GEOMETRY OF COMPLEX MANIFOLDS SIMILAR TO THE CALABI-ECKMANN MANIFOLDS

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In [4] Calabi and Eckmann showed that the product of two odd-dimensional spheres $S^{2p+1} \times S^{2q+1}$ $(p, q \geq 1)$ is a complex manifold. As $S^{2p+1} \times S^{2q+1}$ is not Kaehlerian, the fundamental 2-form $\Omega$ of the Hermitian structure is not closed. However, $d\Omega$ does have a special form on $S^{2p+1} \times S^{2q+1}$; in fact, $S^{2p+1} \times S^{2q+1}$ admits two nonvanishing vector fields which are both Killing and analytic, and whose covariant forms determine $\Omega$. Our purpose here is to study complex manifolds whose complex structures are similar to the complex structure on $S^{2p+1} \times S^{2q+1}$.

In § 1 we review the geometry of the Calabi-Eckmann manifolds. In § 2 we give some elementary properties of vector fields on a Hermitian manifold, and introduce the notion of a holomorphic pair of automorphisms and of a bicontact manifold. § 3 continues the author’s paper [2] on the differential geometry of principal toroidal bundles for the present case. In § 4 we discuss bicontact manifolds and, in particular, the integrable distributions of a bicontact structure on a Hermitian manifold. Finally in § 5 we give some results on the curvatures of a Hermitian manifold admitting a holomorphic pair of automorphisms.

1. The Hermitian structure on the Calabi-Eckmann manifolds

The construction of the complex structure on $S^{2p+1} \times S^{2q+1}$ which we will give is due to Morimoto [6]. It is well known that an odd-dimensional sphere $S^{2p+1}$ carries a contact structure, i.e., a nonvanishing 1-form $\eta$ such that $\eta \wedge (d\eta)^p \neq 0$. Let $G$ be the standard metric on $S^{2p+1}$. Then there exist on $S^{2p+1}$ (see e.g. [8]) a contact form $\eta$, a vector field $\xi$, and a tensor field $\varphi$ of type $(1,1)$ such that

$$\eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \varphi^2 = -I + \eta \otimes \xi,$$

$$G(\xi, X) = \eta(X), \quad G(\varphi X, \varphi Y) = G(X, Y) - \eta(X)\eta(Y),$$

i.e., $S^{2p+1}$ carries an almost contact metric structure. For a contact structure $\eta \wedge (d\eta)^p \neq 0$, $\varphi$, $\xi$ and $G$ may be chosen such that $d\eta(X, Y) = G(\varphi X, Y)$,
as happens in the sphere example. Moreover, the contact metric structure on $S^{2p+1}$ is normal, i.e.,

$$[\varphi, \varphi] + d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$. Thus $S^{2p+1}$ carries a normal contact metric or Sasakian structure.

Now let $(\varphi, \xi, \eta, G)$ and $(\varphi, \bar{\xi}, \bar{\eta}, \bar{G})$ be Sasakian structures on $S^{2p+1}$ and $S^{2q+1}$ respectively. Then define an almost complex structure $J$ on $S^{2p+1} \times S^{2q+1}$ by

$$J(X, \bar{X}) = (\varphi X - \eta(X)\xi, \varphi \bar{X} + \eta(X)\bar{\xi}),$$

and let $g$ be the product metric. Then direct computations show [6] that $J^2 = -I$, $g(J(X, \bar{X}), J(Y, \bar{Y})) = g((X, \bar{X}), (Y, \bar{Y}))$ and, using normality, that $[J, J] = 0$. Thus $S^{2p+1} \times S^{2q+1}$ is a Hermitian manifold.

Defining the fundamental 2-form $\Omega$ of the Hermitian structure by

$$\Omega((X, \bar{X}), (Y, \bar{Y})) = g(J(X, \bar{X}), (Y, \bar{Y})),$$

we find that

$$\Omega = d\eta + d\bar{\eta} + \eta \wedge \bar{\eta},$$

where we view $\eta$ and $\bar{\eta}$ as 1-forms extended to the product. Thus the fundamental 2-form $\Omega$ of the Hermitian structure on $S^{2p+1} \times S^{2q+1}$ satisfies

$$d\Omega = d\eta \wedge \bar{\eta} - \eta \wedge d\bar{\eta}.$$

Finally we remark that from the Hopf fibration $\pi ': S^{2p+1} \to PC^p$ of an odd-dimensional sphere as a principal circle bundle over complex projective space, we obtain a natural fibration $\pi: S^{2p+1} \times S^{2q+1} \to PC^p \times PC^q$ of a Calabi-Eckmann manifold as a principal $T^2$ (2-dimensional torus) bundle over a product of complex projective spaces. In fact the complex coordinates of $S^{2p+1} \times S^{2q+1}$ are essentially those of $PC^p \times PC^q$ together with a fibre coordinate [4], [5].

