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BOUNDED SETS AND FINSLER STRUCTURES FOR MANIFOLDS OF MAPS

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Abstract infinite dimensional manifolds modelled on Banach spaces lack much of the topological structure of both finite dimensional manifolds and their linear Banach space models. In this paper we show that certain manifolds of maps between finite dimensional manifolds, or more generally manifolds of sections of a finite dimensional fiber bundle, have an additional natural structure of sets which we call "intrinsically bounded" which have many of the properties of bounded sets in the linear model. Theorem 1 shows that these sets can be characterized in several different ways. Our results are specifically stated for the Sobolev manifolds $L_k^p(E)$ where E is a fiber bundle over the compact manifold M of dimension less than pk. We also construct a canonical Finsler structure for $L_k^p(E)$ from geometrical structure on E, and find Finsler structures which have intrinsically bounded sets as their bounded sets. The discussion of Finsler structures will be helpful in freeing the use of condition (C) of Palais and Smale in the calculus of variations from the unnaturally arbitrary choice of Finsler structure on the manifolds of maps.

Many of the definitions and formal statements of theorems are due to R. S. Palais. Construction and properties of a weak topology have been obtained by D. Graff. U. Koschorke has developed a more abstract theory for Banach manifolds with specified atlases [4]. J. Dowling has shown that minimizing geodesics exist for the Finsler structures discussed in the second section [1]. A good many of the results of this section were obtained at the same time by H. Eliasson [3] but without the construction of bounded sets. Eliasson also gives the construction in Appendix II in a different style [2].

All manifolds and maps are C^∞ unless otherwise stated. The results are given for sections of a fiber bundle E over a compact base manifold M, with or without bundary, where the fibers are finite dimensional manifolds without boundary. The reader is periodically reminded that $E = M \times N$ is an important case where the sections are merely maps between M and N. Most of the necessary inequalities are of the same general type, which is discussed in Appendix I, and they are therefore not explained in the paper. The construction of the covariant derivatives which are used has been left to Appendix II.

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0. The manifolds

Let $\mathscr{M}(\xi)$ denote a Banach space of functions or distributions for every C^∞ bundle over a compact C^∞ manifold M. Palais has shown that if $\mathscr{M}$ satisfies certain axioms (B§2 and B§5), then $\mathscr{M}(E)$ is a Banach manifold for every C^∞ fiber bundle E over M [5]. As far as the author knows, every Banach space of functions, which depends on a C^∞ structure and is contained in the space of continuous functions, satisfies these axioms, and the results of § 1 are good for spaces which are completely continuous in the continuous functions; no general proofs have been found. We shall use exclusively the spaces $\mathscr{M}(E) = L_k^p(E)$ where $pk > \dim M = n$. $L_k^p(M)$ are the functions whose derivatives up through order k are p integrable. Although a clear and detailed discussion of these manifolds is to be bound in Palais [5], we will find the following information useful.

(0.1) If $E \subseteq \xi$ for a fiber bundle E imbedded in a fiber preserving manner in the vector bundle ξ over M, then $L_k^p(E) = C^\circ(E) \cap L_k^p(\xi)$. For $E = M \times N$ where N is a finite dimensional manifold without boundary, by the Whitney imbedding theorem we can assume there exists an imbedding $N \subseteq R^m$, which gives

$$L_k^p(M, N) = L_k^p(M, R^n) \cap C^\circ(M, N) \ .$$

Since, by patching together such imbeddings, we can assume $E \subseteq M \times R^n$, we can use the above identity as a definition.

(0.2) Let TFE be the tangent bundle along the fiber over E. If $\pi: E \to M$, then TFE is the bundle which is the kernel of $d\pi$. $TFE_y = T_y(E_{\pi(y)})$. If $s: M \to E$ is a section of E, then s^*TFE denotes the bundle over M which is the pull-back by s of TFE. $(s^*TFE)_x = TFE_{s(x)} = T_{s(x)}(E_x)$. The tangent bundle to $L_k^p(E)$ at s is naturally $L_k^p(s^*TFE)$. If the definition of (0.1) is used, the tangent bundle to $L_k^p(E) \subseteq L^p(\xi)$ at the section s of E is identified with $\{u \in L_k^p(\xi) / u(x) \in T_{s(x)}(E_x)\}$.

