J. DIFFERENTIAL GEOMETRY
12 (1977) 473-480

# TOTALLY REAL SUBMANIFOLDS

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## 1. Introduction

Among all submanifolds of a Kaehler manifold $\tilde{M}$, there are two typical classes: one is the class of complex submanifolds and the other is the class of totally real submanifolds. A submanifold $M$ of a Kaehler manifold $\tilde{M}$ is said to be complex (resp. totally real) if each tangent space of $M$ is mapped into itself (resp. the normal space) by the complex structure of $\tilde{M}$.

In [3] Chen and Ogiue studied some fundamental properties of totally real submanifolds. In particular, some characterizations of totally real submanifolds and some classifications of totally real submanifolds in complex space forms are obtained [3]. (See also [7]).

In this paper, we shall continue to study fundamental properties of totally real submanifolds. In particular, we shall obtain two reduction theorems for totally real submanifolds in complex space forms and also study totally real submanifolds with parallel mean curvature vector.

## 2. Basic formulas

Let $\tilde{M}^{2m}$ be a $2m$-dimensional[1] Kaehler manifold with complex structure $J$ and metric tensor $g$. Let $\tilde{\nabla}$ (resp. $\tilde{R}$) be the Levi-Civita connection (resp. the curvature tensor) of $\tilde{M}^{2m}$. We denote by $\nabla$ (resp. $R$) the induced Levi-Civita connection (resp. the curvature tensor) of an $n$-dimensional totally real submanifold $M$. Then the second fundamental form $\sigma$ of the immersion is given by

$$(2.1)\qquad \sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y \ ,$$

where $X, Y, \cdots$, etc. are vector fields in $M$. The *mean curvature vector* $H$ is then given by $H = (1/n) \operatorname{Tr} \sigma$. For a normal vector field $\xi$, we write

$$(2.2)\qquad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \ ,$$

where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X\xi$. Then we have $g(\sigma(X, Y), \xi) = g(A_\xi X, Y)$. Let $R^D$ denote the curva-

Communicated by K. Yano, August 4, 1975. The first author was partially supported by NSF Grant GP-36684.

[1] In this paper we consider only real dimensions of manifolds.

ture tensor of the normal connection $D$, i.e., $R^D(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$. Then the Gauss, Codazzi and Ricci equations are given respectively by

$$g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(Y, Z), \sigma(X, W)) \;, \tag{2.3}$$

$$(\tilde{R}(X, Y)Z)^N = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z) \;, \tag{2.4}$$

$$g(\tilde{R}(X, Y)\xi, \eta) = g(R^D(X, Y)\xi, \eta) - g([A_\xi, A_\eta](X), Y) \;, \tag{2.5}$$

where $(\bar{\nabla}_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$, $(\tilde{R}(X, Y)Z)^N$ is the normal component of $\tilde{R}(X, Y)Z$, and $\xi, \eta, \cdots$, etc. are normal vector fields of $M$ in $\tilde{M}^{2m}$.

A Kaehler manifold $\tilde{M}^{2m}$ is a complex space form of constant holomorphic sectional curvature $c$, denoted by $\tilde{M}^{2m}(c)$, if its curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \frac{c}{4}\{g(\tilde{Y}, \tilde{Z})\tilde{X} - g(\tilde{X}, \tilde{Z})\tilde{Y} + g(J\tilde{Y}, \tilde{Z})J\tilde{X} - g(J\tilde{X}, \tilde{Z})J\tilde{Y} + 2g(\tilde{X}, J\tilde{Y})J\tilde{Z}\} \;, \tag{2.6}$$

where $\tilde{X}, \tilde{Y}, \tilde{Z}, \cdots$, etc. are vector fields in $\tilde{M}^{2m}$. If the ambient space $\tilde{M}^{2m}$ is a complex space form $\tilde{M}^{2m}(c)$, then (2.3), (2.4) and (2.5) reduce respectively to

$$g(R(X, Y)Z, W) = g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)) + \frac{c}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \;, \tag{2.7}$$

$$(\bar{\nabla}_X \sigma)(Y, Z) = (\bar{\nabla}_Y \sigma)(X, Z) \;, \tag{2.8}$$

$$g(R^D(X, Y)\xi, \eta) = g([A_\xi, A_\eta](X), Y) + \frac{c}{4}\{g(JY, \xi)g(JX, \eta) - g(JX, \xi)g(JY, \eta)\} \;. \tag{2.9}$$

