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CONFORMALITY OF RIEMANNIAN MANIFOLDS TO SPHERES

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1. Introduction

Let M be an orientable smooth Riemannian manifold of dimension n with Riemannian metric g_{ij}. Let ∇ be the covariant differentiation operator on M, and K_{hijk}, K_{ij}, r be the Riemann curvature tensor, Ricci curvature tensor, and scalar curvature tensor of M respectively. Let X denote the infinitesimal conformal transformation on M so that we have

$$(1.1) \qquad (\mathscr{L}_x g)_{ij} = \nabla_i X_j + \nabla_j X_i = 2\rho g_{ij} ,$$

where ρ is a function, and $\mathscr{L}_x$ denotes the Lie differentiation with respect to X. Assuming that $\mathscr{L}_x r = 0$ Yano, Obata, Hsiung-Mugridge, Hsiung-Stern (see [1], [2], [6], [8]) have studied the condition for a Riemannian n-manifold M to be conformal to an n-sphere. The purpose of this paper is to relax the condition $\mathscr{L}_x r = 0$ further, that is, to assume $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$, and to obtain conditions for M to be conformal to an n-sphere where $D\rho$ is the vector field associated with the 1-form $d\rho$. Towards this end we prove the following theorems.

Theorem 1.1. *If a compact orientable smooth Riemannian manifold M of dimension $n > 2$ admitting an infinitesimal conformal transformation X: $\mathscr{L}_x g = 2\rho g$, $\rho \neq$ constant with $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$ satisfies $\int_M \left(A_{ij}\rho^i\rho^j + \frac{\alpha}{n^2}\mathscr{L}_x\mathscr{L}_{D\rho} r\right)dv \geq 0$ where $A_{ij} = K_{ij} - (\alpha r/n)g_{ij}$ and $\alpha = 1$, then M is conformal to an n-sphere.*

Theorem 1.2. *Let M be an orientable smooth Riemannian manifold of dimension $n > 2$ admitting an infinitesimal conformal transformation X satisfying (1.1) such that $\rho \neq$ constant, and $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$. Then M is conformal to an n-sphere if $\mathscr{L}_x\mathscr{L}_{D\rho} r \geq 0$ and $\mathscr{L}_x |G|^2 = 0$ where $G_{ij} = K_{ij} - (r/n)g_{ij}$.*

Theorem 1.3. *Let M be an orientable smooth Riemannian manifold of dimension $n > 2$ admitting an infinitesimal conformal transformation X satisfying (1.1) such that $\rho \neq$ constant and $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$. Then M is conformal to an n-sphere if $\mathscr{L}_x\mathscr{L}_{D\rho} r \geq 0$ and $\mathscr{L}_x |W|^2 = 0$ where W is a tensor defined in* § 2.

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It is shown in § 5 that when $\mathscr{L}_x r = 0$, Theorems 1.1 and 1.2 reduce to those of Yano [6], and Theorem 1.3 reduces to that of Hsiung and Stern [2]. Also it is proved that when $r =$ constant, the condition $\alpha = 1$ in Theorem 1.1 may be replaced by $\alpha \geq 1$, and the manifold M would then be isometric to a sphere. The following known theorems are needed in the proofs of our theorems.

Theorem 1.4 (*Obata* [3]). *If a complete Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ such that $\nabla_i \nabla_j \rho = -c^2 \rho g_{ij}$ where c is a positive constant, then M is isometric to an n-sphere of radius $1/c$.*

Theorem (1.5 *Tashiro* [4]**).** *If a complete Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ such that $\nabla_i \nabla_j \rho + (1/n)\Delta \rho g_{ij} = 0$, then M is conformal to an n-sphere.*