(0.3) The canonical neighborhood structure of $L_k^p(E)$ at a section $s \in L_k^p(E)$ consists of $L_k^p(\eta)$ for a vector bundle $\eta \subseteq E$, which is imbedded as an open subbundle of E. $s(x) \in \eta_x \subseteq E_x$. Such a bundle η is called a vector bundle neighborhood of s in E.

(0.4) If $f: E \to F$ is a C^∞ fiber preserving map from E to the fiber bundle F over M, then composition $\circ f: L_k^p(E) \to L_k^p(F)$ is a C^∞ map of Banach manifolds. If E and F actually have vector bundle structures as ξ and η, we have $\|f \circ s\|_{L_k^p(\eta)} \leq C(f, \|s\|_{L_k^p(\xi)})$. This inequality is of the type of Appendix I.

In the above discussion, M may have a boundary. We shall be briefly interested in $L_{k,\partial f}^p(E) \subseteq L_k^p(E)$ which is the closed subbundle agreeing with the section f of E on the boundary through some specified number of derivatives.

For completeness we give the definition of Finsler structure and metric.

(0.5) If B is a bundle with fiber a Banach space b over the space X, then a Finsler structure for B is a real-valued function $\| \ \|$ such that:

(i) $\| \ \|_x$ is a norm on $B_x \simeq b$ for all $x \in X$,

(ii) for every local trivialization $\mathcal{O} \times b \subseteq B$, all $x \in \mathcal{O}$ and all $K > 1$, there exists a neighborhood $\mathcal{U}$ of x such that $(1/K)\|u\|_y \leq \|u\|_x \leq K\|u\|_y$ for $y \in \mathcal{U}$ and $u \in b$.

(0.6) If $\| \ \|$ is a Finsler structure on the tangent bundle to a smooth Banach manifold X (we shall roughly refer to this as a Finsler structure for X), then the length of a differentiable curve σ: $[0, 1] \to X$ is given by $l(\sigma) = \int_0^1 \|\sigma'(t)\|_{\sigma(t)} dt$ where $\delta(x, y) =$ infimum of $l(\sigma)$ such that $\sigma(0) = x$, $\sigma(1) = y$. This distance gives a metric on X, and the Finsler structure is said to be complete if this metric is complete.

1. Intrinsically bounded sets

(1.1) **Definition.** A set $S \subseteq L_k^p(E)$ (E is a C^∞ fiber bundle over the compact manifold M, $pk > n = \dim M$) is intrinsically bounded if S is relatively compact in $C^\circ(E)$, and if any subset of S, which is bounded in $C^\circ(\eta)$ where $\eta \subseteq E$ is an open vector bundle neighborhood in E, is bounded in $L_k^p(\eta)$. Sets in $L_{k,\delta f}^p(E)$ are intrinsically bounded if they are intrinsically bounded as subsets of $L_k^p(E)$. The results for boundary value spaces are not stated, because they follow directly from the results for no boundary values.

In the case where $E = \eta$ is a vector bundle over M, the intrinsically bounded sets are clearly the bounded sets. The closure of an intrinsically bounded set is intrinsically bounded, a subset of an intrinsically bounded set is intrinsically bounded, and the finite union of intrinsically bounded sets is intrinsically bounded. It is only slightly more difficult to show that a set which is relatively compact in $L_k^p(E)$ is intrinsically bounded.

Theorem 1. *S intrinsically bounded (property (1.1)) is equivalent to each of the following*:

(1.2) *S is relatively compact in $C^\circ(E)$, and every element $s \in C^\circ(E)$ lies in a vector bundle neighborhood $\eta \subseteq E$ in which all subsets of S, bounded in $C^\circ(\eta)$, are bounded in $L_k^p(\eta)$.*

(1.3) *There exists a finite number of open vector bundle neighborhoods $\{\xi_i\}$, $1 \leq i \leq R$, such that S is the finite union of sets bounded in $L_k^p(\xi_i)$.*

(1.4) *$E \subseteq F$ is a closed fiber-preserving imbedding, and $S \subseteq L_k^p(E)$ is intrinsically bounded as a subset of $L_k^p(F)$.*

(1.5) *S is a bounded set in one of the special Finsler metrics on $L_k^p(E)$.*

(See Theorems 2a and 2b for clarification and proof of this last equivalence, which will not be discussed in this section.)