2. Some elementary properties of vector fields on a Hermitian manifold

Let $M^{2n}$ be a Hermitian manifold with complex structure $J$ and Hermitian metric $g$. Let $U$ be an analytic vector field on $M^{2n}$, i.e., $\mathcal{L}_U J = 0$ where $\mathcal{L}$ denotes Lie differentiation.

More generally on an almost complex manifold a vector field $U$ is said to be almost analytic if $\mathcal{L}_U J = 0$ and $[J, J(U, X)] = 0$ for all vector fields $X$. 

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[6] David E. Blair, Gerald D. Ludden & Kentaro Yano
Proposition 2.1. If \( U \) is an analytic vector field on \( M^{2n} \), then so is \( V = JU \).

Proof.

\[
0 = [J, J](U, X) = -[U, X] + [V, JX] - J[V, X] - J[U, JX] = -J(\mathbb{L}_U)X + (\mathbb{L}_J)X = (\mathbb{L}_V)X.
\]

Thus, if \( U \) is an infinitesimal automorphism of \( J \), so is \( JU \); but if \( U \) is Killing (an automorphism of \( g \)), \( JU \) is not in general Killing. We therefore make the following definition.

Definition. By a holomorphic pair of automorphisms we mean a unit vector field \( U \) such that \( U \) and \( V = JU \) are infinitesimal automorphisms of the Hermitian structure.

Let \( u \) and \( v \) denote the covariant forms of \( U \) and \( V \) respectively. We begin with some elementary properties of a holomorphic pair of automorphisms \((U, V = JU)\).

Lemma 2.2. \([U, V] = 0\).

Proof. \( 0 = (\mathbb{L}_U)U = [U, JU] - J[U, U] = [U, V] \).

Lemma 2.3. \( du(U, X) = 0, \ dv(U, X) = 0, \ dv(U, X) = 0 \).

Proof. We give the proof for \( du \), the proof for \( dv \) being similar. Since \( U \) is Killing and unit, we have

\[
\begin{align*}
du(U, X) &= (\mathcal{F}_v u)(X) - (\mathcal{F}_x u)(U) = g(\mathcal{F}_v u, X) - g(\mathcal{F}_x U, X) \\
&= -2g(\mathcal{F}_x U, U) = 0,
\end{align*}
\]

where \( \mathcal{F} \) denotes the Riemannian connection of \( g \). Similarly since \([U, V] = 0\) and \( V \) is also Killing, we have

\[
\begin{align*}
du(V, X) &= g(\mathcal{F}_v U, X) - g(\mathcal{F}_x U, V) = g(\mathcal{F}_v V, X) + g(\mathcal{F}_x V, U) = 0.
\end{align*}
\]

Proposition 2.4. At each point of \( M^{2n} \), \( u \) and \( v \) have odd rank, i.e., there exist nonnegative integers \( p \) and \( q \) such that \( u \wedge (du)^p \neq 0 \), \( v \wedge (dv)^q \neq 0 \), \((du)^{p+1} = 0\), \((dv)^{q+1} = 0\).

Proof. First note that \((du)^n = 0\); for evaluating \((du)^n\) on a \( J \)-basis containing \( U \) and \( V \) each term in

\[
(du)^n(U, V, X_1, \ldots, X_{2n})
\]

vanishes by Lemma 2.3; here we have set \( X_1 = U, X_2 = JU = V \) and \( \{X_i\} \) a \( J \)-basis. Suppose now that at \( m \in M^{2n} \), \((du)^p \neq 0\) and \((du)^{p+1} = 0\). Then evaluating \((u \wedge (du)^p)(U, Y_1, \ldots, Y_{2p})\) where \( Y_1, \ldots, Y_{2p} \) are vector fields such that \( du(Y_i, Y_j) \neq 0 \), we have that \( u \wedge (du)^p \neq 0 \). Similarly \( v \) has rank \( 2q + 1 \).