A normal vector field $\xi$ is called a *parallel section* in the normal bundle $T^\perp M$ if $D\xi = 0$. A *unit* normal vector field $\xi$ is called an *isoperimetric section* if $\operatorname{Tr} A_\xi$ is constant. A subbundle $Q$ of the normal bundle $T^\perp(M)$ is *holomorphic* if $Q$ is invariant under $J$, i.e., if $JQ \subset Q$. A subbundle $Q$ of $T^\perp M$ is said to be *parallel* if $Q$ is invariant under parallel translation, i.e., if $D_X\xi$ is also a section in $Q$ for every local section $\xi$ in $Q$. It is clear that a unit normal vector field $\xi$ is parallel if and only if the line bundle generated by $\xi$ is parallel. For a subbundle $Q$ of $T^\perp M$, there exists a unique subbundle $Q^c$ of $T^\perp M$ such that $Q$ and $Q^c$ are orthogonal and $Q \oplus Q^c = T^\perp M$. We called $Q^c$ the *complementary subbundle* of $Q$. It is clear that for a totally real submanifold $M$ in $\tilde{M}$, the complementary subbundle $(J(TM))^c$ of $J(TM)$ is always holomorphic, and a subbundle $Q$ is parallel if and only if its complementary subbunble $Q^c$ is parallel. The complementary subbundles of holomorphic subbundles of $T^{\perp}M$ is called a *coholomorphic subbundles* of $T^{\perp}M$. It is clear that a subbundle $\tilde{Q}$ of $T^{\perp}M$ is coholomorphic if and only if $\tilde{Q}$ is the direct sum of $J(TM)$ and a holomorphic subbundle of $T^{\perp}M$.

## 3. Reduction theorems

Let $M$ be an $n$-dimensional totally real submanifold of a $2m$-dimensional Kaehler manifold $\tilde{M}^{2m}$. If there exists a $2r$-dimensional parallel holomorphic subbundle $Q$ of $T^{\perp}M$, then for any section $\xi$ in $Q$ and vector fields $X$, $Y$ in $M$ we have

$$\begin{aligned} g(\sigma(X,Y),\xi) &= g(\tilde{\nabla}_X Y,\xi) = g(\tilde{\nabla}_X JY, J\xi) \\ &= g(D_X JY, J\xi) = -g(JY, D_X J\xi) = 0 \,, \end{aligned}$$

from which we see that $\sigma|_Q$ (restriction of $\sigma$ to $Q$) vanishes. Thus we have

**Lemma 1.** *Let $M$ be an $n$-dimensional totally real submanifold of a $2m$-dimensional Kaehler manifold $\tilde{M}^{2m}$. If $Q$ is a $2r$-dimensional parallel holomorphic subbundle of $T^{\perp}M$, then $\sigma|_Q \equiv 0$.*

Let $Q$ be a holomorphic subbundle of $T^{\perp}M$. Then the coholomorphic subbundle $Q^c$ contains $J(TM)$ as its subbundle and $Q$ is parallel if and only if $Q^c$ is parallel. Hence from Lemma 1 we obtain

**Lemma 2.** *Let $M$ be an $n$-dimensional totally real submanifold of a $2m$-dimensional Kaehler manifold $\tilde{M}^{2m}$. If $Q$ is a parallel coholomorphic subbundle of $T^{\perp}M$, then* $\operatorname{Im}\sigma \subset Q$, *where* $\operatorname{Im}\sigma = \{\sigma(X,Y) : X, Y \in TM\}$.

In particular, if the ambient space is a complex space form, then from Lemma 2 we have

**Theorem 1.** *Let $M$ be an $n$-dimensional totally real submanifold of a $2m$-dimensional complex space form $\tilde{M}^{2m}(c)$. Then $M$ is a totally real submanifold of a $2(n+s)$-dimensional totally geodesic complex submanifold $\tilde{M}^{2(n+s)}(c)$ of $\tilde{M}^{2m}(c)$ if and only if there exists an $(n+2s)$-dimensional parallel coholomorphic subbudle of $T^{\perp}M$.*

The "only if" part is trivial, and the "if" part follows from Lemma 2 and an argument similar to the proof of Proposition 9 given in [1] with only slight modifications.