(1.6) **Corollary 1.** *Suppose $E \subseteq \xi$ for a vector bundle ξ over M. Then*

$S \subseteq L^p_k(E)$ is intrinsically bounded if and only if $S \subseteq L^p_k(\xi)$ is contained in a ball of finite radius.

This is an application of (1.4).

(1.7) **Corollary 2.** *If S is intrinsically bounded, then the elements $s \in S$ represent only a finite number of homotopy classes of sections of E.*

This is due to (1.3).

We now prove the theorem in a series of steps.

(1.1) $\Rightarrow$ (1.2). This follows from the fact that every continuous section is contained in an open vector bundle neighborhood.

(1.2) $\Rightarrow$ (1.3). For each $s \in \bar{S}$ (closure in $C^\circ(E)$), let $\xi(s)$ be a vector bundle neighborhood, containing the image of s, of the type postulated in (1.2). The infinite union over $s \in \bar{S}$ of open bounded sets in $C^\circ(\xi(s))$ covers $\bar{S}$, so by the compactness of $\bar{S}$ in $C^\circ(E)$ a finite number of sets C° bounded in $C^\circ(\xi(s_i))$, $0 \leq i \leq N$, covers $\bar{S}$. But each subset of S which is bounded in $C^\circ(\xi(s_i))$ is bounded in $L^p_k(\xi(s_i))$ by assumption. Therefore S is contained in the finite union of sets bounded in $L^p_k(\xi_i)$, $1 \leq i \leq N$.

(1.3) $\Rightarrow$ (1.1). A set bounded in $L^p_k(\xi_i)$ is relatively compact in $C^\circ(\xi_i)$ and therefore in $C^\circ(E)$ by the Sobolev imbedding theorem. So, if S satisfies (1.3), then S is relatively compact in $C^\circ(E)$.

Now suppose $S' \subseteq L^p_k(\eta)$ for a vector bundle neighborhood $\eta \subseteq E$, and S' is bounded in $C^\circ(\eta)$ but not in $L^p_k(\eta)$. If $S' \subseteq S$ and S satisfies (1.3), we may extract subsequence $\{s_i\} \subseteq S'$ such that $s_i \in L^p_k(\xi)$, $\|s_i\|_{L^p_k(\xi)} \leq C$, $\|s_i\|_{C^\circ(\eta)} \leq C$ but $\lim_{i\to\infty} \|s_i\|_{L^p_k(\eta)} = \infty$. This can be shown impossible using the type of inequality of Appendix I.

To prove (1.4) is equivalent to the other properties we need the following lemma.

(1.8) **Lemma.** *Let E and F be two fiber bundles over M, and $G\colon E \to F$ a C^∞ fiber preserving map. Then composition with G maps $L^p_k(E)$ smoothly into $L^p_k(F)$, and intrinsically bounded sets in $L^p_k(E)$ to intrinsically bounded sets in $L^p_k(F)$.*

For example, if $E = M \times N$, $F = M \times N'$ and g is a C^∞ map from N to N', this lemma asserts that composition with g, which is a C^∞ map from $L^p_k(M, N)$ to $L^p_k(M, N')$, takes intrinsically bounded sets to intrinsically bounded sets.

Proof. Due to the equivalence of (1.1) with (1.3) we may assume that E is a vector bundle ξ, and S is a bounded set in $L^p_k(\xi)$. Since composition with g is a smooth map from $C^\circ(\xi)$ to $C^\circ(F)$, the image of the relatively compact set S in $C^\circ(\xi)$ is relatively compact in $C^\circ(F)$. We need now show that if $\eta \subseteq F$ is any vector bundle, then a subset of $g(S)$ which is bounded in $C^\circ(\eta)$ is bounded in $L^p_k(\eta)$. This can be done using inequalities from Appendix I again.

