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MORSE THEORY ON QUATERNIONIC GRASSMANNIANS

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Hangan has shown in [4] that one obtains a simple Morse function on a real or complex Grassmann manifold by embedding the manifold in a suitable projective space via the Plücker determinants (see [5, Chapter VII]) and then restricting a natural function on the projective space to the resulting variety. The method does not immediately work for the quaternionic case due to a lack of determinants over skew fields and the fact that $HG(p, q)$ is not a "quaternionic projective variety." We shall show his method may be adapted and extended to include the quaternionic case.

We denote the Grassmann manifold of p-planes in K^{p+q} by $KG(p, q)$, where $K = R, C, H$. $KP(n) = KG(1, n)$ denotes a projective space. We assume a knowledge of Morse theory as may be found in [6].

1. $HG(p, q)$ as a real projective variety

The right H space H^n may be identified with R^{4n} together with three linear operators J_r ($r = 1, 2, 3$) which correspond to right multiplication by i, j, k. For example if $\varphi(a + bi + cj + dk) = (a, b, c, d)$ gives the identification of H^1 with R^4, then J_1 is represented by the matrix $\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

Let $\varphi: H^{p+q} \to R^{4(p+q)}$ be the identification. If $0 \neq v \in H^{p+q}$, then the quaternionic line $\{vq \mid q \in H\}$ has as its φ-image the real 4-plane $\{(aI + bJ_1 + cJ_2 + dJ_3)\varphi(v) \mid a, b, c, d \in R\}$. Similarly we obtain $HG(p, q) \subset RG(4p, 4q) \subset RP(N - 1)$, where $N =$ binomial coefficient $C_{4(p+q),4p}$. The second containment is given by the quadratic p-relations, which are homogeneous equations on $R^N \simeq \Lambda^{4p}(R^{4(p+q)})$. The first containment is given by the homogeneous linear equations $\Lambda^{4p}(J_r)(x) = x$, $x \in \Lambda^{4p}(R^{4(p+q)})$, $r = 1, 2, 3$. These latter equations reflect the statement that a real $4p$-plane is the φ image of a quaternionic p-plane if and only if it is invariant under the J_r. Thus we have $HG(p, q)$ as real projective variety.

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2. The function f and the ordering of the Schubert Symbols

Let S denote the set of Schubert symbols of $4p$ elements in $4(p + q)$-space, and T the set of Schubert symbols of p elements in $p + q$ space. Thus $\sigma \in T$ means that $\sigma = (\sigma_1, \cdots, \sigma_p)$ with $1 \leq \sigma_1 \cdots < \sigma_p \leq p + q$. Two Schubert symbols are said to be neighbors if they have all but one element in common, e.g., $(1, 2, 3)$ and $(1, 3, 4)$ are neighbors.

Let F be the function on $\boldsymbol{RP}(N - 1)$ given by $F([x]) = \Sigma c_\rho (x_\rho)^2 / \Sigma (x_\rho)^2$, where both sums run over all $\rho \in S$ (which will be given a total ordering below), $[x] = [x_1, \cdots, x_N]$ are homogeneous coordinates, and c_ρ is real with $c_\rho < c_\tau$ for $\rho < \tau$. Then we have

Theorem 1. *$f \equiv$ restriction of F to $\boldsymbol{HG}(p, q)$ is a nondegenerate Morse function, and the critical points are the planes spanned by p of the coodinate axes. If $\sigma \in T$ denotes the critical plane spanned by the σ_1-th, $\cdots, \sigma_p$-th axes, then the Morse index at σ is $4d(\sigma) \equiv 4\Sigma(\sigma_i - i)$, and the Poincaré polynomial is $P(\boldsymbol{HG}(p, q); t) = \Sigma t^{4d(\sigma)}$.*

The proof will be given in §§ 3, 4.

To complete the definition of F we will need an ordering on S which differs from the standard lexicographic order. (T *will* be given the lexicographic order.) The new ordering is useful in establishing which points are critical for f. If A, B are subsets of S, then $A < B$ means that $\alpha < \beta$ for all $\alpha \in A$, $\beta \in B$.

