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ISOPERIMETRIC INEQUALITIES FOR MANIFOLDS WITH BOUNDARY

KIT HANES

1. Introduction

It has long been known that if C is a simple closed curve of length L bounding a planar region of area A, then

$$L^2 - 4\pi A \geq 0 ,$$

where equality holds if and only if C is a circle. This is the classical isoperimetric inequality. In 1939 E. Schmidt [8] proved that if S is the $(n-1)$-dimensional measure of the boundary of a solid body M in Euclidean n-space, V is the n-dimensional measure of M, while σ and ν are the corresponding measures, respectively, for the n-ball of radius 1, then

$$(S/\sigma)^n - (V/\nu)^{n-1} \geq 0 ,$$

where equality holds if and only if M is an n-ball.

In 1959 W. T. Reid [7] generalized the classical isoperimetric inequality to regions on a surface. Suppose M is a C^2 image on a surface in Euclidean 3-space of a region in the plane bounded by a simple closed curve. Let A be the area of M, L the length of the boundary ∂M of M, H the mean curvature vector on M, and X the position vector to M. If the origin is an arbitrary point on ∂M, then

$$L^2 - 4\pi\left(A + \int_M X \cdot H d\nu\right) \geq 0 ,$$

where ν denotes 2-dimensional measure on M. In the case of equality, if the unit normal to M is constant on ∂M, then ∂M is a circle of radius $L/(2\pi)$. This result yields a new proof of a theorem originally due to T. Carleman [1]: If M is a minimal surface, then $L^2 - 4\pi A \geq 0$, where equality holds if and only if ∂M is a circle and M is the disk determined by ∂M.

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In 1961 C. C. Hsiung [6] proved that the inequality proved by Reid still holds when M is a 2-dimensional manifold imbedded in Euclidean n-space provided ∂M is diffeomorphic to a circle. In the case of equality if ∂M is contained in a 2-dimensional subspace, then ∂M is a circle of radius $L/(2\pi)$. This result is used to extend the theorem of Carleman as stated above to Euclidean n-space.

The purpose of this paper is to present a generalization of the inequalities of Reid and Hsiung to n-dimensional manifolds with boundary embedded in Euclidean $(n + p)$-space. The boundaries are not necessarily diffeomorphic to spheres. This is done with the aid of an integral formula which is a generalization of one proved by Hsiung and which yields a new proof of a theorem of Minkowski.

An example is given which shows that Schmidt's theorem does not directly generalize to all manifolds with boundary, and which yields some information about possible further generalizations. Finally, Carleman's theorem is generalized to n-dimensional minimal manifolds.

These results are shown to generalize the theorem of Schmidt only for certain types of manifolds.

2. Preliminaries

Let M be an n-dimensional orientable compact Riemannian C^∞ manifold with boundary ∂M. Assume that $n \geq 2$, and let M be embedded in Euclidean $(n + p)$-space E^{n+p}. On each coodinate domain of M let $e_1, \cdots, e_n, \cdots, e_{n+p}$ be C^∞ vector fields such that at each point of M the following are true:

a) $e_1, \cdots, e_n, \cdots, e_{n+p}$ are orthonormal,

b) $e_1, \cdots, e_n$ are tangent to M, and the order of the vectors is coherent with a fixed orientation of M,

c) at each point of ∂M, e_n is the outward normal to ∂M while $e_1, \cdots, e_{n-1}$ are tangent to ∂M, and

d) the determinant $|e_1, \cdots, e_n, \cdots, e_{n+p}|$ is one.

Each vector field e_i is regarded as a derivation on $C^\infty(M)$.