(1.4) $\Leftrightarrow$ (1.1). Since the injection $i\colon E \to F$ is a C^∞ bundle map, sets which are intrinsically bounded in $L^p_k(E)$ are intrinsically bounded in $L^p_k(F)$ according to Lemma 1.8. On the other hand, since $E \subseteq F$ is closed, there exists a C^∞

fiber preserving projection $P\colon \mathcal{O} \to E$ of an open bundle tubular neighborhood $\mathcal{O} \subseteq F$ onto E. Composition with P sends intrinsically bounded sets in $L_k^p(\mathcal{O})$ to intrinsically bounded sets in $L_k^p(E)$, so the proof rests on the triviality of demonstrating that intrinsically bounded sets of $L_k^p(F)$ which are subsets of the closed set $L_k^p(E)$ are intrinsically bounded in the open neighborhood of $L_k^p(E)$ which is $L_k^p(\mathcal{O})$.

(1.9) **Proposition.** *Suppose $k/n - 1/p \geq r/n - 1/q > 0$, $k \geq r$ and $p \geq 1$. An intrinsically bounded set in $L_k^p(E)$ is intrinsically bounded in $L_k^q(E)$ and relatively compact, if $k > r$ and strict inequality holds in the inequality.*

Due to (1.3), this follows directly from the Sobolev imbedding theorems.

2. Finsler structures[1]

There are two more or less natural ways of getting a Finsler structure for $L_k^p(E)$. We may assume $E \subseteq \xi$, where ξ is a vector bundle over M, so $L_k^p(E) \subseteq L_k^p(\xi)$. Since any norm on $L_k^p(\xi)$ is a Finsler structure on $L_k^p(\xi)$ (we call this the flat Finsler structure), this flat metric induces a Finsler structure on $L_k^p(E)$ by restriction:

$$T_s(L_k^p(E)) \subseteq T_s(L_k^p(\xi)) = L_k^p(\xi) \text{ which has a norm.} \tag{2.1}$$

In the second method we assume that tangent bundle along the fiber of E the has a metric and compatible covariant derivative, and that M is a Riemannian manifold. (If $E = M \times N$, we need only a metric on M and N). The tangent space to $s \in L_k^p(E)$ consists of certain sections of the bundle s^*TFE, and the covariant derivative on TFE may be pulled back to s^*TFE. The covariant differentiation on $T^*(M)$ allows us to iterate the covariant derivative $s^*\nabla$ on s^*TFE to get a new section $(s^*\nabla)^j u$ of $\otimes_j T^*(M) \otimes s^*TFE$ for a section u of s^*TFE. These bundles all have induced norms, so we define easily for smooth s and u

$$\|u\|_s = \left(\sum_{j=0}^{k} \int_M |(s^*\nabla)^j u|^p d\mu\right)^{1/p}. \tag{2.2}$$

In appendix II this construction is carried out in more detail, and it is shown that this metric is really defined for all $s \in L_k^p(E)$ and $u \in T_s(L_k^p(E))$, and is a Finsler structure.

(2.3) **Definition.** A set $S \subseteq L_k^p(E)$ is bounded in a Finsler metric if $S = \bigcup_{i=1}^{N} S_i$ such that $\delta(x, y) < K$ for all x and y in S_i. The metric δ is, of course, the metric derived from the Finsler structure (0.6).

Theorem 2a. *If $E \subseteq \xi$ is a closed imbedding, then the Finsler structure (2.1) on $L_k^p(E)$ induced from a flat Finsler structure on $L_k^p(\xi)$*

[1] A more general treatment is given in [7].

(2.4) *is complete, and*
(2.5) *has as its bouded sets exactly the intrinsically bounded sets.*

Theorem 2b. *If M is a Riemannian manifold and a metric and compatible covariant derivative for TFE are given in which the fibers of E are complete ($TFE/E_x = T(E_x)$), then the canonical Finsler metric* (2.2) *induced by this geometry*
(2.6) *has exactly the intrinsically bounded sets as bounded sets, and*
(2.7) *is complete.*

Before we proceed with the proof, one should note the following special case of Theorem 2b. If M and N are Riemannian manifolds and N is complete, then there exists a canonical complete Finsler metric on $L_k^p(M, N)$.

