Z] &= -\sqrt{2} \sum_{j<k} z_{jk}(\alpha_j - \alpha_k) B_{jk} \ , \\
\operatorname{Ad} \exp(-t_0X)Z &= \sum_i z_i A_i + \sum_{j<k} z_{jk}(B_{jk} \sin a_{jk} + C_{jk} \cos a_{jk}) \\
&= \operatorname{Ad} h^{-1}\Big(\sum_i z_i A_i + \sum_{j<k} z_{jk} C_{jk}\Big) \in \mathfrak{M} \ .
\end{aligned}$$

Since $[X, Z] \neq 0$, and $Z \notin \mathfrak{A}$, we get for some $j < k$, $\sin a_{jk} = 0$ with $\alpha_j - \alpha_k \neq 0$, i.e., $\sqrt{2}(\alpha_j - \alpha_k) = \pi m$, where m is a nonzero integer. Thus by the proof of Proposition 3.1, t_0X is a tangential conjugate point of o which is a contradiction. So we have $[X, Z] = 0$. Let $\mathfrak{A}'$ be a Cartan subalgebra which contains X, Z. Then there exists an element $k \in SO(n)$ such that $X = \operatorname{Ad}(k)X$ and $Z = \operatorname{Ad}(k)Y$ for some Y, $Y \neq X$. Then we have

$$\begin{aligned}
\tau_k \operatorname{Exp} t_0X &= \operatorname{Exp} t_0 \operatorname{Ad}(k)X = \operatorname{Exp} t_0X = \operatorname{Exp} t_0Z \\
&= \operatorname{Exp} t_0 \operatorname{Ad}(k)Y = \tau_k \operatorname{Exp} t_0Y \ ,
\end{aligned}$$

and consequently we get $\operatorname{Exp} t_0X = \operatorname{Exp} t_0Y$ for some $Y \in \mathfrak{A}$, $Y \neq X$.

q.e.d.

Now we shall determine the tangential cut locus of o. By Proposition 2.5, it suffices to consider the tangential cut point t_0X of o along a geodesic $t \to \operatorname{Exp} tX$, where $X = \sum \alpha_i A_i \in \mathfrak{A}$, $\sum \alpha_i^2 = 1$. Then by Lemma 4.1, t_0X is a conjugate point of o or there exists a unit vector Y $(\neq X)$ in $\mathfrak{A}$ such that $\operatorname{Exp} t_0X = \operatorname{Exp} t_0Y$. Generally, by a direct calculation, $\operatorname{Exp} tX = \operatorname{Exp} tY$ holds for some unit vector $Y = \sum \beta_i A_i$ if and only if

$$(*) \qquad \sqrt{2}(\alpha_i - \beta_i)t = m_i\pi \qquad \text{for } m_i \in Z$$

holds. So first, for a given X we shall search for the minimum positive number $\bar{t}_0$ such that (*) holds for some unit $Y \in \mathfrak{A}$, $Y \neq X$.

4.2. Lemma. $\bar{t}_0 = \operatorname{Min}_{1\le i\le n} \pi/(2\sqrt{2}\,|\alpha_i|)$.

Proof. We shall use the vector notation ; $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$, $\boldsymbol{m} = (m_1, \cdots, m_n) \in \boldsymbol{Z}^n - \{0\}$. Then (*) is equivalent to

$$(4.1) \qquad t = \frac{\pi\,|\boldsymbol{m}|^2}{2\sqrt{2}\,|\langle \boldsymbol{\alpha}, \boldsymbol{m}\rangle|}\,, \qquad |\boldsymbol{\alpha}| = 1\,,$$

$$(4.2) \qquad \boldsymbol{\beta} = \boldsymbol{\alpha} - \frac{2\langle \boldsymbol{\alpha}, \boldsymbol{m}\rangle}{|\boldsymbol{m}|^2}\boldsymbol{m}\,.$$