Let X be the position vector from the origin to a general point of M, $\omega^1, \cdots, \omega^n$ be the 1-forms on M dual to the vectors $e_1, \cdots, e_n$, and $\omega_i{}^j$ for each $i, j \in \{1, \cdots, n + p\}$ be the connection 1-forms with respect to the basis fields chosen above for the Riemannian connection D. Then DX and De_i are vector valued 1-forms given by

$$D_Y X = \sum_1^n \omega^j(Y) e_j \,, \qquad D_Y e_i = \sum_{j=1}^{n+p} \omega_i{}^j(Y) e_j$$

for each $i \in \{1, \cdots, n + p\}$ and each vector Y tangent to M. It follows that $\omega_i{}^j = -\omega_j{}^i$ for each pair i, j.

For each $r \in \{n+1, \cdots, n+p\}$ the r-th mean curvature vector H_r is defined by

$$H_r = -\frac{1}{n}\left[\omega_r{}^1(e_1) + \omega_r{}^2(e_2) + \cdots + \omega_r{}^n(e_n)\right].$$

Then the mean curvature vector H is defined on M by

$$H = H_{n+1}e_{n+1} + \cdots + H_{n+p}e_{n+p}.$$

Let ν and σ denote n and $n-1$ dimensional measures, respectively. Then $d\nu = \omega^1 \wedge \cdots \wedge \omega^n$ on M, while $d\sigma = \omega^1 \wedge \cdots \wedge \omega^{n-1}$ on ∂M, where $\wedge$ denotes the exterior product of differential forms. The Hodge star mapping of forms is denoted by $*$. It is linear, and if $\alpha = \omega^{i_1} \wedge \cdots \wedge \omega^{i_r}$ then $*\alpha$ is the $(n-r)$-form such that $\alpha \wedge *\alpha = d\nu$. Finally, let ρ be the radius of the $(n-1)$-dimensional sphere whose σ measure equals $\sigma(\partial M)$.

3. The integral formula

The following integral formula yields equation (2.14) of [6] as a special case.

Theorem 1. $$\int_{\partial M} X \cdot e_n d\sigma = n\left[\nu(M) + \int_M X \cdot H d\nu\right].$$

Proof. From Green's formula,

$$\int_{\partial M} e_n(X \cdot X) d\sigma = \int_M \Delta(X \cdot X) d\nu, \tag{1}$$

where Δ is the Laplace-Beltrami operator given by

$$\Delta\varphi = *d*d\varphi = \sum_{i=1}^n \left[e_i(e_i(\varphi)) + \sum_{j=1}^n e_j(\varphi)\omega_j{}^i(e_i)\right].$$

(See [5, pp. 386–393].) For each i,

$$e_i(X \cdot X) = 2(D_{e_i}X \cdot X) = 2(e_i \cdot X), \tag{2}$$

so

$$\begin{aligned} e_i(e_i(X \cdot X)) &= 2e_i(X \cdot e_i) = 2[(D_{e_i}X \cdot e_i) + (X \cdot D_{e_i}e_i)] \\ &= 2\left[1 + \sum_{j=1}^{n+p} \omega_i{}^j(e_i)(X \cdot e_j)\right]. \end{aligned}$$

Hence

$$\sum_{i=1}^{n} e_i(e_i(X\cdot X)) = 2\left[n + \sum_{i=1}^{n}\sum_{j=1}^{n+p} (X\cdot e_j)\omega_i{}^j(e_i)\right].$$

Then

$$\begin{aligned}
\Delta(X\cdot X) &= 2\left[n + \sum_{i=1}^{n}\sum_{j=1}^{n+p} (X\cdot e_j)\omega_i{}^j(e_i)\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} 2(X\cdot e_j)\omega_j{}^i(e_i) \\
&= 2\left[n + \sum_{i=1}^{n}\left(\sum_{j=1}^{n+p} (X\cdot e_j)\omega_i{}^j(e_i) - \sum_{j=1}^{n} (X\cdot e_j)\omega_i{}^j(e_i)\right)\right] \qquad (3) \\
&= 2\left[n + n\sum_{j=n+1}^{n+p} (X\cdot e_j)H_j\right] = 2n(1 + X\cdot H).
\end{aligned}$$

Therefore, by combining (1), (2), and (3),

$$\int_{\partial M} X\cdot e_n d\sigma = \int_M n(1 + X\cdot H)d\nu = n\left[\nu(M) + \int_M X\cdot H d\nu\right]. \qquad \text{q.e.d.}$$

An immediate consequence of this theorem is the following theorem of Minkowski.