The proof of (2.4) of Theorem 2a is given by Palais [5]. Since distance is shorter in the larger space $L_k^p(\xi)$ than in $L_k^p(E)$, sets bounded in the Finsler metric in $L_k^p(E)$ are bounded in $L_k^p(\xi)$ and hence intrinsically bounded by Corollary 1.6. This proves one direction of (2.5) of Theorem 2a. To go in the other direction, we use (1.3) and assume S is bounded in $L_k^p(\eta)$, $\eta \subseteq E \subseteq \xi$. Let $g\colon \eta \subseteq \xi$ be the fiber preserving imbedding, so $dg_{s(x)}\colon T_{s(x)}(\eta_x) \to \xi_x$. Since all metrics in $L_k^p(\xi)$ are equivalent, we choose the metric

$$\|f\|_{L_k^p(\xi)} = \left(\int_M \sum_{j=0}^k |\nabla_\xi^j f|_\xi^p \, d\mu\right)^{1/p} ,$$

where ∇_ξ, $|\ |_\xi$ and $d\mu$ are some choice of covariant derivative, metric on ξ and measure on M. The length of the linear path in $L_k^p(\eta)$ from 0 to s in the induced metric is $\int_0^s \|dg_{ts}\cdot s\|_{L_k^p(\xi)} dt$. But $\|dg_{ts} s\|_{L_k^p(\xi)}$ is equal to $\left(\int_M \sum_{j=0}^k |\nabla_\xi^j (dg_{ts}\cdot s)|_\xi^p \, d\mu\right)^{1/p}$ $\le C(\|s\|_{L_k^p(\eta)})$ using Appendix I. This proves that intrinsically bounded sets are bounded in this induced Finsler structure. In fact, this type of inequality shows that intrinsically bounded sets are bounded in any Finsler structure given by differential operators.

Part of Theorem 2b, that intrinsically bounded implies bounded in the geometric Finsler structure, is proved exactly like the equivalent statement in Theorem 2a. The rest of Theorem 2b is easier after Proposition 2.8.

(2.8) **Proposition.** *There are global Sobolev inequalities between the canonical geometric Finsler structures. Let $\|u\|_{s,L_t^q}$ denote the canonical geometric Finsler structure* (2.2) *for L_t^q, and $\|u\|_{s,C^0} = \max_{x\in M} |u(x)|_{s(x)}$. If $k/n - 1/p > r/n - 1/q > 0$ and $k \ge r$, $p \ge 1$, then*

$$\|u\|_{s,C^0} \le C_1 \|u\|_{s,L_r^q} \le C_2 \|u\|_{s,L_k^p} ,$$

where C_1 and C_2 are independent of u and s.

The proof of Proposition 2.8 can be done by induction on k and r, so we assume $k - r = 1$. If ζ is any bundle with inner product (,) and compatible covariant derivative ∇, and if $1/n - 1/p + 1/q \geq 0$ $(1 \leq p < q)$, then we can let $1/l = 2/q$ and $1/s = 1/p - 1/q$ so $1/n - 1/s + 1/l \geq 0$. Thus for all smooth sections v of the bundle ζ, using ordinary Sobolov inequalities,

$$\left(\int_M (|v|^2)^l d\mu\right)^{1/l} \leq C\left(\int_M (|\nabla |v|^2|^s + |v|^{2s} d\mu)\right)^{1/s}$$
$$= C\left(\int_M (|2(\nabla v, v)|^s + |v|^{2s} d\mu)\right)^{1/p}$$
$$\leq 2\bar{C}\left(\int_M (|\nabla v|^p + |v|^p d\mu)\right)^{1/p}\left(\int_M |v|^{2l} d\mu\right)^{1/2l}.$$

Dividing both sides by $\left(\int_M |v|^{2l} d\mu\right)^{1/2l}$ we get $\left(\int_M |v|^q d\mu\right)^{1/q} \leq 2\bar{C}\left(\int_M (|\nabla v|^p + |v|^p d\mu)\right)^{1/p}$. For $1/n - 1/p > 0$ we have in a similar calculation

$$\max_{x \in M} |v(x)|^2 \leq C\left(\int_M (\nabla |v(x)|^2)^p + |v(x)|^{2p} d\mu)\right)^{1/p}$$
$$\leq 2c \max_{x \in M} |v(x)| \left(\int_M (|\nabla v(x)|^p + |v(x)|^p d\mu)\right)^{1/p},$$

which gives $\max_{x \in M} |v(x)| \leq 2C\left(\int_M (|\nabla v(x)|^p + |v(x)|^p d\mu)\right)^{1/p}$. Proposition (2.8) can be proved by induction with this process, as all metrics and covariant derivatives used are compatible. However, the general inequality seems worth stating as a separate proposition.