So, if we determine $\boldsymbol{m}_0 \neq 0$ such that the value of t defined by (4.1) takes the minimum positive value, then $\boldsymbol{\beta}$ is automatically determined by (4.2). Now we put $\alpha := \operatorname{Max}_{1\le i\le n}|\alpha_i| = \operatorname{Max}_{|\boldsymbol{m}|=1}|\langle \boldsymbol{\alpha}, \boldsymbol{m}\rangle|/|\boldsymbol{m}|^2$. Then we get

$$\frac{|\langle \boldsymbol{\alpha}, \boldsymbol{m}\rangle|}{|\boldsymbol{m}|^2} \le \frac{|\alpha_1||m_1| + \cdots + |\alpha_n||m_n|}{m_1^2 + \cdots + m_n^2} \le \alpha\frac{|m_1| + \cdots + |m_n|}{m_1^2 + \cdots + m_n^2} \le \alpha\,.$$

So $\operatorname{Max}_{m\in Z^n-\{0\}}|\langle \boldsymbol{\alpha}, \boldsymbol{m}\rangle|/|\boldsymbol{m}|^2 = \alpha$, and the equality holds only in the following case: Let $\alpha = |\alpha_{i_1}| = \cdots = |\alpha_{i_k}|$, then $m_{i_1} = \varepsilon_1 \operatorname{sgn} \alpha_{i_1}, \cdots, m_{i_k} = \varepsilon_k \operatorname{sgn} \alpha_{i_k}$ $(\varepsilon_1, \cdots, \varepsilon_k = 0 \text{ or } 1)$ and other m_i's are equal to zoro. Thus we have $\bar{t}_0 = \operatorname{Min}_{m\in Z^n-\{0\}} \pi\,|\boldsymbol{m}|^2/(2\sqrt{2}\,|\boldsymbol{\alpha}, \boldsymbol{m}|) = \pi/(2\sqrt{2}\,\alpha)$ with $\alpha = \operatorname{Max}_{1\le i\le n}|\alpha_i|$.

4.3. Remark. If $\alpha = |\alpha_{i_1}| = \cdots = |\alpha_{i_k}|$, then $\boldsymbol{\beta}$ which is determined by (4.2) with above m_i's are given by $\boldsymbol{\beta} = (\alpha_1, \cdots, \underset{i_1}{\pm\alpha_{i_1}}, \cdots, \underset{i_j}{\pm\alpha_{i_j}}, \cdots, \underset{i_k}{\pm\alpha_{i_k}}, \cdots, \alpha_n)$. So there exist $2^k - 1$ $Y = \sum \beta_i A_i (\neq X)$ such that $\operatorname{Exp} \bar{t}_0X = \operatorname{Exp} \bar{t}_0Y$ holds.

Now the tangential cut point t_0X of o along $t \to \operatorname{Exp} tX$ is given by

$t_0 := \operatorname{Min}\{t > 0 \mid tX$ is the first tangential conjugate point of o along $t \to \operatorname{Exp} tX$ or there exists a unit $Z \in \mathfrak{M}$ $(Z \neq X)$ such that $\operatorname{Exp} tX = \operatorname{Exp} tZ\}$

$= \operatorname{Min}\{t > 0 \mid tX$ is the first tangential conjugate point of o along $t \to \operatorname{Exp} tX$ or there exists a unit $Y \in \mathfrak{A}$ $(Y \neq X)$ such that $\operatorname{Exp} tX = \operatorname{Exp} tY\}$.

Since tX is the first tangential conjugate point of o along $t \to \operatorname{Exp} tX$ if and only if $t = \operatorname{Min}_{j<k} \pi/(\sqrt{2}\,|\alpha_j - \alpha_k|)$ and obviously $\operatorname{Min}_{j<k} \pi/(\sqrt{2}\,|\alpha_j - \alpha_k|) \geq \operatorname{Min} \pi/(2\sqrt{2}\,|\alpha_i|)$ holds, the tangential cut point t_0X of o along $t \to \operatorname{Exp} tX$ ($X = \sum \alpha_i A_i$) is given by t_0 $(=\bar{t}_0) = \operatorname{Min}_i \pi/(2\sqrt{2}\,|\alpha_i|)$. Note that the cut point t_0X is the first conjugate point if and only if $\operatorname{Max}_{j<k}|\alpha_j - \alpha_k| = 2\operatorname{Max}|\alpha_i|$, i.e., there exist some $j < k$ such that $|\alpha_j| = |\alpha_k| = \alpha$ and $\alpha_j + \alpha_k = 0$ hold. Thus we get