2. Notations and formulas

The raising and lowering of the indices are, as usual, carried out respectively with g^{ij} and g_{ij}. The tensors thus obtained are called associated tensors. Let S, T be covariant tensors of order s with local components $S_{i_1 \cdots i_s}$ and $T_{i_1 \cdots i_s}$ respectively. The associated contravariant components of T are $T^{i_1 \cdots i_s}$. We define the inner product of S and T by $S_{i_1 \cdots i_s} T^{i_1 \cdots i_s}$ and denote it by $\langle S, T \rangle$. If $S = T$ we write $|S|^2$ for $\langle S, S \rangle$. For the sake of easy reference we list some known formulas; for details see Yano [7]:

$$\mathscr{L}_x r = 2(n-1)\Delta \rho - 2r\rho \,, \tag{2.1}$$

$$\mathscr{L}_x g^{ij} = -2\rho g^{ij} \,, \tag{2.2}$$

$$\mathscr{L}_x K_{hijk} = 2\rho K_{hijk} - g_{hk} \nabla_j \rho_i + g_{hj} \nabla_i \rho_k - g_{ij} \nabla_h \rho_k + g_{ik} \nabla_h \varphi_j \,, \tag{2.3}$$

$$\mathscr{L}_x K_{ij} = g_{ij} \Delta \rho - (n-2) \nabla_i \rho_j \,, \tag{2.4}$$

$$\nabla_k \nabla_i Y^j - \nabla_i \nabla_k Y^j = K_{kih}{}^j Y^h \,, \qquad g^{kj}(\nabla_k \nabla_i Y_j - \nabla_i \nabla_k Y_j) = K_i^h Y_h \,, \tag{2.5}$$

where Δ is the Laplace-Beltrami operator on M, and Y is any differentiable vector field on M. If the associated 1-form of a vector field Y is ξ, the components of ΔY and $\Delta \xi$ are given by

$$\Delta Y\colon -g^{kj} \nabla_k \nabla_j Y^i + K_h^i Y^h \,, \qquad \Delta \xi\colon -g^{kj} \nabla_k \nabla_j Y_i + K_i^h Y_h \,. \tag{2.6}$$

If d is the exterior differentiation operator on M, and f is any function on M, then we denote the associated vector field of the 1-form df by Df.

Write $f_i = \nabla_i f$, and $f^i = g^{ij} f_j$, and define the tensors Z and W by

$$Z_{hijk} = K_{hijk} - \frac{r}{n(n-1)}(g_{hk} g_{ij} - g_{hj} g_{ik}) \,, \tag{2.7}$$

$$W_{hijk} = aZ_{hijk} + b_1 g_{hk} G_{ij} - b_2 g_{hj} G_{ik} + b_3 g_{ij} g_{hk} - b_4 g_{ik} G_{hj} + b_5 g_{hi} G_{jk} - b_6 g_{jk} G_{hi} , \tag{2.8}$$

where $a, b_1, \cdots, b_6$ are any constants.

3. Lemmas

Lemma 3.1. *Let M be a compact orientable Riemannian manifold of dimension $n \geq 2$. For any vector field Y and a differentiable function f we have*

$$\int_M (\nabla_i Y^i) dv = 0 , \qquad \int_M \Delta f dv = 0 .$$

The first equation is the well known Green's formula, and the second follows as a consequence of the first.

Lemma 3.2 (*Yano and Sawaki* [9]). *Let M be a compact oriented Riemannian manifold of dimension $n > 2$ admitting an infinitesimal non-isometric conformal transformation X satisfying* (1.1). *Then for any function f on M we have*

$$\int_M \rho f dv = -\frac{1}{n} \int_M \mathscr{L}_x f dv .$$

Lemma 3.3. *For a manifold M having the same properties as in Lemma* 3.2, *we have*

$$\int_M (\Delta\rho)^2 dv = \int_M \rho^i \nabla_i \Delta\rho dv = \int_M (K_{ij}\rho^j - g^{kj}\nabla_k \nabla_j \rho_i)\rho^i dv . \tag{3.1}$$

Furthermore, if $r =$ constant, then

$$\int_M (\Delta\rho)^2 dv = \frac{r}{n-1} \int_M \rho^i \rho_i dv \tag{3.2}$$

Proof. $\nabla_i(\rho^i \Delta\rho) = \rho^i \nabla_i \Delta\rho - (\Delta\rho)^2 = (K_{ij}\rho^j - g^{kj}\nabla_k\nabla_j\rho_i)\rho^i - (\Delta\rho)^2$ by (2.5). Integrating and using Lemma 3.1 we get (3.1).