(2.9) **Proposition** (*nonlinear Sobolev inequalities*). *Let M be a Riemannian manifold of dimension n, and ξ a bundle over M with a metric $|\ |$ and compatible covariant derivative. If the iterated covariant derivative ∇^j is interpreted properly* (*see Appendix* II), *and $1/n - 1/p + 1/q > 0$, then*

$$\left(\int_M |u|^q d\mu\right)^{1/q} \leq C\left(\int_M \sum_{j=0}^k |\nabla^j u|^p d\mu\right)^{1/p} \qquad \text{for all } u \in L_k^p(\xi),$$

where C is independent of the bundle, the metric and the covariant derivative. Likewise, if $k/n - 1/p > 0$, then

$$\max_{x \in M} |u(x)| \leq C\left(\int_M \sum_{j=0}^k |\nabla^j u|^p d\mu\right)^{1/p},$$

where C does not depend on ξ.

We now proceed with the rest of the proof of Theorem 2b. In fact, the proof that $L_k^p(E)$ is complete in this Finsler metric and the proof that bounded sets are intrinsically bounded depend on the same inequalities. We prove the theorem by induction on k. For $k = 1$, assume $E \subseteq M \times R^m$ is a closed imbedding.

If S is bounded in the L_1^p Finsler metric, then it is bounded in the C° Finsler metric, and therefore there exists K such that $|s(x)|_{R^m} \leq K$ since the imbedding $E \subseteq M \times R^m$ is closed.

The metric induced by the geometry on TFE will be denoted $|\ |_s$, and the covariant derivative $s^*\nabla$, whereas the flat metric from the imbedding $E \subseteq M \times R^m$ will be $|\ |$, and the covariant derivative is just the ordinary gradient on m functions.

$$\begin{aligned}\|u\|_{s,L_1^p} &= \left(\int_M (|s^*\nabla u|_s^p + |u|_s^p d\mu)\right)^{1/p} \\ &\geq C\left(\int_M (|\text{grad}\, u + (\text{grad}\, s)\cdot\Gamma(s)\cdot u|^p + |u|^p d\mu)\right)^{1/p} \\ &\geq C\left(\int_M (|\text{grad}\, u|^p + |u|^p d\mu)\right)^{1/p} - \bar{C}\left(\int_M (|\text{grad}\, s|^p)\right)^{1/p} \max|u(x)|\,,\end{aligned}$$

where $\bar{C}$ depends on the form of the connection for Γ and $\max|s(x)|$.

Let s_t be a path in $L_1^p(E)$ and $(d/dt)s_t = u_t$. Then

$$\begin{aligned}\frac{d}{dt}\|s_t - s_0\|_{L_1^p(M,R^m)} &\leq \|u_t\|_{L_1^p(M,R^m)} \\ &\leq \frac{1}{C}\|u_t\|_{s_t,L_1^p} + \bar{C}(\|s_0\|_{L_1^p(M,R^m)} \\ &\quad + \|s_t - s_0\|_{L_1^p(M,R^m)})\|u_t\|_{s_t,C}\,.\end{aligned}$$

Divide by $1 + \|s_t - s_0\|_{L_1^p}$, integrate by t from 0 to 1 and take the exponential of both sides to get

$$\|s_1 - s_0\|_{L_1^p(M,R^m)} \leq e^{QL} - 1 \leq e^{QL}\cdot QL\,, \tag{2.10}$$

where $L = \int_0^1 \|u_t\|_{s_t,L_1^p} dt$ is the length of the path in the Finsler metric, and $Q = C_1 + C_2\|s_0\|_{L_1^p(M,R^m)}$. The theorem for $k = 1$ follows easily from this inequality (2.10).

