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A REMARK ON THE GROUP OF AUTOMORPHISMS OF A FOLIATION HAVING A DENSE LEAF

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Introduction

When $\mathfrak{f}$ is a C^∞ foliation of a compact manifold, the connected component of the identity of the group $D_f(M)$ of automorphisms of the foliation admits the structure of an infinite dimensional manifold (see [3]). The purpose of this note is to prove the following theorem: Let $D_{I\mathfrak{f}}(M)$ be the connected component of the identity of the closed normal subgroup of $D_\mathfrak{f}(M)$ which leaves the leaves of f fixed, that is, $f(L) = L$ for each $f \in D_{I\mathfrak{f}}(M)$ and each $L \in \mathfrak{f}$. If there exists a finite number of leaves $L_1, \cdots, L_n \in \mathfrak{f}$ such that $L_1 \cup \cdots \cup L_n$ is dense in M, then $D_\mathfrak{f}(M)/D_{I\mathfrak{f}}(M)$ is a finite dimensional Lie group.

At the end of this note we consider two examples, namely, an irrational linear flow on the two dimensional torus, and the case of an Anosov flow. The above theorem can be extended to C^k, $k \geq 2$, foliations as we shall point out in a remark.

While preparing this note the author had several useful conversations with M. Hirsch and S. Smale. Smale suggested applying the results to Anosov flows.

Preliminaries

Let M be a compact smooth (i.e., C^∞) manifold of dimension n, and $\mathfrak{f}$ a smooth foliation of M by leaves of dimension k. $\mathfrak{f}$ determines canonically an integrable G-structure on M, where $G = \{(a_{ij})\}$ is the subgroup of $GL(n)$ given by $a_{ij} = 0$ when $i \leq k < j$. The reduction of the frame bundle on M to G is given by choosing the first k vectors in each frame at $x \in M$ to be tangent to the leaf through x. An automorphism of this G-structure is precisely a diffeomorphism of M which maps leaves of $\mathfrak{f}$ onto (in general) other leaves of $\mathfrak{f}$. Thus in veiw of [3] the group $D_\mathfrak{f}(M)$ of diffeomorphisms which maps leaves onto leaves admits the structure of a Lie group modelled on its Lie algebra $\mathscr{D}_\mathfrak{f}(M)$.

In the terminology of [3], $D_\mathfrak{f}(M)$ is a first order classical subgroup of Diff (M), that is, there exist a smooth locally trivial bundle $E \to M$ and a bundle map $f\colon \mu_1 \to E$, where μ_1 is the fiber space of invertible 1-jets of smooth endomor-

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phisms of M, so that setting $D = f_* \circ j_1$, where j_1: Diff $(M) \to \mu_1$ is the canonical map, we have

i) $D^{-1}(D(e)) = D_f(M)$, ii) $D(gh) = D(h)$ for $g \in D_f(M)$.

In our case E is the bundle weakly associated to the principal bundle $\mu_1 \to M$ with fiber $GL(n)/G_k$, and f: $\mu_1 \to E$ is the canonical map.

Let $\Gamma(M)$ be the Lie algebra of smooth vector fields on M, and $\mathscr{D}_f(M) = \{g \in \Gamma(M) | T_e D(g) = 0\}$ be the Lie algebra of $D_f(M)$. $D_f(M)$ is characterized by the fact that exp $(tX) \in D_f(M)$ for all $t \in R$, where $\exp_{\mathscr{L}}$: $|(M) \to M$ is the Lie exponential.

The modelling of $D_f(M)$ on $\mathscr{D}_f(M)$ is given as follows: consider a connection on the principal bundle $P \to M$ of frames whose first k vectors are tangent to the leaf through the point in question, and let Exp_G be the exponential map associated to this connection. Then $\psi(X) = \mathrm{Exp}_G \circ X$ gives a local diffeomorphism of a neighborhood of 0 in $\mathscr{D}_f(M)$ onto a neighborhood of the identity in $D_f(M)$.