Definition. We say that a differentiable manifold \( M^{2n} \) is bicontact if it admits 1-forms \( u \) and \( v \) such that \( u \wedge v \wedge (du)^p \wedge (dv)^q \neq 0 \), \((du)^{p+1} = 0\)
and \((dv)^{q+1} = 0\) with \(p + q + 1 = n\). \(M^{2n}\) is called a Hermitian bicontact manifold if \(M^{2n}\) is both Hermitian and bicontact, and the 1-forms \(u\) and \(v\) are the covariant forms of a holomorphic pair of automorphisms.

**Lemma 2.5.** If \(du\) is of bidegree \((1,1)\) with respect to the complex structure \(J\), then so is \(dv\).

**Proof.** Recall that the Nijenhuis torsion of a vector-valued 1-form \(h\) is given by its action on a 1-form \(\theta\). This action is

\[
[h, h]_\theta = -h^{(2)} d\theta + h^{(1)} d(\theta \circ h) - d(\theta \circ h^2),
\]

where for a 2-form \(\theta\),

\[
(h^{(1)} \theta)(X, Y) = \theta(hX, Y) + \theta(X, hY), \quad (h^{(2)} \theta)(X, Y) = \theta(hX, hY).
\]

\(h^{(1)} \theta\) is often denoted by \(\theta \propto h\). Now since \(v = -u \circ J\) and \(du\) is of bidegree \((1,1)\), we have

\[
0 = ([J, J] u)(X, Y) = -du(JX, JY) - dv(JX, Y) - dv(X, JY) + du(X, Y) = -dv(JX, Y) - dv(X, JY),
\]

and hence \(dv\) is of bidegree \((1,1)\).

**Remark.** The above proof also shows that if \(du = dv\), then \([J, J] = 0\) implies that \(du (= dv)\) is of bidegree \((1,1)\). The authors have studied certain manifolds admitting independent 1-forms \(u\) and \(v\) with \(du = dv\), [1], [2].

**Proposition 2.6.** If \(M^{2n}\) is Kaehlerian, then \(du = dv = 0\).

**Proof.** First since \(V\) is analytic, we have

\[
0 = (\square_v J) X = \square_v JX - \square_J JX - J\square_v JX + J\square_J X = -\square_J JX + J\square_J X .
\]

Now since \(V\) is Killing,

\[
du(X, Y) = g(\nabla_X U, Y) - g(\nabla_Y U, X) = g(-\nabla_J JY, Y) - g(-\nabla_J YJ, X)
\]

\[
= g(\nabla_J J, JY) + g(\nabla_J Y, X) = -g(\nabla_J Y, JX) + g(\nabla_J V, V) = 0 .
\]

Similarly one can show that \(dv = 0\).

In [9] one of the authors introduced the notion of an \(f\)-structure on a differentiable manifold, i.e., the manifold admits a tensor field \(f \neq 0\) of type \((1, 1)\) satisfying \(f^2 + f = 0\) (see also [1], [7]).

**Proposition 2.7.** Let \((M^{2n}, J, g)\) be an almost Hermitian manifold admitting a nonvanishing vector field \(U\), then \(U, V = JU, u, v\) (the covariant forms of \(U\) and \(V\)) and \(f = J + v \otimes U - u \otimes V\) define an \(f\)-structure with complemented frames and rank \((f) = 2n - 2\) on \(M^{2n}\), i.e., we have
\[
f^* = -I + u \otimes U + v \otimes V , \quad fU = fV = 0 , \quad u \circ f = v \circ f = 0 , \\
u(U) = v(V) = 1 , \quad u(V) = v(U) = 0 .
\]

The proof of this proposition is a straightforward computation and will be omitted.

An \( f \)-structure with complemented frames \((f, U, V, u, v)\) is said to be \textit{normal} if the tensor \( S \) defined by

\[
S(X, Y) = [f, f](X, Y) + du(X, Y)U + dv(X, Y)V
\]
vanishes. Computing \( S \) in our case gives

\[
S(X, Y) = \langle J, J \rangle(X, Y) - (du \wedge J)(X, Y) - (dv \wedge J)(X, Y) \\
+ u(X)(2vJ)Y - u(Y)(ZvJ)X.
\]

Thus we have the following result.