We now proceed by induction for $k > 1$. Using Proposition (2.8) we may assume that if S is bounded (or a Cauchy sequence) in the geometric Finsler metric $\|\ \|_{s,L_k^p}$, then it is bounded (or Cauchy) in the metric $\|\ \|_{s,L_{k-1}^q}$ for all

$1/n - 1/p + 1/q > 0$. The hypothesis of the induction is that the theorem is true in $L^q_{k-1}(E)$ for $(k-1)/n - 1/q > 0$, so we may assume S to be intrinsically bounded (convergent) in $L^q_{k-1}(E)$. We separate out the terms with k derivatives in them in the Finsler structure.

$$\begin{aligned}\| \ \|_{s,L^p_k} &\geq C\Big(\int_M \Big(\sum_{j=0}^{k} |(s^*\nabla)^j u|^p d\mu\Big)\Big)^{1/p} \\ &\geq C\Big(\int_M |(\text{grad})^k u|^p d\mu\Big)^{1/p} - \bar{C}\Big(\int_M (\text{grad})^k s d\mu\Big)^{1/p} \|u\|_{s,C^0} \\ &\quad - \Big(\int_M G(s,u) d\mu\Big)^{1/p},\end{aligned}$$

where there are no k-th order derivatives in $G(s, u)$. We go through the same process as to get equation (2.10), and the contributions from $G(s, u)$ can be estimated using the induction hypothesis for $L^q_{k-1}(E)$ for $q = pk/(k-1)$. We get again that

$$\|s_1 - s_0\|_{L^p_k(M,R^m)} \leq e^{QL}\cdot QL\,, \tag{2.11}$$

where $L = \int_0^1 \|u_t\|_{s_t,L^p_k} dt$ is the length of the path s_t in the Finsler metric for L^p_k, and Q depends on $\|s_0\|_{L^p_k(M,R^m)}$ and $\|s_t\|_{L^q_k(M,R^m)}$.

The two kinds of Finsler structures (2.1) and (2.2) belong to the following class of Finsler structures.

Definition. A Finsler structure $\| \ \|$ defined on $L^p_k(E)$ or $L^p_{k,\delta f}(E)$ is admissible, if over every vector bundle neighborhood $\eta \subseteq E$ and $K > 0$, there exists an N such that

$$N^{-1}\|u\|_s \leq \|u\|_{L^p_k(\eta)} \leq N\|u\|_s\,,$$

for all $\|s\|_{L^p_k(\eta)} \leq K$, $s \in L^p_{k,\delta f}(E)$ and u in the tangent bundle at s. It is immediately clear that if $\| \ \|$ and $\| \ \|'$ are two admissible Finsler structures, then for every intrinsically bounded set S there exists an N such that

$$N^{-1}\|u\|'_s \leq \|u\|_s \leq N\|u\|'_s$$

for all $s \in S$ and $u \in T_s L^p_{k,\delta f}(E)$. Any natural method of putting a Finsler structure on $L^p_k(E)$ or $L^p_{k,\delta f}(E)$ is likely to give an admissible Finsler structure. The last theorem explains why it is unnecessary to specify Finsler structures for condition (C) on manifolds of maps.

Theorem 3. *Let f be a real-valued C^1 function on the smooth Banach manifold $L^p_{k,\delta f}(E)$, and suppose that the inverse image of compact sets in R is intrinsically bounded. Then f satisfies condition (C) in all admissible Finsler structures, if it satisfies this condition in any admissible Finsler structure.*

3. Appendices

I. All the inequalities necessary to complete the proofs in this paper follow from

(1) the formula for a mixed partial derivative of the product of several functions,

(2) the formula for a mixed partial derivative of the composition of two functions,

(3) the Sobolev imbedding theorems,

(4) Hölder inequalities of the form:

$$\int_M s_0 \cdot D^{\alpha_1} s_1 \cdot D^{\alpha_2} s_2 \cdot \cdots \cdot D^{\alpha_n} s_n d\mu \leq \prod_{i=1}^{n} \left(\int_M |D^{\alpha_i} s_i|^{q_i} d\mu \right)^{1/q_i} \max_{x \in M} |s_0(x)| \ ,$$

where $q_i = \left(\sum_{j=1}^{m} |\alpha_j| \right) \Big/ |\alpha_i|$.

See for example the proof of the change of coordinates for $L_k^p(E)$ in Palais [5].

II. Covariant derivatives [6]. A covariant derivative on a bundle η over M is a differential operator ∇, which maps sections of η into sections of $\eta \otimes T^*(M)$, and which locally over trivializations has the form grad $+\Gamma$, where $\Gamma \otimes L(\eta, \eta \otimes T^*(M))$. In order to define the Finsler structure (2.2) we need:

(1) the formula for pulling back a covariant derivative from a bundle η over E to $s^*\eta$ over M if $s\colon M \to E$,

(2) a method for getting a covariant derivative $n \otimes_j T^*(M) \otimes \eta$ given one on $T^*(M)$ and η so we can iterate these successively, and

(3) the ability to handle non-smooth covariant derivatives, which arise when $s\colon M \to E$ is not smooth.