4.4. Proposition. *The tangential cut point t_0X of o along a geodesic $t \to \operatorname{Exp} tX$, where $X = \sum \alpha_i A_i$ and $g(X, X) = 1$, is given by $t_0 = \operatorname{Min}_{1 \leq i \leq n} \pi/(2\sqrt{2}\,|\alpha_i|)$.*

4.5. Theorem. *For $X = \sum \alpha_i A_i$, where $\sum \alpha_i^2 = 1$, we put $\alpha(X) := \operatorname{Max}_{1 \leq i \leq n} |\alpha_i|$ and $t_0(X) := \operatorname{Min}_{1 \leq i \leq n} \pi/(2\sqrt{2}\,|\alpha_i|) = \pi/(2\sqrt{2}\,\alpha(X))$. Then $\Lambda^k(n) := \{\lambda \in \Lambda(\boldsymbol{R}^{2n}) \mid \dim(\lambda \cap \mu_0) = k\}$ is given by $\{\operatorname{Exp} \operatorname{Ad}(SO(n))t_0(X)X \mid X = \sum \alpha_i A_i,\ \sum \alpha_i^2 = 1$, with $\alpha(X) = |\alpha_{i_1}| = \cdots = |\alpha_{i_k}|\}$. In particular, we have $\bigcup_{k=1}^n \Lambda^k(n) =$ Cut locus of o, $\Lambda^0(n) =$ Interior set of o.*

Proof. First we shall show that for a unit vector $X \in \mathfrak{A}$ with $\alpha(X) = |\alpha_{i_1}| = \cdots = |\alpha_{i_k}|$, $\dim((\operatorname{Exp} t_0(X)X) \cap \mu_0)$ is equal to k. In fact, we may assume $\alpha(X) = |\alpha_1| = \cdots = |\alpha_k| > |\alpha_{k+1}| \geq \cdots \geq |\alpha_n|$. Then we have

$$
\begin{aligned}
\operatorname{Exp} t_0(X)X &= \exp t_0(X)X \cdot \lambda_0 \\
&= \left\{ (p, q) \mid p = \begin{pmatrix} x_1 \cos \sqrt{2}\,\alpha_1 t_0(X) \\ \vdots \\ x_n \cos \sqrt{2}\,\alpha_n t_0(X) \end{pmatrix},\right. \\
&\qquad \left. q = \begin{pmatrix} x_1 \sin \sqrt{2}\,\alpha_1 t_0(X) \\ \vdots \\ x_n \sin \sqrt{2}\,\alpha_n t_0(X) \end{pmatrix};\ (x_1, \cdots, x_n) \in \boldsymbol{R}^n \right\} \\
&= \left\{ (p, q) \mid p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k+1} \cos(\alpha_{k+1}\pi/2\alpha(X)) \\ \vdots \\ x_n \cos(\alpha_n \pi/2\alpha(X)) \end{bmatrix},\right. \\
&\qquad \left. q = \begin{bmatrix} \pm x_1 \\ \vdots \\ \pm x_k \\ x_{k+1} \sin(\alpha_{k+1}\pi/2\alpha(X)) \\ \vdots \\ x_n \sin(\alpha_n \pi/2\alpha(X)) \end{bmatrix};\ (x_1, \cdots, x_n) \in \boldsymbol{R}^n \right\}.
\end{aligned}
$$

From this our assertion is obvious. Since τ_h $(h \in SO(n))$ leaves μ_0 invariant, we get

$$\Big\{\operatorname{Exp} \operatorname{Ad}(SO(n))t_0(X)X \,|\, X = \sum_i \alpha_i A_i \Big(\sum_i \alpha_i^2 = 1\Big) \text{ with } \alpha(X) = |\alpha_{i_1}| = \cdots = |\alpha_{i_k}|\Big\} \subset \Lambda^k(n) \,.$$

Similary it is easy to show that

$$\operatorname{Int}(o) = \{\operatorname{Exp} \operatorname{Ad}(SO(n))tX \,|\, 0 \leq t < t_0(X), X \in \mathfrak{A}, g(X, X) = 1\} \subset \Lambda^0(n).$$

But by Propositions 1.4 and 4.4,

$$\begin{aligned} M = \bigcup_{k=0}^{n} \Lambda^k(n) \supset \bigcup_{X \in \mathfrak{A}, |X|=1} & (\{\operatorname{Exp} \operatorname{Ad}(SO(n))t_0(X)X\} \\ & \cup \{\operatorname{Exp} \operatorname{Ad}(SO(n))tX \,|\, 0 \leq t < t_0(X)\}) \\ = \text{Cut locus of } o \cup \operatorname{Int}(o) = M. & \end{aligned}$$

Thus the proof is completed.

