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PREASSIGNING CURVATURE OF POLYHEDRA HOMEOMORPHIC TO THE TWO-SPHERE

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In [2] it was shown that PL Riemannian two-manifolds exist with arbitrarily preassigned curvature satisfying the Gauss-Bonnet formula

$$(*) \qquad \sum_{\mathring{M}} \text{curvature} + \sum_{\partial M} \text{exterior angles} = 2\pi \cdot \text{Euler characteristic}.$$

A related problem is that of finding PL manifolds *embedded in* a Euclidean n-space $\boldsymbol{R}^n$ with preassigned curvature satisfying (*). Naturally an embedding theorem for PL Riemannian manifolds, analogous to the Nash theorem in the smooth category, would suffice here. Unfortunately, as of this date the isometric embedding problem in PL Riemannian geometry remains unsolved.

One embedding theorem is known: Alexandrov has shown [1] that an abstract PL Riemannian two-sphere whose curvature is everywhere nonnegative can be realized in $\boldsymbol{R}^3$ as the boundary of a convex set. Ironically, this result may not be applied to the spheres constructed in [2] to yield embedded spheres, since Alexandrov's theorem excludes the special case of the double of a convex polygon (it appears as a degenerate case, the "boundary" of a convex set with volume 0).

In this note we demonstrate the existence of embedded spheres with arbitrarily preassigned positive curvatures. More precisely:

Theorem 1. *Let $p_1, \cdots, p_r$ be points on the two-sphere S, and $k_1, \cdots, k_r$ real numbers such that*

1) $0 < k_i < 2\pi$ *for all* i,
2) $\sum_1^r k_i = 4\pi$.

Then there exists an embedding of S into $\boldsymbol{R}^n$ whose image is a polyhedral two-sphere, such that the induced PL Riemannian metric on S has curvatures k_i at the points p_i and is flat elsewhere.

Corollary 2. *The embedded sphere in Theorem* 1 *may be chosen to be the boundary of a convex linear three-cell in $\boldsymbol{R}^3$.*

(Note that this will follow from Alexandrov's theorem once it has been verified that the Riemannian metric on S is not induced from the double of a convex polygon. In fact, by a different method Robert Connelly has found an explicit construction of a convex linear cell in $\boldsymbol{R}^3$ with the desired curvature

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data; it is achieved as a polyhedron circumscribed about the unit sphere.)

As in [2], the homogeneity of manifolds implies that it will suffice to find a polyhedral sphere M in $\boldsymbol{R}^n$ with points $p_1', \cdots, p_r'$ such that the curvature is k_i at p_i' and is zero at all other points.

The proof of Theorem 1 depends on a basic result about tetrahedra.

Theorem 3. *Let $k_1, \cdots, k_4$ be positive numbers with $\sum_1^4 k_i = 4\pi$, and T a triangle with vertices V_1, V_2, V_3, and denote by a_i the interior angle at V_i. If $2a_i < 2\pi - k_i, 1 \leq i \leq 3$, then there is a tetrahedron M with vertices W_1, W_2, W_3, W_4 such that the linear map $T \to M$ sending V_i to W_i is isometric and the curvature at W_i is k_i. Furthermore, such tetrahedra are unique up to congruence or symmetry.*

1. Proof of Theorem 1

Assuming Theorem 3, the proof of Theorem 1 proceeds by induction on r. The case $r = 4$ follows immediately from Theorem 3. Assume inductively that one can construct a sphere with $r - 1$ vertices and preassigned curvatures, and furthermore that any three specified curvatures can be made to appear at the vertices of a flat triangular face. Let $k_1, \cdots, k_r$ be given. Suppose that a sphere is demanded with these curvatures, and that k_1, k_2 and k_3 are to appear at the vertices of a flat triangular face. Since $k_1 + k_2 + k_3 + k_r \leq 4\pi$, we may choose numbers $\varepsilon_i > 0$, $1 \leq i \leq 3$ such that

1) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = k_3$,

2) $\varepsilon_1 + k_1 < 2\pi$, $\varepsilon_2 + k_2 < 2\pi$, $\varepsilon_3 + k_r < 2\pi$.