**Proposition 2.8.** On a Hermitian manifold with a nonvanishing analytic vector field \( U \), if \( du \) is of bidegree \((1,1)\), then the \( f \)-structure \((f, U, V, u, v)\) is normal.

It is well known (see e.g. [7]) that for a normal \( f \)-structure with complemented frames, we have

\[
\mathcal{L}_u f = 0 , \quad \mathcal{L}_U u = 0 , \quad \mathcal{L}_V v = 0 , \quad \mathcal{L}_U = 0 , \quad \mathcal{L}_V = 0 , \quad \mathcal{L}_f = 0 , \quad [U, V] = 0 .
\]

Thus a straightforward computation shows that \( S = 0 \) implies \([J, J] = 0\).

Now if \( g \) is the Hermitian metric on \( M^{2m} \), then

\[
g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y) ,
\]

that is, \((f, g, u, v)\) defines a metric \( f \)-structure with complemented frames.

Finally we define the fundamental 2-forms \( \Omega \) and \( F \) of the structures by

\[
\Omega(X, Y) = g(JX, Y) , \quad F(X, Y) = g(fX, Y) .
\]

Then a short computation gives

\[
F = \Omega - u \wedge v .
\]

### 3. Fibering by a holomorphic pair of automorphisms

In [2] the authors proved the following result.

**Theorem.** Let \( M^{2m+2} \) be a compact connected manifold with a regular normal \( f \)-structure of rank \( 2m \). Then \( M^{2m+2} \) is the bundle space of a principal toroidal bundle over a complex manifold \( N^{2m} \).
Now if a complex manifold $M^{2n}$ admits a regular analytic vector field $U$ (i.e., every point $m \in M^{2n}$ has a neighborhood such that the integral curve of $U$ through $m$ passes through the neighborhood only once), the vector field $V = JU$ is not necessarily regular. Thus we say that a holomorphic pair of automorphisms is regular if both $U$ and $V$ are regular vector fields. Then using the above theorem and Proposition 2.8 we can prove the following result.

**Theorem 3.1.** If a compact Hermitian manifold $(M^{2n}, J, g)$ admits a regular holomorphic pair of automorphisms $(U, V = JU)$ with $du$ of bidegree $(1, 1)$, then $M^{2n}$ is a principal $T^2$ bundle over a Hermitian manifold $N^{2n-2}$.

**Proof.** From the above theorem and Proposition 2.8 we obtain the desired fibration. Thus we shall only exhibit the Hermitian structure on $N^{2n-2}$. As $U$ and $V$ are analytic, $J$ is projectable and we define $J'$ on $N^{2n-2}$ by

$$J'X = \pi_* J\bar{\pi}X,$$

where $\bar{\pi}$ denotes the horizontal lift with respect to the Riemannian connection of $g$ (in the nonmetric case one can use the pair $(u, v)$ as a Lie algebra valued connection form to determine $\bar{\pi}$ [2]). Then it is easy to check that $J'^2 = -I$. Moreover we have

$$[J', J'](X, Y) = -[\pi_* \bar{\pi}X, \pi_* \bar{\pi}Y] - [\pi_* J\bar{\pi}X, \pi_* \bar{\pi}Y] - \pi_* J\bar{\pi}X, \pi_* J\bar{\pi}Y] = \pi_* [J, J](\bar{\pi}X, \bar{\pi}Y) = 0.$$

Finally as $U$ and $V$ are Killing, the metric $g$ is projectable to a metric $g'$ on $N^{2n-2}$ given by $g'(X, Y) \circ \pi = g(\bar{\pi}X, \bar{\pi}Y)$. Then

$$g'(J'X, J'Y) \circ \pi = g(J\bar{\pi}X, J\bar{\pi}Y) = g(\bar{\pi}X, \bar{\pi}Y) = g'(X, Y) \circ \pi,$$

and hence the structure on $N^{2n-2}$ is Hermitian.