Let $\rho = (\rho_{11}, \rho_{12}, \rho_{13}, \rho_{14}, \rho_{21}, \rho_{22}, \cdots, \rho_{p4}) \in S$. Define S_i by $S_i = \{\rho \in S \mid 4i - 3 \leq \rho_{11} \leq 4i\}$, and set $S_i < S_j$ if $i < j$. For fixed i define ${}^0S_i = \{\rho \in S_i \mid \rho_{14} = 4i\}$, ${}^1S_i = \{\rho \in S_i \mid \rho_{13} \leq 4i < \rho_{14}\}$, ${}^2S_i = \{\rho \in S_i \mid \rho_{12} \leq 4i < \rho_{13}\}$, ${}^3S_i = \{\rho \in S_i \mid \rho_{11} \leq 4i < \rho_{12}\}$, and set ${}^0S_i < {}^1S_i < {}^2S_i < {}^3S_i$. For $r = 1, 2, 3$ give ${}^rS_i$ the lexicographic order. For ${}^0S_i$ we repeat the process by considering $\rho_{21}, \rho_{22}, \rho_{23}, \rho_{24}$.

${}^0S_i$ is partitioned into sets $S_{i,i+1}, S_{i,i+2}, \cdots$. Each S_{ij} is partitioned into sets ${}^rS_{ij}$, and each ${}^0S_{ij}$ is further partitioned. The process ends at the stage ${}^0S_{i_1 \cdots i_p}$ since this latter set has only one element. Thus we get our desired ordering.

3. The critical points of f

Let $\pi \in \boldsymbol{HG}(p, q)$. We may choose a basis $X_1, \cdots, X_p$ of π over $\boldsymbol{H}$ so that if the X_i are the rows of a matrix, then the matrix is in row echelon form $(*)$. $\varphi(\pi)$ is spanned by the real vectors whose matrix is $(**)$, where $[a + bi + cj + dk]$ denotes the 4×4 matrix
$$\begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}.$$

$$(*)\qquad \begin{array}{c} \qquad\qquad\sigma_1 \qquad\qquad\quad \sigma_2 \qquad\qquad\quad \sigma_3 \qquad\qquad \cdots \\ \begin{bmatrix} 0\cdots 0 & 1 & q_{1,\sigma_1+1} & \cdots 0 & q_{1,\sigma_2+1} & \cdots 0 & q_{1,\sigma_3+1} & \cdots \\ 0\cdots 0 & 0 & 0 & \cdots 1 & q_{2,\sigma_2+1} & \cdots 0 & q_{2,\sigma_3+1} & \cdots \\ 0\cdots 0 & 0 & 0 & \cdots 0 & 0 & \cdots 1 & q_{3,\sigma_3+1} & \cdots \\ \vdots & & & \vdots & & \vdots & & \vdots \\ \cdots & & & \cdots & & \cdots & & \cdots \end{bmatrix} \end{array}.$$

$$(**)\qquad \begin{bmatrix} \begin{matrix} 0\cdots 0\,1\,0\,0\,0 \\ 0\cdots 0\,0\,1\,0\,0 \\ 0\cdots 0\,0\,0\,1\,0 \\ 0\cdots 0\,0\,0\,0\,1 \end{matrix} & \begin{bmatrix} \\ q_{1,\,\sigma_1+1} \\ \\ \end{bmatrix} & \begin{bmatrix} \\ q_{1,\,\sigma_1+2} \\ \\ \end{bmatrix} & \cdots & \begin{matrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{matrix} & \begin{bmatrix} \\ q_{1,\,\sigma_2+1} \\ \\ \end{bmatrix} & \cdots \\ \begin{matrix} 0\cdots\cdots\cdots \\ 0\cdots\cdots\cdots \\ 0\cdots\cdots\cdots \\ 0\cdots\cdots\cdots \end{matrix} & \begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} & \begin{bmatrix} \\ q_{2,\,\sigma_2+1} \\ \\ \end{bmatrix} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \cdots & \cdot & \cdot & \cdot & \cdot & & \cdots \end{bmatrix}.$$

If $\tau = (\tau_1, \cdots, \tau_{4p}) \in S$, let $v_\tau(\pi)$ be the τ-th Plücker determinant of $(**)$, that is, the determinant of the submatrix of $(**)$ consisting of the τ_1-th, τ_2-th, $\cdots$ columns. (Recall, these determinants give the embedding of $\boldsymbol{RG}(4p, 4q)$ in $\boldsymbol{RP}(N-1)$—the N-tuple $[\cdots, v_\tau(\pi), \cdots]$ satisfies the quadratic p-relations, and if any other choice of basis of π is made, then the resulting N-tuple from the Plücker determinants is a nonzero multiple of the one above.)