In the following theorem, we shall give some necessary and sufficient conditions for the codimension of totally real submanifolds which can be reduced to minimal one.

**Theorem 2.** *Let $M$ be an $n$-dimensional totally real submanifold of a $2m$-dimensional complex space form $\tilde{M}^{2m}(c)$. Then $M$ is contained in a $2n$-dimensional totally geodesic complex submanifold $\tilde{M}^{2n}(c)$ of $\tilde{M}^{2m}(c)$ if and only if one of the following statements holds:*

(i) *$J(TM)$ is parallel.*

(ii) $\operatorname{Im}\sigma \subset J(TM)$.

(iii) *$(J(TM))^c$ is parallel.*

*Proof.* Since $J(TM)$ is coholomorphic, Theorem 1 implies that $M$ is a totally real submanifold of a $2n$-dimensional totally geodesic complex submanifold $\tilde{M}^{2n}(c)$ of $\tilde{M}^{2m}(c)$ if and only if statement (i) holds. Hence it suffices to prove the equivalences of (i), (ii) and (iii). The equivalence of (i) and (iii) are clear. Moreover, Lemma 1 shows that (iii) implies (ii). Thus the theorem follows from

**Lemma 3.** "$\operatorname{Im}\sigma \subset J(TM)$" *implies* "$(J(TM))^c$ *is parallel.*"

*Proof.* Let $\xi$ be any section of the holomorphic subbundle $(J(TM))^c$ and $\operatorname{Im}\sigma \subset J(TM)$. Then we have $\sigma|_{(J(TM))^c} = 0$ and hence

$$\begin{aligned} 0 &= g(\sigma(X, Y), J\xi) = g(\tilde{\nabla}_X Y, J\xi) \\ &= g(\tilde{\nabla}_X JY, -\xi) = -g(D_X JY, \xi) = g(JY, D_X \xi) \ . \end{aligned}$$

Since this is true for all $X$ and $Y \in TM$, $(J(TM))^c$ is parallel. q.e.d.

As an application of Theorems 1 and 2, we have the following.

**Corollary 1.** *Let $M$ be an $n$-dimensional totally real, totally umbilical submanifold ($n \geq 2$) of a $2m$-dimensional complex space form $\tilde{M}^{2m}(c)$, $c \neq 0$.*

(i) *If $H \equiv 0$, then $M$ is contained in a $2n$-dimensional totally geodesic complex submanifold $\tilde{M}^{2n}(c)$ of $\tilde{M}^{2m}(c)$.*

(ii) *If $H \not\equiv 0$, then $M$ is contained in a $2(n+1)$-dimensional totally geodesic complex submanifold $\tilde{M}^{2(n+1)}(c)$ of $\tilde{M}^{2m}(c)$.*

*Proof.* Since $M$ is totally real and totally umbilical in $\tilde{M}^{2m}(c)$, Theorem 1 of [4] implies either (1) $M$ is totally geodesic in $\tilde{M}^{2m}(c)$ or (2) the mean curvature vector $H$ is nonzero and parallel. If Case (1) holds, then Theorem 2 shows that $M$ is contained in a $2n$-dimensional totally geodesic complex submanifold $\tilde{M}^{2n}(c)$ of $\tilde{M}^{2m}(c)$. If Case (2) holds, then $H$ and $JH$ span a holomorphic plane subbundle of $T^\perp M$, say, $V$. From (2.9) of Ricci, we find that $V$ is perpendicular to $J(TM)$. Hence $J(TM) \oplus V$ is an $(n+2)$-dimensional coholomorphic subbundle of $T^\perp M$. Now, since $H$ is parallel, for any vector fields $X$, $Y$ in $M$ we have

$$J\nabla_X Y + J\sigma(X, Y) = -A_{JY}(X) + D_X(JY) \ , \tag{3.1}$$

$$-A_{JH}(X) + D_X(JH) = -JA_H(X) \ . \tag{3.2}$$

On the other hand, by the total umbilicity of $M$ we find $\sigma(X, Y) = g(X, Y)H$. Thus (3.1) and (3.2) give

$$D_X(JH), D_X(JY) \in J(TM) \oplus V \ . \tag{3.3}$$

From these, we see that the subbundle $J(TM) \oplus V$ is a parallel coholomorphic subbundle of $T^\perp M$. Hence, by Theorem 1, $M$ is contained in a $2(n+1)$-dimensional totally geodesic complex submanifold $M^{2(n+1)}(c)$ of $\tilde{M}^{2m}(c)$.