We now compute the fundamental 2-form $F$ of the $f$-structure $(J, U, V, u, v)$ on $M^{2n}$. First of all it is clear that $F(U, X) = 0$ and $F(V, X) = 0$. Thus it is enough to compute $F$ on vector fields of the form $\bar{\pi}X, \bar{\pi}Y$ where $X$ and $Y$ are vector fields on $N^{2n-2}$.

$$F(\bar{\pi}X, \bar{\pi}Y) = g(J\bar{\pi}X, \bar{\pi}Y) = g(\bar{\pi}X, \bar{\pi}Y) = g'(X, Y) \circ \pi,$$

where $\Omega'$ is the fundamental 2-form on $N^{2n-2}$. Hence we have $F = \pi^* \Omega'$. Now $dF = d\pi^* \Omega' = \pi^* d\Omega'$ and $dF = d\Omega - du \wedge v + u \wedge dv$, from which we get the following result.

**Theorem 3.2.** The base manifold $(N^{2n-2}, J', g')$ of the above fibration is Kaehlerian if and only if
\[ d\Omega = du \wedge v - u \wedge dv \]
on \(M^{2n}\).

Note also that by Proposition 2.6, \(d\Omega = 0\) implies \(du = dv = 0\) and hence \(dF = 0\). Thus we have the following result.

**Proposition 3.3.** If \(M^{2n}\) is Kaehlerian, then the base manifold \(N^{2n-2}\) is also Kaehlerian.

### 4. Hermitian bicontact manifolds

We begin with the following elementary result on the topology of a compact bicontact manifold.

**Theorem 4.1.** Let \(M^{2n}\) be a compact bicontact manifold, and let \(2p + 1\) and \(2q + 1\) denote the ranks of the bicontact forms \(u\) and \(v\) Then the betti numbers \(b_{2p+1}\) and \(b_{2q+1}\) are nonzero.

**Proof.** As \((2p + 1) + (2q + 1) = 2n\) it suffices to show that \(b_{2p+1}\) is nonzero. We shall show that \(u \wedge (du)^p\) has nonzero harmonic part. Suppose \(u \wedge (du)^p\) has no harmonic part, then as \((du)^{p+1} = 0\), \(u \wedge (du)^p\) is exact, say \(da\). Now on a bicontact manifold \(u \wedge (du)^p \wedge v \wedge (dv)^q\) is a volume element, hence, since \((dv)^{q+1} = 0\), we have

\[ 0 \neq \int_M u \wedge (du)^p \wedge v \wedge (dv)^q = \int_M d\alpha \wedge v \wedge (dv)^q = \int_M d(\alpha \wedge v \wedge (dv)^q) = 0 , \]
a contradiction.

We shall now digress briefly to introduce the notion of a semi-invariant submanifold [3]. Let \(M^{2n}\) be an almost complex manifold with a vector field \(U\) and a 1-form \(u\) with \(u(U) = 1\), and set \(V = JU\), \(v = -u \circ J\). Let \(\iota: \tilde{M} \rightarrow M^{2n}\) be a submanifold of \(M^{2n}\) such that 1) the transform of a vector tangent to \(M\) by \(\iota\) is in the space spanned by the tangent space of \(\tilde{M}\) and the vector \(U\), 2) \(V\) is tangent to \(\tilde{M}\), and 3) \(u \circ \iota = 0\); we then say that \(\tilde{M}\) is semi-invariant with respect to \(U\). Note that \(U\) is never tangent to \(\tilde{M}\), for if it were, then \(U = \iota \circ U\), and \(1 = u(U) = u(\iota(U)) = 0\), a contradiction.

Now define a tensor field \(\varphi\) of type \((1, 1)\), a vector field \(\xi\), and a 1-form \(\eta\) on \(\tilde{M}\) by

\[ J_{\iota \circ X} = \iota \circ \varphi X - \gamma(X)U , \quad V = \iota \circ \xi . \]

We then have

\[ -\iota \circ X = \iota \circ \varphi \circ X - \gamma(\varphi(X))U - \gamma(X)\iota \circ \xi , \]
from which it follows that

\[ \gamma^2 = -I + \gamma \otimes \xi , \quad \gamma \circ \varphi = 0 . \]
Also

\[-U = JV = J\xi = \xi \varphi \xi = \eta(\xi)U,\]

giving

\[\varphi \xi = 0, \quad \eta(\xi) = 1.\]

Thus we have the following result.

**Proposition 4.2.** A submanifold of \(M^{2n}\), which is semi-invariant with respect to \(U\), admits an almost contact structure.