By hypothesis there is a sphere S' in some $\boldsymbol{R}^n$ with curvatures $k_1 + \varepsilon_1, k_2 + \varepsilon_2, k_r + \varepsilon_3, k_4, \cdots, k_{r-1}$ at the vertices, the first three at the vertices V_1, V_2, V_3 of the triangular face T. If the angles of T are a_1, a_2, a_3, then it is easy to see that

$$a_1 < \tfrac{1}{2}(2\pi - k_1 - \varepsilon_1)\ , \quad a_2 < \tfrac{1}{2}(2\pi - k_2 - \varepsilon_2)\ , \quad a_3 < \tfrac{1}{2}(2\pi - k_r - \varepsilon_3)$$

(this is in fact the "triangle inequality" for angles around a vertex). By the lemma there exists a tetrahedron W with base congruent to T and curvatures $2\pi - 2a_1 - \varepsilon_1, 2\pi - 2a_2 - \varepsilon_2, 2\pi - 2a_3 - \varepsilon_3$, and k_3 at vertices W_1, W_2, W_3, W_4. Choose a point V such that the join W' of V and T is congruent to W and disjoint from $S' \backslash T$; this is certainly possible in $\boldsymbol{R}^{n+1}$. Let $S = \mathrm{cl}\,[(S' \cup \partial W') \backslash T]$, the connected sum of S' aud $\partial W'$. One easily checks that S is the required sphere. For example, at V_1 the angle sum is $2\pi - k_1 - \varepsilon_1$ in S', and is $2a_1 + \varepsilon_1$ in $\partial W'$. Therefore in S the angle sum at V_1 is $(2\pi - k_1 - \varepsilon_1) + (2a_1 + \varepsilon_1) - 2a_1 = 2\pi - k_1$.

2. Proof of Corollary 2

In order to apply Alexandrov's theorem, we must show that the sphere S is not isometric to a double of polygon. This is easily demonstrated, as follows.

The point W can be joined to each of the points V_1, V_2, V_3 by a unique geodesic in $\boldsymbol{R}^{n+1}$; since these geodesics lie in S, they are unique shortest paths from W to V_1, V_2, and V_3. But in a doubled polygon, any vertex of positive curvature can be joined to only two other vertices by unique shortest paths.

3. Outline of the proof of Theorem 3

Suppose a triangle T is given, situated in the plane $\boldsymbol{R}^2 \subset \boldsymbol{R}^3$. Any tetrahedron with base T is determined up to congruence (or symmetry) by a point V in open upper half space H, namely, by forming the *join* T^*V. Thus we may think of H as the *space of tetrahedra with base* T; it is homeomorphic to an open three-cell. If the vertices of T are V_1, V_2, V_3, and the lengths of the edges VV_1, VV_2, VV_3 are x, y, z respectively, then there is a well defined map $h\colon H \to \boldsymbol{R}^3$ given by $h(V) = (x, y, z) = (x(V), y(V), z(V))$. The map h is a diffeomorphism onto its image H' which is a reparametrization of the space of tetrahedra with base T.

Given a point X in H', that is, a tetrahedron, there is a well-defined triple (k_1, k_2, k_3) of numbers in $\boldsymbol{R}^3$ defined by $k_i =$ the curvature of the tetrahedron X at the vertex V_i. Thus there is a well-defined map $\varphi\colon H' \to K \subset \boldsymbol{R}^3$, where K consists of all triples (k_1, k_2, k_3) satisfying

1) $0 < k_i < 2\pi - 2a_i$, $1 \le i \le 3$,
2) $\sum_1^3 k_i < 4\pi$.

K is evidently an open convex linear cell.

Theorem 3 can now be restated: $\varphi\colon H' \to K$ is a homeomorphism onto. This will be proved in two steps. First, φ is differentiable; we compute the Jacobian $J(\varphi)$ and show that it is never zero. It follows that φ is an open map. Second, by a compactification argument it will be shown that φ is extendable to a map from a closed cell with interior H' to $\operatorname{cl} K$ which sends boundary points to boundary points. It will then follow that φ is surjective, and in fact a homeomorphism onto.