1. Integrable subalgebras of $D(M)$

Definition. A sequence of C^∞ Banach manifolds $\{X_r\}$ is called an inverse limit Banach system (or an I.L.B. system) when

(i) $X_{r+1} \subset X_r$,

(ii) there is a Banach chain $\{B_r\}$ so that for each $x \in X_\infty$ there exist charts at x

$$Y_r\colon U_r \to X\ , \qquad \text{with open } U_r \subset B_r\ .$$

An I.L.B. system is called an inverse limit Banach system of groups (or an I.L.B.G. system) when multiplication and inversion define smooth maps in the category of Banach chains. In the category of Banach chains (see [2]) a mapping f: $U \to B^2_\infty$ with open $U \subset B^1_\infty$ is said to be C^r when there exists a strictly monotone increasing sequence of integers k such that f can be extended to C^r mappings f_k: $U_k \to B^2_{\lambda(k)}$ where $U_k \subset B^1_k$ is open and $U = B^1_\infty \cap U_k$. C^r in the category of Banach chains is a stronger notion than that in the category of nuclear spaces (see [3]).

It is known (see [2]) that Diff (M) is an I.L.B.G.

Definition. A subalgebra $\mathscr{A}$ of $D(M)$ is said to be *integrable* when there exists an I.L.B. subgroup $\{G_s\}$ of $D(M)$ such that $T_e G_s$ is isomorphic to the closure of $\mathscr{A}$ in $\mathscr{D}_s(M)$ = the Lie algebra of C^s vector fields on M.

By the methods of [2] one can prove

Theorem. *If $\mathscr{A}$ is an integrable subalgebra of $\Gamma(M)$, and $\mathscr{B}$ is a finite or cofinite dimensional subalgebra of $\mathscr{A}$, then $\mathscr{B}$ is integrable. Further, if G is the connected subgroup of D(M) having $\mathscr{A}$ as its Lie algebra, and G is locally modelled on $\mathscr{A}$, then the subgroup corresponding to $\mathscr{B}$ is locally modellable on $\mathscr{B}$.*

Let $G \subset \mathrm{Diff}\,(M)$ be a classical group determined by the canonical map $f\colon \mu_1 \to \mu_1/H$ where H is a subgroup of $GL(n)$ (see page 602). In this case its Lie algebra is the space of sections of a sheaf $T_eD^{-1}(0)$.

Definition. A totally bounded element $\alpha \in B$ of a Banach chain is an element so that there exists a real number c with $\|\alpha\|_s < c$ for all s.

Theorem. *Let H be the group of automorphisms of a G-structure on M, and suppose that H is an* I.L.B.G. *with Lie algebra $\{\mathscr{H}_s\}$. Suppose that there exists a covering $\{U_i\}$ of M such that $\mathscr{H}_\infty(U_i)$ contains a set of totally bounded elements generating $\mathscr{H}_\infty(U_i)$. Let $\mathscr{K}$ be a subsheaf of $\mathscr{H}_\infty$ which is finite codimensional and suppose $\mathscr{K}(U_i)$ also contains a set of totally bounded elements generating $\mathscr{K}(U_i)$. If $\mathscr{K}$ is an ideal of the sheaf $\mathscr{H}_\infty$, then the group K having $\mathscr{K}$ as its Lie algebra is normal in H.*

Proof. For $x \in \mathscr{H}$ and $Y \in \mathscr{K}$ set $\sigma = \exp_{\mathscr{L}}(X)$ and $\tau = \exp_{\mathscr{L}}(Y)$. Putting $X \cdot Y = [X, Y]$ we have that the Taylor series of $\tau_\sigma(g) = \sigma g \sigma^{-1}$ is given by

$$(*) \qquad \textstyle\sum_{n\geq 0} (1/n!) X^n \cdot Y \qquad \text{at } \tau\ .$$

As $\|X^n \cdot Y\|_k \leq (k+1)!\,2^n \|X\|_{k+1}^n \|Y\|_{k+1}$, it follows that for X and Y totally bounded with X sufficiently small $(*)$ converges, and thus $\sigma\tau\sigma^{-1} \in K$. Since $\exp_{\mathscr{L}}\colon \mathscr{H} \to H$ is continuous in $\mathscr{H}$, and $\mathscr{K}$ and the totally bounded elements are locally dense, we have $\exp_{\mathscr{L}}(X)\exp_{\mathscr{L}}(Y)\exp_{\mathscr{L}}(X)^{-1} \in K$ for all $X \in \mathscr{H}$, $Y \in \mathscr{K}$.