4.5. Corollary. *Diameter of* $M = \sqrt{n}\,\pi/(2\sqrt{2})$. *Injectivity radius of M* $= \pi/(2\sqrt{2}) =$ (Diameter of $M)/\sqrt{\operatorname{rank} M}$.

Proof. Diameter of $M = \operatorname{Max}_{\sum \alpha_i^2 = 1} \operatorname{Min}_{1 \leq i \leq n} \pi/(2\sqrt{2}\,|\alpha_i|) = \sqrt{n}\,\pi/(2\sqrt{2})$. Injectivity radius of $M = \operatorname{Min}_{\sum \alpha_i^2 = 1} \operatorname{Min}_{1 \leq i \leq n} \pi/(2\sqrt{2}\,|\alpha_i|) = \pi/(2\sqrt{2})$.

q.e.d.

5. Closed geodesics

5.1. Theorem. *For* $\boldsymbol{m} := (m_1, \cdots, m_n) \in \boldsymbol{Z}^n - \{0\}$, *we put* $X(\boldsymbol{m}) := \sum_i (m_i/|\boldsymbol{m}|)A_i \in \mathfrak{A}$. *Then each of the following holds*:

(i) $c(t): t \to \operatorname{Exp} tX(\boldsymbol{m})$, $0 \leq t \leq |\boldsymbol{m}|\pi/\sqrt{2}$, *is a closed geodesic of length* $|\boldsymbol{m}|\pi/\sqrt{2}$ *with the initial point o. Its multiplicity is equal to the greatest common divisor of* $m_1, \cdots, m_n$.

(ii) *Every closed geodesic of M with the initial point o may be expressed in the form* $t \to \operatorname{Exp} t \operatorname{Ad}(h)X(\boldsymbol{m})$, *where* $h \in SO(n)$ *and* $X(\boldsymbol{m}) = \sum_i (m_i/|\boldsymbol{m}|)A_i$, $\boldsymbol{m} \in \boldsymbol{Z}^n - \{0\}$.

(iii) *The intersection number of a clesed geodesic* $t \to \operatorname{Exp} t \operatorname{Ad}(h)X(m)$, $0 \leq t \leq |\boldsymbol{m}|\,\pi/\sqrt{2}$, *with the oriented codimension one cycle* $\bigcup_{k=1}^{n} \Lambda^k(n)$ *is given by* $\sum m_i$.

Proof. 1°. $c(t): t \to \operatorname{Exp} t(\sum \alpha_i A_i)$, where $\sum \alpha_i^2 = 1$ and $0 \leq t \leq t_1$, is a geodesic loop $\Leftrightarrow \exp t_1(\sum \alpha_i A_i) \in O(n)$

$$\begin{aligned} &\Leftrightarrow ((\cos \sqrt{2}\,\alpha_i t_1 + \sqrt{-1} \sin \sqrt{2}\,\alpha_i t_1)\delta_{ij}) \in O(n) \Leftrightarrow \sqrt{2}\,\alpha_i t_1 = m_i,\ m_i \in \boldsymbol{Z} \\ &\Leftrightarrow t_1 = \pi m_i/(\sqrt{2}\,\alpha_i) = \pi\,|\boldsymbol{m}|/\sqrt{2}\ , \\ &\qquad \alpha_i = m_i/|\boldsymbol{m}| \text{ with } \boldsymbol{m} = (m_1, \cdots, m_n) \in \boldsymbol{Z}^n - \{0\}\ . \end{aligned}$$

Next, since $\exp(\pi|\boldsymbol{m}|X(\boldsymbol{m})/\sqrt{2}) = ((\cos m_i\pi)\delta_{ij}) \in O(n)$, we get

$$\begin{aligned}\dot{c}(\pi|\boldsymbol{m}|/\sqrt{2}) &= d\tau_{\exp(\pi|\boldsymbol{m}|X(\boldsymbol{m})/\sqrt{2})}d\pi(\textstyle\sum \alpha_i A_i)\\ &= d\pi(\mathrm{Ad}\,((\cos m_i\pi)\delta_{ij}))(\textstyle\sum \alpha_i A_i)\\ &= d\pi(\textstyle\sum \alpha_i A_i) = \dot{c}(0)\ ,\end{aligned}$$

that is, $c(t)$, $0 \leq t \leq \pi|\boldsymbol{m}|/\sqrt{2}$ is a closed geodesic.