4. Computation of $J(\varphi)$

Suppose a triangle is given with sides of lengths a, b, c and opposite angles A, B, C, respectively. The law of cosines gives

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = \cos^{-1} u .$$

Viewing C as a function of a, b, and c we have

$$\frac{\partial C}{\partial u} = -(1 - u^2)^{-1/2} = \frac{-1}{\sin C} = \frac{-b}{c \sin B} = \frac{-a}{c \sin A} ,$$

these last by the law of sines. We can also easily have

$$\frac{\partial u}{\partial a} = \frac{a^2 + c^2 - b^2}{2a^2b} = \frac{c \cos B}{ab} ,$$

and similarly,

$$\frac{\partial u}{\partial b} = \frac{c \cos A}{ab} , \quad \text{while} \quad \frac{\partial u}{\partial c} = \frac{-c}{ab} .$$

Therefore we derive the formulas

$$\frac{\partial C}{\partial a} = \frac{-\cot B}{a} , \quad \frac{\partial C}{\partial b} = \frac{-\cot A}{b} , \quad \frac{\partial C}{\partial c} = \frac{\csc B}{a} = \frac{\csc A}{b} .$$

Using these formulas, we compute $J(\varphi)$. Let the fixed tetrahedron K have vertices V_1, V_2, V_3, V_4, faces Q, R, S, T opposite these vertices respectively, edges $x = V_1V_4$, $y = V_2V_4$, $z = V_3V_4$, $q = V_2V_3$, $r = V_3V_1$, and $s = V_1V_2$ (Fig. 1). A face angle of K will be denoted by a letter determining the face and a subscript determining the vertex, Thus Q_2 is the angle at V_2 on the triangle Q, etc.

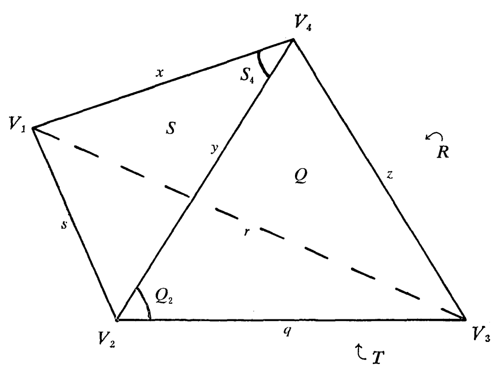

Fig. 1

Now $\varphi(x, y, z) = (k_1, k_2, k_3)$ where

$$k_1 = 2\pi - R_1 - S_1 - T_1 , \quad k_2 = 2\pi - Q_2 - S_2 - T_2 ,$$
$$k_3 = 2\pi - Q_3 - R_3 - T_3 .$$

Recalling that the angles T_i are constant, while the other angles depend on x, y, z, the Jacobian matrix of φ is

$$\begin{bmatrix} -\frac{\partial R_1}{\partial x}-\frac{\partial S_1}{\partial x} & -\frac{\partial S_1}{\partial y} & -\frac{\partial R_1}{\partial z} \\ -\frac{\partial S_2}{\partial x} & -\frac{\partial Q_2}{\partial y}-\frac{\partial S_2}{\partial y} & -\frac{\partial Q_2}{\partial z} \\ -\frac{\partial R_3}{\partial x} & -\frac{\partial Q_3}{\partial y} & -\frac{\partial Q_3}{\partial z}-\frac{\partial R_3}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x}(\cot R_4 + \cot S_4) & -\frac{1}{x}\csc S_4 & -\frac{1}{x}\csc R_r \\ -\frac{1}{y}\csc S_4 & \frac{1}{y}(\cot Q_4 + \cot S_4) & -\frac{1}{y}\csc Q_4 \\ -\frac{1}{z}\csc R_4 & -\frac{1}{z}\csc Q_4 & \frac{1}{z}(\cot Q_4 + \cot R_4) \end{bmatrix}.$$