If π is spanned by p basis vectors $X_i = \boldsymbol{e}_{\sigma_i}$ (and the $q_{ij} = 0$ in $(*)$), by abuse of notation, we shall denote π by σ. $\varphi(\sigma) \in \boldsymbol{RG}(4p, 4q)$ is spanned by the $4\sigma_j - 4 + k$ axes ($j = 1, \cdots, p$; $k = 1, \cdots, 4$). $\varphi(\sigma) = \rho = (4\sigma_1 - 3, 4\sigma_1 - 2, \cdots) \in S$. $\pi = \sigma$ is critical for f since $v_\tau(\pi) = 0$ for $\tau \neq \rho$, and every N-tuple with all but one entry zero is critical for F.

Suppose π is not spanned by coordinate axes. Then there is a least integer i such that X_i is not a basis vector $\boldsymbol{e}_{\sigma_i}$, and for that choice of i there is a least integer $j > \sigma_i$ such that $q_{ij} \neq 0$. Let these i, j be fixed in the discussion below.

Define a path in $\boldsymbol{HG}(p, q)$ by $Y_k = X_k$ if $k \neq i$, and $Y_i = (1 + t)X_i - t\boldsymbol{e}_{\sigma_i}$, (the $\boldsymbol{e}_k$ are the standard basis vcetors of $\boldsymbol{H}^{p+q}$). We set $\pi(t)$ to be the plane spanned by the Y_k, and prove below that $(d/dt)(f \circ \pi(t))\,|_{t=0} \neq 0$, and hence π is not a critical plane since $\pi(0) = \pi$.

Denoting $v_\tau(\pi)$ by v_τ and $v_\tau(\pi(t))$ by w_τ, we compute that $df/dt = 2[(\Sigma c_\tau w_\tau w'_\tau)(\Sigma(w_\eta)^2) - (\Sigma c_\eta(w_\eta)^2)(\Sigma w_\tau w'_\tau)]/(\Sigma(w_\tau)^2)^2$. Hence we need to know $w'_\tau(0)$. By choice of i we have $w_\tau \equiv v_\tau = 0$ unless $\tau \in S_{\sigma_1\cdots\sigma_{i-1}\lambda}$ for some $\lambda \geq \sigma_i$. Thus we have five cases for possible nonzero terms:

(0) $\tau \in {}^0S_{\sigma_1\cdots\sigma_i}$, $w_\tau \equiv v_\tau$, $w'_\tau(0) = 0$;

(1) $\tau \in {}^1S_{\sigma_1\cdots\sigma_i}, w_\tau = (1+t)v_\tau, w'_\tau(0) = v_\tau$;
(2) $\tau \in {}^2S_{\sigma_1\cdots\sigma_i}, w_\tau = (1+t)^2v_\tau, w'_\tau(0) = 2v_\tau$;
(3) $\tau \in {}^3S_{\sigma_1\cdots\sigma_i}, w_\tau = (1+t)^3v_\tau, w'_\tau(0) = 3v_\tau$;
(4) $\tau \in S_{\sigma_1\cdots\sigma_{i-1},\lambda}, \lambda > \sigma_i, w_\tau = (1+t)^4v_\tau, w'_\tau(0) = 4v_\tau$.

Note that (0) < (1) < (2) < (3) < (4). If we let $\Sigma_{m,n}$ denote $\Sigma(c_\tau - c_\eta)v_\tau^2v_\eta^2$, the sum running over all $\tau \in (m)$, $\eta \in (n)$, then a simple calculation yields

$$(df/dt)(0) = 2\sum_{0\le s<r\le 4}(r-s)\Sigma_{r,s}/(\Sigma v_\tau^2)^2 .$$

Each term in the numerator is nonnegative, so $f'(0) \ge 0$. Now $\rho \in (0)$ and $v_\rho(\pi) = 1$. For $k = 1, \cdots, 4$, let $\rho_k \in (1)$ be the neighbor of ρ having $4j - k + 1$ instead of $4\sigma_i$. Then $v_{\rho_k} = \pm q_{ij}^k$, where $q_{ij} = q_{ij}^1 + q_{ij}^2\boldsymbol{i} + q_{ij}^3\boldsymbol{j} + q_{ij}^4\boldsymbol{k}$. Since $q_{ij} \ne 0$, we have that one of the $q_{ij}^k \ne 0$, the term $(c_{\rho_k} - c_\rho)v_{\rho_k}^2v_\rho^2 > 0$ in $\Sigma_{1,0}$, and $f'(0) > 0$.

Hence π is not critical, and the only critical points are those planes spanned by p of the coordinate axes.