Now computing \([J, J](\xi)(X, \xi)(Y)\) we have

\[
\begin{align*}
[J, J](\xi)(X, \xi)(Y) &= \xi \varphi(X, Y) + \eta(X)\xi - \eta(Y)\xi + (\xi)(\xi)(Y) - (\eta)(\eta)(Y)(X) - \eta(Y)(\xi)(X)U,
\end{align*}
\]

from which we obtain the following result.

**Proposition 4.3.** If a submanifold is semi-invariant with respect to an analytic vector field \(U\) on a complex manifold \(M^{2n}\), then its almost contact structure is normal.

Returning to the bicontact case, we assume for the remainder of this section that \(M^{2n}\) is a Hermitian bicontact manifold as defined in § 2. We define a distribution \(\mathcal{U}\) of dimension \(2q + 1\) by

\[\mathcal{U} = \{X | i(X)u = 0, i(X)du = 0\},\]

where \(i\) denotes the interior product operator. We shall show that \(\mathcal{U}\) is integrable. Let \(X\) and \(Y\) be vector fields belonging to \(\mathcal{U}\). Then

\[0 = du(X, Y) = Xu(Y) - Yu(X) - u([X, Y]) = -u([X, Y]).\]

Also for any \(Z\)

\[0 = du(X, Z) = Xu(Z) - u([X, Z]) = (\xi)(u)(Z),\]

and therefore

\[du([X, Y], Z) = [X, Y]u(Z) - Zu([X, Y]) = u([X, Y], Z)\]

\[= (\xi)(\xi)(u)(Z) = ((\xi)(\xi)(Y) - (\eta)(\xi)(Y)u)(Z) = 0.\]

Similarly the complementary distribution \(\mathcal{V} = \{X | i(X)v = 0, i(X)dv = 0\}\) of dimension \(2p + 1\) is integrable.

**Theorem 4.4.** A Hermitian bicontact manifold \(M^{2n}\) with \(du\) of bidegree \((1, 1)\) is locally the product of two normal contact manifolds \(M^{2p+1}\) and \(M^{2q+1}\).

**Proof.** As noted above the distributions \(\mathcal{U}\) and \(\mathcal{V}\) are complementary and integrable. Thus \(M^{2n}\) is locally the product of the respective maximal integral
submanifolds $M^{2q+1}$ and $M^{2p+1}$. We shall show that the integral submanifold $M^{2q+1}$ of $\mathcal{U}$ is semi-invariant with respect to $U$. Let $\iota: M^{2q+1} \to M^{2n}$ denote the immersion, and let $X$ be tangent to $M^{2q+1}$, i.e., $\iota_*X \in \mathcal{U}$. Set $Y = J\iota_*X + v(\iota_*X)U$. Then

$$u(Y) = u(J\iota_*X) + v(\iota_*X) = -v(\iota_*X) + v(\iota_*X) = 0,$$

and

$$du(Y, Z) = du(J\iota_*X + v(\iota_*X)U, Z) = du(J\iota_*X, Z) = -du(\iota_*X, JZ) = 0$$

since $du$ is of bidegree $(1, 1)$. Thus $Y \in \mathcal{U}$ so that $M^{2q+1}$ is semi-invariant with respect to $U$, and hence by Proposition 4.3 its almost contact structure is normal. Finally as

$$\eta(X) = -g(J\iota_*X, U) = g(\iota_*X, V) = v(\iota_*X),$$

we have that $\eta \wedge (d\eta)^n \neq 0$ on $M^{2q+1}$. Similarly, $M^{2p+1}$ is semi-invariant with respect to $V$, and is thus a normal contact manifold completing the proof.

Now let $P$ and $Q$ denote the projection maps to the tangent spaces of $M^{2p+1}$ and $M^{2q+1}$ respectively. We note for later use that $\iota(P - u \otimes U) = (P - u \otimes U)J$ as is easily verified, and hence that

$$JP = PJ + u \otimes V + v \otimes U.$$ 

We now compute the Lie derivative of $P$ with respect to $U$ and $V$. First note that

$$(\mathcal{L}_U P)X = [U, PX] - P[U, X].$$

Thus, if $X$ is $U$ or $V$, we clearly have $(\mathcal{L}_U P)X = 0$. If $X$ is orthogonal to $U$ but also tangent to $M^{2p+1}$, then $PX = X$ and $[U, X]$ is again tangent to $M^{2p+1}$ so that

$$(\mathcal{L}_U P)X = [U, X] - [U, X] = 0.$$ 

Finally, if $X$ is orthogonal to $V$ and tangent to $M^{2q+1}$, then $PX = 0$. Let $Y$ be arbitrary. Then as $U$ is Killing and $P$ symmetric, we have

$$g((\mathcal{L}_U P)X, Y) = -g(P[U, X], Y) = -g(V_PX, PY) + g(V_XU, PY)$$

$$= g(X, V_PY) - g(X, V_PYU) = g(X, [U, PY]) = 0.$$ 

Similarly $\mathcal{L}_VP = 0$, and thus $P$ and $Q = I - P$ are projectable by the fibration of § 3.