We will assume the bundle are trivial, $\eta = E \times R^m$ since the formulas can be patched together. Denote the gradient or flat covariant derivatives by D.

(1) The pull-back $s^*\nabla$ on $M \times R^m$ of ∇ on $E \times R^m$ by $s\colon M \to E^m$ should obey the rule $(s^*\nabla)f \circ s = \nabla f \circ ds$. If $\nabla = D + \Gamma$, then $s^*\nabla = D + \Gamma(s) \circ (ds)^*$. Since $\Gamma_{s(x)} \in L(R^m, R^m \otimes T^*_{s(x)}(M))$, we have $\Gamma \circ d_{s(x)} \in L(R^m, R^m) \otimes T^*_x(M))$.

(2) If $\nabla_1 = D + \Gamma_1$ is a covariant derivative on $M \times R^m$, and $\nabla_2 = D + \Gamma_2$ is a covariant derivative on $M \times R^n$, then the only covariant derivative which obeys the usual product rule for $M \times R^m \otimes R^n$ is $\nabla_1 \otimes \nabla_2 = D + \Gamma_1 \otimes I + I \otimes \Gamma_2$, [6].

(3) We say a connection ∇ is of class L_r^q if $\nabla = D + \Gamma$ where $\Gamma \in L_r^q \cdot (L(R^m, R^m \otimes T^*(M))$. In the case of a cross-product bundle, the connections of class L_r^q form a linear space and in general they form an affine space. Given the C^∞ connection $\nabla = D + \Gamma$ on $E \times R^m$, the map $s \to s^*\nabla$ is a C^∞ map from $L_k^p(M, E)$ to L_{k-1}^p connections on $M \times R^m$. Given a C^∞ connection $\hat{\nabla}$ on $T^*(M)$, the map $s \to s^*\nabla \to \otimes_j \hat{\nabla} \otimes s^*\nabla$ is a C^∞ map from $L_k^p(M, E)$ to L_{k-}^p

connections on $\otimes_j T^*(M) \otimes R^m$. If we only then prove that an L^p_{k-1} connection maps $L^p_{k-i}(M, R^l)$ to $L^p_{k-i-1}(M, R^l \otimes T^*(M))$, $0 \leq 1 \leq k-1$, and this map depends smoothly on the connection, then we have shown that $(s^*\nabla)^j u = \otimes_{j-1} \hat{\nabla} \otimes s^*\nabla((s^*\nabla)^{j-1}u)$ (etc.) maps $u \in L^p_k(M, R^m)$ linearly into $L^p_0(M_j \otimes_j T_*(M) \otimes R^m)$, and the dependence is C^∞ in $s \in L^p_k(M, E)$. This outline shows how to handle the Finsler structures (2.2). We have only neglected to show that the several covariant derivatives are compatible with the proper metrics for Proposition 2.8.

References

[1] J. Dowling, *Finsler geometry on Sobolev manifolds,* Global Analysis (Proc. Sympos. Pure Math., XV, Berkley, Calif., 1968), Amer. Math. Soc. 1970, 1–6.

[2] H. Eliasson, *On the geometry of manifolds of maps,* J. Differential Geometry **1** (1967) 167–194.

[3] ——, *Variational integrals in fibre bundles,* Global Analysis (Proc. Sympos. Pure Math., XVI, Berkeley, Calif., 1968), Amer. Math. Soc. 1970, 67–90.

[4] U. Koschorke, *Pseudo-compact subsets on infinite-dimensional manifolds,* J. Differential Geometry **5** 1971) 127–134.

[5] R. S. Palais, *Foundations of global non-linear analysis,* Benjamin, New York, 1968.

[6] ——, *Seminar on the Atiyah-Singer index theorem,* Annals of Math. Studies No. 57, Princeton University Press, Princeton, 1965, p. 88.

[7] K. Uhlenbeck, *The calculus of variations and global analysis,* Brandeis thesis, 1968.

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