4. The Hessian of f

Let $\sigma \in T$ correspond to a critical point $\rho = \varphi(\sigma)$. A neighborhood of σ is given by all matrices of the form (♯). There are pq arbitrary quaternionic entries in (♯), and hence $4pq$ real coordinates. Under φ, (♯) goes over to a similar display (♯♯) which we shall omit.

$$(\sharp)\qquad \begin{array}{c} \begin{matrix}\sigma_1 & \sigma_2 & \cdots & \sigma_p\end{matrix} \\ \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ * \,.\, * & . & * \cdots * & . \, * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} \end{array} .$$

Consider the function v_τ on (♯♯): it is a homogeneous polynomial in the real coordinates. $v_\rho = 1$, while v_τ is linear if and only if τ is a neighbor of ρ. Let $\{j_1, \cdots, j_q\}$ be the set of indices complementary to the elements of σ arranged in increasing order, and $\rho_{a,m,b,n}$ be the neighbor of ρ where $\rho_{\sigma_a,m}$ is replaced by $4j_b + n - 4$, $(m, n \le 4)$. If ‡ denotes the product on the Klein 4-group on the symbols $1, \cdots, 4$, with 1 as identity, then it is easy to compute that $(v_{\rho_{a,m,b,n}})^2 = (q_{ab}^{m\ddagger n})^2$.

Now $f = g/h$ where g and h are polynomials with no linear terms. If x and y represent two coordinates of the form q_{ab}^s and q_{cd}^r, then one has $f_{xy}(0) = [g_{xy}(0)h(0) - g(0)h_{xy}(0)]/(h(0))^2$. Furthermore, $g(0) = c_\rho$, $h(0) = 1$, and the second order terms of g and h are squares of coordinates by the previous paragraph. Hence $f_{xy}(0) = 0$ for $x \ne y$, and $f_{xx}(0) = 2\Sigma(c_{\rho_{a,m,b,n}} - c_\rho)$ where the sum runs over all $m\ddagger n = s$. Note that the order of $\rho_{a,m,b,n}$ and ρ does not depend on m, n so that $f_{xx}(0) \neq 0$ and σ is a nondegenerate critical point.

The index of f at σ is the number of q^s_{ab} such that $\rho_{a,m,b,n} < \rho$ for all $m \ddagger n = s$. This is the same as four times the number of pairs (a, b) such that $j_b < \sigma_a$, which is the same as four times the number of neighbors of σ which are less than σ. Hence the index $\lambda_\sigma = 4d(\sigma)$. Since the indices are all even, the Morse inequalities are equalities and $\boldsymbol{H}G(p, q)$ has torsion-free homology. Its Poincaré polynomial for any field of coefficients is thus given by $P(\boldsymbol{H}G(p, q); t) = \Sigma t^{4d(\sigma)}$. Hence Theorem 1 is proved.

5. The case of critical manifolds

By changing F so that certain of the $c_\tau = 1$ and the rest $= 0$, we can arrange it so that there are two submanifolds of critical points—one consists of all p-planes containing the basis vector $\boldsymbol{e}_1$, the other all p-planes orthgonal to $\boldsymbol{e}_1$. These critical submanifolds are nondegenerate in the sense of Bott [1]. This same alteration can also be carried out with Hangan's function in the real and complex cases.

The Morse-Bott inequalities [2, p. 323] and [3, p. 44] are equalities in the cases $\boldsymbol{C}G$ and $\boldsymbol{H}G$ by induction since the indices are even. In the case of $\boldsymbol{R}G$ one applies a technique due to Frankel [3], namely, to combine the Morse-Bott inequalities with opposing inequalities derived by Floyd in the study of fixed points of involutions, to prove equality as long as the coefficient field is $\boldsymbol{Z}_2$. Thus, following Bott, one has

Theorem 2. *$\boldsymbol{K}G(p, q)$ has the same homotopy type as $\boldsymbol{K}G(p-1, q)$ with a dp-dimensional vector bundle over $\boldsymbol{K}G(p, q-1)$ attached, $(d = \dim_{\boldsymbol{R}} \boldsymbol{K})$. $P(\boldsymbol{K}G(p, q); t) = P(\boldsymbol{K}G(p-1, q); t) + t^{dp}P(\boldsymbol{K}G(p, q-1); t)$, for $\boldsymbol{Z}_2$ coefficients if $\boldsymbol{K} = \boldsymbol{R}$, and for any field of coefficients if $\boldsymbol{K} = \boldsymbol{C}, \boldsymbol{H}$.*

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