## 4. Submanifolds with parallel mean curvature vector

In this section we shall assume that $M$ is a $2n$-dimensional Kaehler manifold $\tilde{M}^{2n}$. Then from (2.1) and (2.2) we immediately have

$$D_X(JY) = J\nabla_X Y\ , \qquad JA_{JY}X = \sigma(X, Y)\ , \tag{4.1}$$

$$\nabla_X(J\xi) = JD_X\xi\ . \tag{4.2}$$

From the first equation of (4.1) we obtain

$$R^D(X, Y)(JZ) = JR(X, Y)Z\ . \tag{4.8}$$

Moreover, (4.1) implies

**Lemma 4.** *Let $M$ be an $n$-dimensional totally real submanifold of a $2n$-dimensional Kaehler manifold $\tilde{M}^{2n}$. Then the normal bundle $T^\perp M$ admits a parallel nontrivial (local) section if and only if the tangent bundle admits a parallel nontrivial (local) section.*

**Remark 1.** Lemma 4 implies that the sectional curvature of every plane section containing $J\xi$ vanishes if $\xi$ is parallel. In particular, if $M$ is of constant sectional curvature, then the normal bundle admits no nontrivial parallel sections unless $M$ is flat.

In the following, we shall assume that the ambient space $\tilde{M}^{2n}$ is a complex space form $\tilde{M}^{2n}(c)$. If $\xi$ is a parallel section in $T^\perp M$, then (2.8) of Codazzi reduces to

$$(\nabla_X A_\xi)Y = (\nabla_Y A_\xi)X\ . \tag{4.4}$$

Let $e_1, \cdots, e_n, \xi_1, \cdots, \xi_n$ be a local field of orthonormal vectors in $\tilde{M}^{2n}(c)$, defined along $M$, such that $e_1, \cdots, e_n$ are tangent to $M$ (and hence $\xi_1, \cdots, \xi_n$ are normal to $M$). We put $h^k_{ij} = g(A_k e_i, e_j)$, where $A_k = A_{\xi_k}$. Let $K_{ij}$ denote the sectional curvature of the plane section $\pi(e_i, e_j)$. Then $K_{ij} = g(R(e_i, e_j)e_j, e_i)$. By (2.7) of Gauss, (4.4) of Codazzi and an argument similar to Smyth [6] we may prove

**Lemma 5.** *Let $M$ be an $n$-dimensional totally real submanifold of $\tilde{M}^{2n}(c)$. If $M$ admits a parallel isothermal section $\xi$, then*

$$\tfrac{1}{2}\Delta(\operatorname{Tr} A_\xi^2) = \sum_{i<j}\left\{K_{ij} + \sum_k (h^k_{ij})^2\right\}(\lambda_i - \lambda_j)^2 + \|\nabla A_\xi\|^2\ , \tag{4.5}$$

*where $\Delta$ is the Laplacian on $M$, $e_1, \cdots, e_n$ orthonormal eigenvectors of $A_\xi$ with eigenvalues $\lambda_1, \cdots \lambda_n$, and $K_{ij}$ the sectional curvature of the plane section $\pi(e_i, e_j)$.*