Setting $\mathscr{L}_x r = 0$ in (2.1) and using the result in (a) we obtain (3.2).

Lemma 3.4. *Let M be a manifold having the same properties as in Lemma* 3.2 *and satisfying the condition $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$. Then*

$$\int_M (r\rho^i\rho_i) dv = (n-1) \int_M (\Delta\rho)^2 dv + \frac{1}{n} \int_M \mathscr{L}_x \mathscr{L}_{D\rho} r dv . \tag{3.3}$$

Furthermore, if $\mathscr{L}_x r = 0$, then

$$(3.4)\qquad \frac{1}{n}\int_M \mathscr{L}_x\mathscr{L}_{D\rho}r\,dv = \int_M r\rho_i\rho^i dv - \frac{1}{n-1}\int_M r^2\rho^2 dv\ .$$

Proof. From (2.1) we have

$$\begin{aligned} 0 = \mathscr{L}_{D\rho}\mathscr{L}_x r &= 2\mathscr{L}_{D\rho}((n-1)\Delta\rho - \rho r) \\ &= 2[(n-1)\rho^i\nabla_i\Delta\rho - \rho\rho^i\nabla_i r - r\rho_i\rho^i]\ . \end{aligned}$$

Integrating and using Lemmas 3.2 and 3.3 we get (3.3). If $\mathscr{L}_x r = 0$, then $(n-1)\Delta\rho = \rho r$. Substituting this in (3.3) we obtain (3.4).

4. Proofs of Theorems

Proof of Theorem 1.1. For an arbitrary vector field Y, by writing $\nabla^j = g^{ji}\nabla_i$ and using (2.5) we find that

$$\begin{aligned} &\nabla^j\Big(\nabla_j Y_i + \nabla_i Y_j - \frac{2\alpha}{n}g_{ij}\nabla_t Y^t\Big)Y^i \\ &= \Big(g^{jk}\nabla_k\nabla_j Y_i + \nabla_i\nabla_j Y^j + K_{jih}{}^j Y^h - \frac{2\alpha}{n}\nabla_i\nabla_t Y^t\Big)Y^i + \frac{2}{n}\alpha(1-\alpha)(\nabla_t Y^t)^2 \\ &\quad + \frac{1}{2}\Big(\nabla_j Y_i + \nabla_i Y_j - \frac{2\alpha}{n}g_{ij}\nabla_t Y^t\Big)\Big(\nabla^j Y^i + \nabla^i Y^j - \frac{2\alpha}{n}g^{ij}\nabla_t Y^t\Big)\ . \end{aligned}$$

Putting $Y^i = \rho^i$, integrating the above equation, using Lemmas 3.1 and 3.3, and setting $K_{ij} = A_{ij} + (r\alpha/n)g_{ij}$ we get

$$\begin{aligned} &\int_M A_{ij}\rho^i\rho^j dv + \frac{1}{n}(-n + 2\alpha - \alpha^2)\int_M (\Delta\rho)^2 dv + \frac{\alpha}{n}\int_M r\rho_i\rho^i dv \\ &\qquad + \int_M \Big|\nabla\nabla\rho + \frac{\alpha}{n}g\Delta\rho\Big|^2 dv = 0\ . \end{aligned}$$

Substituting (3.3) in the above equation and simplifying we obtain finally

$$(4.1)\qquad \begin{aligned} &\int_M \Big(A_{ij}\rho^i\rho^j + \frac{\alpha}{n^2}\mathscr{L}_x\mathscr{L}_{D\rho}r\Big)dv \\ &\qquad + \int_M \Big|\nabla\nabla\rho + \frac{1}{n}(1 + \sqrt{(\alpha-1)(n-1)})g\Delta\rho\Big|^2 dv = 0\ . \end{aligned}$$

Hence Theorem 1.1 follows from Theorem 1.5 and the integral formula (4.1).