2°. Let k be the greatest common divisor of $m_1, \cdots, m_n$; i.e., $\boldsymbol{m} = k\boldsymbol{p}$, $\boldsymbol{p} := (p_1, \cdots, p_n) \in \boldsymbol{Z}^n - \{0\}$ and $p_1, \cdots, p_n$ are relatively prime. Then we get by 1°, $c(\pi|\boldsymbol{p}|/\sqrt{2}) = c(0)$, $\dot{c}(\pi|\boldsymbol{p}|/\sqrt{2}) = \dot{c}(0)$, and consequently $c(t)$, $0 \leq t \leq \pi|\boldsymbol{m}|/\sqrt{2}$, is a closed geodesic of multiplicity k. Conversely let $c(t): t \to \mathrm{Exp}\, tX(\boldsymbol{m})$, $0 \leq t \leq t_1 = \pi|\boldsymbol{m}|/\sqrt{2}$, $\boldsymbol{m} \in \boldsymbol{Z}^n - \{0\}$, be a closed geodesic of multiplicity k. Then from $c(t_1/k) = c(0)$, $\dot{c}(t_1/k) = \dot{c}(0)$, we get $\pi|\boldsymbol{m}|/(\sqrt{2}k) = \pi|\boldsymbol{p}|/\sqrt{2}$, $p_i/|\boldsymbol{p}| = m_i/|\boldsymbol{m}|$ for some $\boldsymbol{p} \in \boldsymbol{Z}^n - \{0\}$ with relatively prime $p_1, \cdots, p_n$. That is $\boldsymbol{m} = k\boldsymbol{p}$ and the greatest common divisor of $m_1, \cdots, m_n$ is equal to k. Thus we have shown (i). (ii) is obvious from 1° and the fact that $\mathfrak{M} = \mathrm{Ad}\,(SO(n))\mathfrak{A}$.

3°. Let $c(t): t \to \mathrm{Exp}\, t\,\mathrm{Ad}\,(h)X(\boldsymbol{m})$, $0 \leq t \leq \pi|\boldsymbol{m}|/\sqrt{2}$, be a closed geodesic. To show (iii), it suffices to consider the case $h = e$. Then the intersection number of $c(t)$ with the oriented cycle $\bigcup_{k=1}^n A^k(n)$ is given by

$$\textstyle\sum_{c(t)\cap\mu_0\neq\{0\}} \mathrm{sgn}\, q_{c(t)\cap\mu_0}\dot{c}(t)\ ,$$

where $q_{c(t)\cap\mu_0}\dot{c}(t)$ is the following symmetric form on a subspace $c(t) \cap \mu_0$ of $c(t)$: Let $q_{c(t)}\dot{c}(t)$ be the symmetric bilinear form on $c(t)$ defined by

$$\begin{aligned}&q_{c(t)}\dot{c}(t)\left(\begin{pmatrix}x_i \cos\sqrt{2}\alpha_i t\\ x_i \sin\sqrt{2}\alpha_i t\end{pmatrix}, \begin{pmatrix}y_i \cos\sqrt{2}\alpha_i t\\ y_i \sin\sqrt{2}\alpha_i t\end{pmatrix}\right)\\ &\quad = (\sqrt{2}/|\boldsymbol{m}|)\textstyle\sum m_i x_i y_i \sin^2(\sqrt{2}m_i t/|\boldsymbol{m}|)\ , \qquad (\alpha_i := m_i/|\boldsymbol{m}|)\ .\end{aligned}$$