If $M$ is compact and of nonnegative sectional curvatures, then Lemma 2, together with Hopf's lemma, gives

$$\left\{K_{ij} + \sum_k (h^k_{ij})^2\right\}(\lambda_i - \lambda_j) = 0\ , \qquad i \neq j\ , \tag{4.6}$$

$$\nabla A_\xi = 0 \ . \tag{4.7}$$

Without loss of generality we may assume that

$$\lambda_1 = \cdots = \lambda_{v_1} = \rho_1 \ , \qquad \lambda_{v_1+1} = \cdots = \lambda_{v_1+v_2} = \rho_2 \ ,$$
$$\cdots, \lambda_{v_1+\cdots+v_{r-1}+1} = \cdots = \lambda_n = \rho_r \ ,$$

where $\rho_1, \cdots, \rho_r$ are all distinct. We put

$$[\rho_t] = \{v_1 + \cdots + v_{t-1} + 1, \cdots, v_1 + \cdots + v_{t-1} + v_t\} \ , \quad t = 1, \cdots, r \ .$$

From (4.6) it follows that

$$h^k_{ij} = K_{ij} = 0 \ , \qquad k = 1, \cdots, n, \ i \in [\rho_t], \ j \in [\rho_s], \ t \neq s \ . \tag{4.8}$$

Thus with respect to the eigenvectors $e_1, \cdots, e_n$ of $A_\xi$ we find

$$A_k = \begin{pmatrix} B^k_1 & 0 & \cdots & 0 \\ 0 & B^k_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^k_r \end{pmatrix} , \tag{4.9}$$

where $B^k_t$ is a $v_t \times v_t$-matrix.

Now let us consider the orthonormal frame field given by

$$e_1, \cdots, e_n, Je_1 = \xi_1, \cdots, Je_n = \xi_n \ , \tag{4.10}$$

where $e_1, \cdots, e_n$ are still eigenvectors of $A_\xi$. With respect such frame fields, $A_k$'s still have the forms (4.9). Moreover $A_k$'s satisfy [3]

$$h^i_{jk} = h^k_{ij} = h^j_{ik} \ , \qquad i, j, k = 1, \cdots, n \ . \tag{4.11}$$

By using (4.8) and (4.11) we obtain

**Lemma 6.** *Let $M$ be a compact totally real submanifold in $\tilde{M}^{2n}(c)$. If $M$ has nonnegative sectional curvature and admits a parallel isoperimetric section $\xi$, then with respect to the frame field* (4.10), *$A_k$'s are given in the following forms*:

$$A_k = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & 0 \\ & & \ddots & & \\ \vdots & \vdots & & B^k_s & \vdots \\ & & & \ddots & \\ 0 & 0 & \cdots & & 0 \end{pmatrix} ,$$

$$v_1 + \cdots + v_{s-1} < k \leq v_1 + \cdots + v_s,\ 1 \leq s \leq r\ ,$$

*where* $B_s^k$ *is a* $v_s \times v_s$*-matrix.*

By using Lemma 6 we may prove

**Theorem 3.** *Let $M$ be a compact $n$-dimensional manifold with nonnegative sectional curvature immersed in a $2n$-dimensional complex space form $\tilde{M}^{2n}(c)$ as a totally real submanifold. Then the normal bundle $T^{\perp}M$ admits no parallel isoperimetric section unless $\tilde{M}^{2n}(c)$ is flat.*

*Proof.* We assume that $\xi$ is a parallel isoperimetric section. We shall divide the proof into two cases.

*Case* a. $\lambda_1 = \cdots = \lambda_n = \lambda$. In this case, $M$ is umbilical with respect $\xi$. Hence we may choose $e_1, \cdots, e_n$ in such a way that $Je_1 = \xi$. Lemma 4 then implies that $K_{1j} = 0$ for $j > 1$. On the other hand, (2.7) of Gauss gives $K_{1j} = c/4 + \sum_k \{h_{11}^k h_{jj}^k - (h_{1j}^k)^2\}$. Thus by (4.11) we find $c = 0$.

*Case* b. $\lambda_1 \neq \lambda_i$ *for some* $i$. In this case, (4.8) implies $K_{1j} = 0$ for $j \in [\lambda_i]$. On the other hand, Lemma 6 gives $\sum_k \{h_{11}^k h_{jj}^k - (h_{1j}^k)^2\} = 0$. Thus (2.7) of Gauss implies $c = 0$.

**Corollary 2.** *Under the hypothesis of Theorem* 3, *if the mean curvature vector $H$ is parallel, then either* (i) $c > 0$ *and $M$ is minimal in $\tilde{M}^{2n}(c)$ or* (ii) $\tilde{M}^{2n}(c)$ *is flat*, i.e., $c = 0$.