Proof of Theorem 1.2. From (2.2) and (2.4) we easily get

$$(4.2)\qquad \langle G, \nabla\nabla\rho\rangle = -\frac{2\rho}{n-2}|G|^2 - \frac{1}{2(n-2)}\mathscr{L}_x|G|^2\ .$$

On the other hand,

$$\nabla^i(G_{ij}\rho\rho^j) = G_{ij}\rho^i\rho^j + \rho\langle G, \nabla\nabla\rho\rangle + \frac{n-2}{2n}\rho(\rho^i\nabla_i r) . \tag{4.3}$$

Multiply (4.2) by ρ and integrate, integrate (4.3), and eliminate $\int_M \rho\langle G, \nabla\nabla\rho\rangle dv$ from the two resulting equations so that we have the integral formula

$$\begin{aligned}&\int_M \left(G_{ij}\rho^i\rho^j + \frac{1}{n^2}\mathscr{L}_x\mathscr{L}_{D\rho}r\right)dv \\ &\quad = \frac{2}{n-2}\int_M \left((\rho^2|G|^2 + \frac{1}{4}\rho\mathscr{L}_x|G|^2\right)dv + \frac{1}{2n}\int_M \mathscr{L}_x\mathscr{L}_{D\rho}r dv .\end{aligned} \tag{4.4}$$

Hence Theorem 1.2 follows from Theorem 1.1 and the integral formula (4.4).

Proof of Theorem 1.3. From (2.7), (2.8), (2.3), (2.4) and (2.2) we get (for details see [2])

$$\langle\mathscr{L}_x W, W\rangle = 2\rho|W|^2 - c\langle G, \nabla\nabla\rho\rangle , \tag{4.5}$$

where c is a constant given by

$$\begin{aligned}\frac{c-4a^2}{n-2} &= 2a\sum_{i=1}^{4} b_i + \left(\sum_{i=1}^{6}(-1)^{i-1}b_i\right)^2 \\ &\quad - 2(b_1b_3 + b_2b_4 - b_5b_6) + (n-1)\sum_{i=1}^{6} b_i^2 .\end{aligned}$$

Here $c \geq 0$. Use of (2.2) yields

$$\mathscr{L}_x|W|^2 = 2\langle\mathscr{L}_x W, W\rangle - 8\rho|W|^2 \tag{4.6}$$

Thus from (4.3), (4.5) and (4.6) we obtain

$$\begin{aligned}&c\int_M \left(G_{ij}\rho^i\rho^j + \frac{1}{n^2}\mathscr{L}_x\mathscr{L}_{D\rho}r\right)dv \\ &\quad = 2\int_M \rho^2|W|^2 dv + \frac{1}{2}\int_M \rho\mathscr{L}_x|W|^2 dv + \frac{c}{2n}\int_M \mathscr{L}_x\mathscr{L}_{D\rho}r dv .\end{aligned} \tag{4.7}$$

Hence Theorem 1.3 follows from Theorem 1.1 and the integral formula (4.7).

5. Special cases

1. Let $\alpha = 1$ and $\mathscr{L}_x r = 0$. The condition for conformality in Theorem 1.1 reduces, by (3.4), to

$$\int_M \left(K_{ij}\rho^i\rho^j - \frac{r^2\rho^2}{n(n-1)}\right)dv \geq 0 .$$

Also we have

$$\mathscr{L}_x|G|^2 = \mathscr{L}_x|R|^2 , \qquad \mathscr{L}_x|W|^2 = a^2\mathscr{L}_x|K|^2 + \frac{c-4a^2}{n-2}\mathscr{L}_x|R|^2 ,$$

where $|K|^2 = K_{hijk}K^{hijk}$ and $|R|^2 = K_{ij}K^{ij}$. The condition $\mathscr{L}_x\mathscr{L}_{D\rho}r \geq 0$ for M implies by (3.4) that

$$\int_M \left(r\rho_i\rho^i - \frac{r^2\rho^2}{n-1}\right)dv \geq 0 .$$

With these, Theorem, 1.1 and 1.2 reduce to results due to Yano [6], and Theorem 1.3 reduces to that due to Hsiung and Stern [2].

2. Let $\alpha \geq 1$ and $r =$ constant. From (4.1) it follows that M is isometric to a sphere if

$$\int_M A_{ij}\rho^i\rho^j dv \geq 0;$$

when $\alpha = 1$, this is a known condition [5]

$$\int_M G_{ij}\rho^i\rho^j dv \geq 0$$

for M to be isometric to a sphere.

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