$f\colon \mathscr{H} \xrightarrow{\exp} \mathscr{L}H \xrightarrow{\pi} H/K$ defines a smooth map from $\mathscr{H}$ to the finite dimensional manifold H/K so that $T_0 f$ is subjective. It follows that $f(\mathscr{H})$ contains a neighborhood of $f(0)$, (see [2]).

H is generated by K, and $\exp_{\mathscr{L}}(\mathscr{H})$ which suffices to establish K is normal in H.

2. The group of automorphisms of a foliation

Proposition. *Let $\mathfrak{f}$ be a foliation determined by a Lie algebra $\mathfrak{X} \subset \mathscr{D}_{\mathfrak{f}}(M)$. If $\mathfrak{f}$ has a finite number of leaves $L_1, \cdots, L_l$ such that $\overline{L_1 \cup \cdots \cup L_l} = M$, then $\mathfrak{X}$ is of finite codimension in $\mathscr{D}_{\mathfrak{f}}(M) \leq l(\dim(M) - k)$.*

Proof. Let $a = \{U_i, f_i\}$ be a Reeb atlas for the foliation such that $x_{R+1} =$ constant, $\cdots, x_n =$ constant determine the local leaves for each chart. With respect to the Reeb atlas $\mathscr{D}_f(M)$ may be represented locally by $\sum_{i=1}^n f_i \partial/\partial x_i$, where $\partial f_i/\partial x_j = 0$ for $i > k \geq j$, since $\exp(tX) \in D_{\mathfrak{f}}(M)$ characterizes $\mathscr{D}_{\mathfrak{f}}(M)$. Let $\mathfrak{f}M$ be the subbundle of $TM \to M$ determined by the foliation, that is, $\mathfrak{f}M_x$ is the space of vectors tangent to the leaf of $\mathfrak{f}$ through x. $\mathfrak{X}$ is the space of sections of $\mathfrak{f}M \to M$. By the canonical projection of TM on TM/fM, $\mathscr{D}_{\mathfrak{f}}(M)$ defines a subspace S of $\Gamma(TM/M)$. Setting $\Sigma_t = \tau_L^*(S)$ we see that if $X \in \Sigma_k$ is such that $X(x) = 0$, $x \in L_k$, then there exists a neighborhood V of x in L_k such that $X|V \equiv 0$. L_k being connected this implies $\dim(\Sigma_t) \leq \dim((\Sigma_t)_x) \leq n - k$. Since $\xi, \eta \in S$ are equal if and only if $\tau^*_{L_1\cup\cdots\cup L_l}(\xi) = \tau^*_{L_1\cup\cdots\cup L_l}(\eta)$, it follows that $\prod_{j=1}^l \tau_{L_j}^*\colon S \to \prod_{i=1}^l \Sigma_j$ is a monomorphism.

Corollary to the proof. *Under the hypotheses of the above theorem $\mathfrak{X}$ is an ideal in $\mathscr{D}_{\mathfrak{f}}(M)$ of finite codimension.*

Proposition. *There exists a normal subgroup $H \subset D_{\mathfrak{f}}(M)$ having $\mathfrak{X}$ as its Lie algebra, and further $H \subset D_{I\mathfrak{f}}(M)$.*

Proof. The first part of the proposition follows from § 1, since in terms of a Reeb chart both $\mathfrak{X}$ and $\mathscr{D}_{\mathfrak{f}}(M)$ contain locally the dense subspace generated by the $\Sigma f_i(x)\partial/\partial x_i$ where $f_i(x)$ are polynomials in $x_1, \cdots, x_n$. The second part follows from the fact that $\exp_G$ determines a local modelling at the identity of H on $\mathfrak{X}$ as the leaves of $\mathfrak{f}$ are totally geodesic with respect to a G-connection, since parallel translation takes tangent vectors of a leaf to tangent vectors of another leaf.