*Proof.* Since $\operatorname{Tr} A_H = |H|^2$, the parallelism of $H$ implies that either $H = 0$ or $H/|H|$ is a parallel isoperimetric section. If $H = 0$, then $M$ is minimal in $\tilde{M}^{2n}(c)$, and the sectional curvatures of $M$ is $\leq \frac{1}{4}c$. Thus by the hypothesis we have $c \geq 0$. If $H \neq 0$, and $H/|H|$ is an isoperimetric section, then Theorem 3 implies that $\tilde{M}^{2n}(c)$ is flat.

**Remark 2.** If $\tilde{M}^{2n}(c)$ is flat, then there exist compact submanifolds of $\tilde{M}^{2n}(c)$ which satisfy the assumptions of Theorem 3 and also admit parallel isoperimetric section. For example, let $S^1$ be a unit circle in the complex plane $C^1$. Then $S^1 \times S^1$ is a such totally real surface in $C^2$.

In view of Theorem 3, it is interesting to study totally real submanifolds of the complex number space $C^n$ which admits a parallel isoperimetric section. The proofs of the following two theorems are similar to that of Theorem 2 in [6]. So we just only give the necessary outlines of the proofs.

**Theorem 4.** *Let $M$ be a compact $n$-dimensional totally real submanifold imbedded in $C^n$. If $M$ has nonnegative sectional curvature, and it admits a parallel isoperimetric section $\xi$, then $M$ is a product submanifold $M_1 \times \cdots \times M_r$, where $M_t$ is a compact $v_t$-dimensional totally real submanifold imbedded in some $C^{v_t}$, and $M_t$ is contained in a hypersphere of $C^{v_t}$.*

*Outline of proof.* The assumption of the theorem implies that $\nabla A_\xi = 0$. Thus the distinct eigenspaces $T_1, \cdots, T_r$ of $A_\xi$ define parallel distributions of $M$. By the de Rham decomposition theorem, $M$ is a product of Riemannian manifold $M_1 \times \cdots \times M_r$, where the tangent bundle of $M_s$ corresponds to $T_s$. By Lemma 6 and a lemma of Moore [5] we see that $M = M_1 \times \cdots \times M_r$

is a product submanifold imbedded in $C^n = C^{v_1} \times \cdots \times C^{v_r}$. Moreover, Lemma 6 implies that each of $M_t$'s is a totally real submanifold imbedded in some $C^{v_t}$:

$$M_1 \times \cdots \times M_r \overset{\text{imbedding}}{\hookrightarrow} C^{v_1} \times \cdots \times C^{v_r} .$$

Let $\pi_t(\xi)$ be the component of $\xi$ in the subspace $C^{v_t}$. Then $\pi_t(\xi)$ is a parallel normal section of $M_t$ in $C^{v_t}$, and $M_t$ is umbilical with respect to $\pi_t(\xi)$. From these it follows that $M_t$ is contained in a hypersphere of $C^{v_t}$ (see, for instance, [2]).

**Theorem 5.** *Let $M$ be a compact n-dimensional totally real submanifold imbedded in $C^n$. If $M$ has nonnegatve sectional curvature and parallel mean curvature vector $H$, then $M$ is a product submanifold $M_1 \times \cdots \times M_r$, where $M_t$ is a compact $v_t$-dimensional totally real submanifold imbedded in some $C^{v_t}$, and $M_t$ is also a minimal submanifold of a hypersphere in $C^{v_t}$.*

*Outline of proof.* Since the mean curvature vector $H$ is parallel and there exists no compact minimal submanifold in $C^n$, $H/|H|$ is a parallel isoperimetric section. By Theorem 4, $M$ is a product submanifold $M_1 \times \cdots \times M_r$ such that $M_t$ is totally real in some $C^{v_t}$ and $M_t$ is umbilical with respect to the component $\pi_t(H)$ of $H$ in the subspace $C^{v_t}$. Since $\pi_t(H)$ is parallel and is the mean curvature vector of $M_t$ in $C^{v_t}$, $M_t$ is a minimal submanifold of a hypersphere in $C^{v_t}$.

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