Therefore

$$\begin{aligned} J(\phi) = \frac{1}{xyz}[&\cot^2 R_4(\cot Q_4 + \cot S_4) + \cot^2 S_4(\cot Q_4 + \cot R_4) \\ &+ \cot^2 Q_4(\cot R_4 + \cot S_4) + 2\cot Q_4 \cot R_4 \cot S_4 \\ &- 2\csc Q_4 \csc R_4 \csc S_4 - \csc^2 R_4(\cot Q_4 + \cot S_4) \\ &- \csc^2 S_4(\cot Q_4 + \cot R_4) - \csc^2 Q_4(\cot R_4 + \cot S_4)] \\ = \frac{1}{xyz}[&2\cot Q_4 \cot R_4 \cot S_4 - 2\csc Q_4 \csc R_4 \csc S_4 \\ &- (\cot Q_4 + \cot S_4) - (\cot Q_4 + \cot R_4) - (\cot R_4 + \cot S_4)] \\ = \frac{2}{xyz \sin Q_4 \sin R_4 \sin S_4}&[\cos Q_4 \cos(R_4 + S_4) \\ &- 1 - \sin Q_4 \sin(R_4 + S_4)] \\ = -2\frac{1-\cos(Q_4+R_4+S_4)}{xyz \sin Q_4 \sin R_4 \sin S_4} &< 0\,, \end{aligned}$$

since $Q_4 + R_4 + S_4 < 2\pi$. This proves that φ is a local homeomorphism and an open map.

5. Proof of Theorem 3: conclusion

In order to verify that φ is a homeomorphism, it suffices to show that $\varphi \circ h \colon H \to K$ is a homeomorphism.

We first compactify H as follows. Compactify $\boldsymbol{R}^3 \to B^3$ to a three-cell in the usual way, that is, every point in $B^3 \setminus \boldsymbol{R}^3$ corresponds to a direction of a ray from a fixed point 0 in $\boldsymbol{R}^3$. Remove the points V_1, V_2, V_3 from B^3, yielding a manifold with three ends. Cap off each end with a sphere; a point on such a sphere corresponds to a direction in $\boldsymbol{R}^3$ of a ray emanating from the deleted point. The resulting space is a three-cell with three holes; the closure $\bar{H}$ of H in this space is clearly homeomorphic to a three-cell.

Any point P in $\bar{H} \backslash H$ is a limit of points in H; it should be thought of as the limit of tetrahedra, a degenerate tetrahedron. The value $\varphi(h(P))$ is defined to be the limit of $\varphi(h(P_j))$ for $P_j \to P$. That this makes sense derives from the fact that as $P_j \to P$, the direction of the line segment from V_i to P_j in $\boldsymbol{R}^3$ approaches a limiting value. This argument would fail in B^3, because as $P_j \to V_1$, for example, the rays from V_1 to P_j would not necessarily converge in direction. Thus a degenerate tetrahedron with two identical vertices does not have well-defined curvatures. Any other degenerate tetrahedron *does* have well-defined curvatures; for example a vertex at infinite distance has curvature 2π, while a vertex lying inside the triangle T has curvature 0.

The map $\varphi \circ h \colon \bar{H} \to \operatorname{cl} K$ is a continuous map between compact spaces (three-cells) and therefore takes closed sets to closed sets. Also, $\varphi \circ h$ takes $\bar{H} \backslash H$ to ∂K, so $\varphi \circ h \colon H \to K$ is also a closed map; hence $\varphi \circ h$ is surjective. In fact, it is easy to see $\varphi \circ h$ is a homeomorphism, as follows. Inverse images of compact sets are compact, so in particular point inverses are finite. If $\{P_1, \cdots, P_n\} = (\varphi \circ h)^{-1}(w)$, $w \in K$, then choosing a sufficiently small neighborhood O of w we may find neighborhoods U_i of P_i mapping homeomorphically onto O. There cannot be points in O arbitrarily close to w whose inverse images are not contained in $\bigcup_1^n U_j$; for then one could find a sequence in $\bar{H}$ whose images converged to w but which could not have a limit point (such a point would map to w). It now follows that $\varphi \circ h$ is a covering map, hence a homeomorphism. This completes the proof of Theorem 3.

References

[1] A. D. Alexandrov, *Konvexe polyeder,* Akad. Verlag, Berlin, 1958.
[2] H. Gluck, K. Krigelman & D. Singer, *The converse to the Gauss-Bonnet theorem in* PL, J. Differential Geometry **9** (1974) 601–616.

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