Then $q_{c(t)\cap\mu_0}\dot{c}(t)$ is defined as the restriction of $q_{c(t)}\dot{c}(t)$ to the subspace $c(t) \cap \mu_0$ of $c(t)$, [4]. Now $c(t) \cap \mu_0 \neq \{0\}$ if and only if $\cos\sqrt{2}\alpha_i t = 0$ for at least one α_i. Now put $T := \{(m_i, r) \mid 1 \leq i \leq \mathrm{n}, 1 \leq r \leq |m_i|, r \text{ integer}\}$ and consider the following equivalence relation "$\sim$" on T: $(m_i, r) \sim (m_j, s) \Leftrightarrow t_{i,r}(:= \pi|\boldsymbol{m}|/(\sqrt{2}|m_i|)\cdot\frac{1}{2}(2r-1)) = t_{j,s}(:= \pi|\boldsymbol{m}|/(\sqrt{2}|m_j|)\cdot\frac{1}{2}(2s-1))$. We denote by $[(m_i, r)]$ the equivalence class of (m_i, r) with respect to "$\sim$". Then $c(t) \cap \mu_0 \neq \{0\}$ holds if and only if $t = t_{i,r}$ sor some $(m_i, r) \in T$, and $c(t_{i,r}) \cap \mu_0 = \{(0, q) \mid q = {}^t(0, \cdots, \pm x_{i_1}, \cdots, \pm x_{i_k}, \cdots, 0)$, where $i_1, \cdots, i_k$ are determined by $[(m_i, r)] = \{(m_{i_1}, r_1), \cdots, (m_{i_k}, r_k)\}$. Thus we have $q_{c(t_{i,r})\cap\mu_0}\dot{c}(t_{i,r}) := (\sqrt{2}/|\boldsymbol{m}|)\sum_{j=1}^k m_{i_j}x_{i_j}y_{i_j}$, so that $\mathrm{sgn}\, q_{c(t_{i,r})\cap\mu_0}\dot{c}(t_{i,r}) = \sum_{j=1}^k \mathrm{sgn}\, m_{i_j}$, and consequently the intersection number is equal to $\sum_{[(m_i,r)]\in T/\sim}\sum_{j=1}^k \mathrm{sgn}\, m_{i_j} = \sum_{i=1}^n (\mathrm{sgn}\, m_i)|m_i| = \sum_{i=1}^n m_i$, because of $\sharp T = \sum_{i=1}^n |m_i|$.

5.2. Corollary. *Two closed geodesics* $t \to \mathrm{Exp}\, t\,\mathrm{Ad}\,(h)X(\boldsymbol{m})$, $0 \leq t \leq$

$|\boldsymbol{m}|\pi/\sqrt{2}$ *and* $t \to \operatorname{Exp} t \operatorname{Ad}(k)X(\boldsymbol{n})$, $0 \leq t \leq |\boldsymbol{n}|\pi/\sqrt{2}$, *where* $h, k \in SO(n)$, *are homotopically equivalent if and only if* $\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} n_i$.

References

[1] V. I. Arnold, *On a characteristic class entering in quantization condition*, Funct. Anal. Appl. **1** (1967) 1–13.

[2] R. Crittenden, *Minimum and conjugate points in symmetric spaces*, Canad. J. Math. **14** (1962) 320–328.

[3] J. J. Duistermaat, *Fourier integral operators*, Lecture Notes, Courant Institute, New York, 1973.

[4] ——, *On the Morse index in variational calculus*, Advances in Math. **21** (1976) 173–195.

[5] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

Added in proof. After the present note had been submitted, the following articles on cut loci of compact symmetric spaces appeared.

[6] H. Naitoh, *On cut loci and first conjugate loci of the irreducible symmetric* R-*spaces and the irreducible compact hermitian symmetric spaces*, Hokkaido Math. J. **6** (1977) 230–242.

[7] T. Sakai, *On cut loci of compact symmetric spaces*, Hokkaido Math. J. **6** (1977) 136–161.

[8] ——, *On the structure of cut loci in compact riemannian symmetric spaces*, to appear in Math. Ann.

[9] ——, *Cut loci of compact symmetric spaces*, to appear in Proc. United States-Japan Sem. Differential Geometry, 1977.

[10] M. Takeuchi, *On conjugate loci and cut loci of compact symmetric spaces*, I, II, preprints, 1977.

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