On the base manifold $N^{2n-2}$ of the fibration we define an almost product structure as follows.
\[ P'X = \pi_* P\hat{\pi}X, \quad Q'X = \pi_* Q\hat{\pi}X. \]

Then a direct computation shows that
\[
P'^2 = P', \quad Q'^2 = Q', \quad P'Q' = Q'P' = 0, \quad P' + Q' = I.
\]

Moreover as both the distributions \( \mathcal{U} \) and \( \mathcal{V} \) are integrable, \( [P, P] = 0 \) so that
\[
[P', P'][X, Y] = \pi_* P^2\hat{\pi}[\pi_* \hat{\pi}X, \pi_* \hat{\pi}Y] + [\pi_* P\hat{\pi}X, \pi_* P\hat{\pi}Y]
- \pi_* P\hat{\pi}[\pi_* \hat{\pi}X, \pi_* \hat{\pi}Y] - \pi_* P\hat{\pi}[\pi_* \hat{\pi}X, \pi_* \hat{\pi}Y]
- \pi_* [P, P](\hat{\pi}X, \hat{\pi}Y) = 0.
\]

Thus the induced almost product structure on \( N^{2n-2} \) is integrable, and so \( N^{2n-2} \)
is locally the product of two manifolds \( N^{2p} \) and \( N^{2q} \).

We have already seen that \( J \) is projectable since \( U \) and \( V \) are analytic, and that \( (J' = \pi_* J, g') \) is a Hermitian structure on \( N^{2n-2} \). Now let \( \iota : N^{2p} \rightarrow N^{2n-2} \)
denote the immersion of \( N^{2p} \) in \( N^{2n-2} \), and let \( X \) be a vector field on \( N^{2p} \). Then using \( J'P = PJ + u \otimes V + v \otimes U \), we have
\[
J'\iota_* X = \pi_* J\iota_* P\iota_* X = \pi_* P\iota_* X = \pi_* PJ\iota_* X
= \pi_* P\hat{\pi}J\iota_* X = P'J\iota_* X,
\]
and hence \( N^{2p} \) is an invariant submanifold of \( N^{2n-2} \) and consequently is a
Hermitian submanifold of \( N^{2n-2} \). Moreover, if \( N^{2n-2} \) is Kaehlerian, so is \( N^{2p} \)
and similarly \( N^{2q} \). Also, if each of the induced structures on \( N^{2p} \) and \( N^{2q} \) are
Kaehlerian, so is the structure on \( N^{2n-2} \). Thus using Theorems 3.1 and 4.4
and Proposition 3.2 we have

**Theorem 4.5.** Let \( M^{2n} \) be a regular Hermitian bicontact manifold with \( du \)
of bidegree \((1,1)\). Then the base manifold \( N^{2n-2} \) of the induced fibration
is locally the product of two Hermitian manifolds. Moreover, \( N^{2n-2} \) is locally
the product of two Kaehler manifolds if and only if the fundamental 2-form \( \Omega \) on
\( M^{2n} \) satisfies \( d\Omega = du \wedge v - u \wedge dv \).

## 5. Curvature

In this section we give some results on the curvature of a Hermitian mani-
fold admitting a holomorphic pair of automorphisms.

**Proposition 5.1.** Let \((M^{2n}, J, g)\) be a Hermitian manifold admitting a
holomorphic pair of automorphisms \((U, V = JU)\). Then the sectional curvature
of a section spanned by \( U \) and \( V \) vanishes.