It follows that

Proposition. *If $\mathfrak{X}$ is finite codimensional in $\mathscr{D}_{\mathfrak{f}}(M)$, then H is the connected component of the identity of $D_{I\mathfrak{f}}(M)$.*

Proof. Let K be the connected component of the identity of $D_{I\mathfrak{f}}(M)/K$ which is the quotient of the finite dimensional Lie group $D_{\mathfrak{f}}(M)/H$ by the closed subgroup K/H. Then the canonical map $D_{\mathfrak{f}}(M) \to D_{\mathfrak{f}}(M)/K$ is a submersion onto a finite dimensional manifold, so that K is a Lie subgroup of $\mathscr{D}_{\mathfrak{f}}(M)$. Let $\mathscr{S}: I \to K$ be a smooth arc such that $\mathscr{S}(0) =$ identity. Then $\mathscr{S}(t)(x)$ is on the leaf through x for every x, and $\mathscr{S}'(0) \in \mathfrak{X}$. Hence $H = K$.

In summary, we have

Theorem. *Let $\mathfrak{f}$ be a foliation such that the sections of its tangent bundle is of finite codimension in the Lie algebra $\mathfrak{X} \subset \Gamma(M)$ determining the foliation. Then the connected component of the identity of the group $\mathscr{D}_{\mathfrak{f}}(M)$ leaving the leaves fixed is a normal Lie subgroup of $\mathscr{D}_{\mathfrak{f}}(M)$ having $\mathfrak{X}$ as its Lie algebra.*

Corollary. *If $\mathfrak{f}$ has a finite number of leaves $L_1, \cdots, L_l$ such that $\overline{L_1 \cup \cdots \cup L_l} = M$, then $D_{\mathfrak{f}}(M)/D_{I\mathfrak{f}}(M)$ is a Lie group of dimension $\leq l(n-k)$.*

Example 1. On a two-dimensional torus T^2:

$$x(\omega, \mathscr{S}) = ((R + r)\cos\omega)\cos\mathscr{S} ,$$
$$y(\omega, \mathscr{S}) = ((R + r)\cos\omega)\sin\mathscr{S} ,$$
$$z(\omega, \mathscr{S}) = r\sin\omega ,$$

consider an irrational linear flow given by

$$d\omega/dt = a , \qquad d\mathscr{S}/dt = b ,$$

where a and b are linearly independent over Q:

$$f_c(x(\omega, \mathscr{S}), y(\omega, \mathscr{S}), z(\omega, \mathscr{S})) = (x(\omega + c, \mathscr{S}), y(\omega + c, \mathscr{S}), z(\omega + c, \mathscr{S})) ,$$
$$f^c(x(\omega, \mathscr{S}), y(\omega, \mathscr{S}), z(\omega, \mathscr{S})) = (x(\omega, \mathscr{S} + c), y(\omega, \mathscr{S} + c), z(\omega, \mathscr{S} + c)) .$$

$c \in [0, 2\pi]$ determines a subgroup of $\mathscr{D}_{\mathfrak{f}}(M)$ isomorphic to $S^1 \times S^1$ which is transitive on the leaves of $\mathfrak{f}$. Thus $D_{\mathfrak{f}}(M)/D_{I\mathfrak{f}}(M) \approx S^1$ being of dimension ≤ 1.

Example 2. Let $\mathscr{S}_t : M \to M$ be an Anosov flow with an integral invariant. Then $\mathscr{S}_t$ has a dense leaf, and the periodic orbits are dense and countable. Thus $D_{\mathfrak{f}}(M) = D_{I\mathfrak{f}}(M)$; for, if $\dim (D_{\mathfrak{f}}(M)/D_{I\mathfrak{f}}(M)) \geq 1$, then choose $g \in D_{\mathfrak{f}}(M)$ near the identity, let γ be a closed orbit with $x \in \gamma$, and let g_t be a smooth arc in $D_{\mathfrak{f}}(M)$ such that $g_1 = g$ and $g_0 =$ identity. Thus $g_t(x)$ is on a closed orbit for each t. From the fact that there is only a countable number of closed orbits it follows that $g_t(x) \in \gamma$ for each t, and therefore that g leaves the closed orbits fixed. Since g is closed to the identity and leaves the closed orbits fixed, using a Reeb chart we may conclude that g leaves all orbits fixed. Hence in this case we have $D_{\mathfrak{f}}(M) = \mathfrak{X}$.

Remark. When M is a C^s manifold, $2 \leq s < \infty$, using the notion of a Banach pseudo—Lie group of [2, p. 295] one is able to establish analogous results.

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