**Proof.** Since \( U \) is Killing, from \( g(\nabla_V U, X) - g(\nabla_X U, V) = 0 \) which
was derived in the proof of Lemma 2.3 it follows that \( 2g(\nabla_U X, U) = 0 \) and hence
that \( \nabla_U U = 0 \). Moreover as \( U \) is a unit vector field, we have \( 0 = g(\nabla_X U, U) = -g(\nabla_U X, U) \)
giving \( \nabla_U U = 0 \). Thus \( g(R_{UV} U, V) = 0 \), where \( R \) is the
curvature tensor of $g$, and hence the sectional curvature of a section spanned by $U$ and $V$ vanishes.

**Theorem 5.2.** If the Hermitian manifold $M^{2n}$ of Theorem 3.1 has nonnegative sectional curvature, then the base manifold $N^{2n-2}$ also has nonnegative curvature.

**Proof.** First we note some relations.

$$[\pi X, \pi Y] = \pi[X, Y] + u([\pi X, \pi Y])U + v([\pi X, \pi Y])V.$$

Since $U$ and $V$ are Killing, we have

$$g(\nabla_{\pi X}\pi Y, U) = -g(\pi Y, \nabla_{\pi X}U) = -\frac{1}{2}du(\pi X, \pi Y),$$

$$g(\nabla_{\pi X}\pi Y, V) = -g(\pi Y, \nabla_{\pi X}V) = -\frac{1}{2}dv(\pi X, \pi Y),$$

and hence

$$\nabla_{\pi X}\pi Y = \pi\nabla'_{X}Y - \frac{1}{2}du(\pi X, \pi Y)U - \frac{1}{2}dv(\pi X, \pi Y)V,$$

where $\nabla'$ is the Riemannian connection of $g'$. Also, since $[U, \pi X]$ is vertical, $g(\nabla U, \pi X) = g(\nabla_{\pi X}U + [U, \pi X], \pi Y) = \frac{1}{2}du(\pi X, \pi Y)$, and similarly $g(\nabla_{\pi X}U, \pi Y) = \frac{1}{2}dv(\pi X, \pi Y)$.

We now compute the curvature.

$$g(R_{\pi X, \pi Y}, \pi Y) = g(\nabla_{\pi X}\nabla'_{\pi X}Y - \nabla_{\pi X}\nabla'_{\pi Y}X - \nabla_{\pi X, \pi Y}\pi X, \pi Y)$$

$$= g(\nabla_{\pi X}(\pi\nabla'_{X}Y - \frac{1}{2}du(\pi Y, \pi X)U - \frac{1}{2}dv(\pi Y, \pi X)V)$$

$$- \nabla_{\pi X}\pi\nabla'_{X}Y - \nabla_{\pi X, \pi Y}\pi X, \pi Y)$$

$$= g(\pi\nabla'_{X}Y, \pi Y) - \frac{1}{2}du(\pi Y, \pi X)g(\nabla_{\pi X}U, \pi Y)$$

$$- \frac{1}{2}dv(\pi Y, \pi X)g(\nabla_{\pi X}V, \pi Y)$$

$$- g(\pi\nabla'_{X,Y}X, \pi Y) - u([\pi X, \pi Y])g(\nabla_{\pi X}U, \pi Y)$$

$$- v([\pi X, \pi Y])g(\nabla_{\pi X}V, \pi Y)$$

$$= g'\nabla'_{X, Y}Y \circ \pi + \frac{1}{2}du(\pi X, \pi Y)^2 + \frac{1}{2}dv(\pi X, \pi Y)^2$$

since $du(\pi X, \pi Y) = \pi Xu(\pi Y) - \pi Y u(\pi X) - u([\pi X, \pi Y]) = -u([\pi X, \pi Y]).$

Now for the sectional curvature $K$ we have

$$K(\pi X, \pi Y) = -\frac{g(R_{\pi X, \pi Y}, \pi Y)}{g(\pi X, \pi Y)g(\pi Y, \pi Y) - g(\pi X, \pi Y)^2}.$$

Thus, if $K \geq 0$, then $g(R_{\pi X, \pi Y}, \pi Y) \leq 0$ and hence

$$-g'(\nabla'_{X,Y}X, Y) \circ \pi \geq \frac{1}{4}(du(\pi X, \pi Y)^2 + dv(\pi X, \pi Y)^2),$$

from which it follows that the sectional curvature $K'(X, Y) \geq 0.
References

[1] D. E. Blair, Geometry of manifolds with structural group \( \mathfrak{g}(n) \times \mathfrak{g}(\mathfrak{s}) \), J. Differential Geometry 4 (1970) 155–167.


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