Theorem 2. *If M is a closed manifold, then*

$$\nu(M) = -\int_M X\cdot H d\nu .$$

4. An eigenvalue problem

Before taking up the isoperimetric inequality it is necessary to consider an eigenvalue problem on ∂M. Let (,) denote the inner product on p-forms on ∂M defined by

$$(\alpha, \beta) = \int_{\partial M} \alpha \wedge *\beta .$$

If $f \in C^\infty(\partial M)$, then

$$df = \sum_{i=1}^{n-1} e_i(f)\omega^i , \qquad \operatorname{grad} f = \sum_{i=1}^{n-1} e_i(f)e_i .$$

Note that if $f \in C^\infty(\partial M)$ then, since $\omega^i \wedge *\omega^j = 0$ when $i \neq j$,

$$(df, df) = \int_{\partial M} \sum_{i=1}^{n-1} (e_i(f))^2 d\sigma = \int_{\partial M} |\operatorname{grad} f|^2 d\sigma . \qquad (4)$$

Consider now the eigenvalue problem

$$\Delta\varphi + \lambda\varphi = 0 \tag{5}$$

for functions $\varphi \in C^\infty(\partial M)$, where Δ is the Laplace-Beltrami operator on ∂M. The following theorem describes the solutions to this problem. (See [9, p. 685].)

Theorem 3. *The eigenvalues for the above problem are nonnegative and form a discrete set whose only limit point is ∞. The eigenfunctions are orthonormal with respect to the inner product $(\ , \ )$, and span $L^2(\partial M)$. The only eigenfunctions corresponding to the eigenvalue* 0 *are the constant functions.*

Since the eigenvalues form a discrete set with no finite limit point, let λ denote the smallest positive eigenvalue. The proof of the next theorem is standard and depends upon the completeness of the eigenfunctions and the existence of λ.

Theorem 4. *If f is any nonconstant function in $C^\infty(\partial M)$, then*

$$(df, df) \geq \lambda(f, f) ,$$

where equality holds if and only if $f = c\varphi$, c being a constant and φ an eigenfunction corresponding to λ.

5. The isoperimetric inequality

One of the aims of this paper is to establish the following inequality for selected values of c:

$$\sigma(\partial M) - cn\left[\nu(M) + \int_M X\cdot H d\nu\right] \geq 0 , \tag{6}$$

where c is a parameter depending upon ∂M.

If $c = 1/\rho$ and $p = 0$, then in the notation of the introduction (6) becomes $S - nV/\rho \geq 0$. Since $S = \rho^{n-1}\sigma$ and $\sigma = n\nu$, it follows that this last inequality is just that proved by Schmidt for solid bodies. If $c = 1/\rho$, $n = 2$ and the origin is a point of ∂M, then in the notation of the introduction $\rho = L/(2\pi)$ and $nc = 4\pi/L$. In this case, when $p = 1$ the inequality (6) becomes that proved by Reid, and when p is arbitrary it becomes that proved by Hsiung.

By means of Theorem 1, (6) is readily seen to be equivalent to the following inequality:

$$\int_{\partial M} (1 - cX\cdot e_n)d\sigma \geq 0 . \tag{7}$$

Suppose the origin is P, a point on ∂M. Then define

$$\delta_P = (d\,|X|, d\,|X|)/(|X|, |X|) , \qquad \delta = \min\{\delta_P : P \in \partial M\} .$$

Thus the groundwork has been laid for the following theorem.

Theorem 5. *Let n and p be arbitrary, and P be on ∂M. Then the inequalities* (6) *and* (7) *hold where c is any constant such that $c \leq 2\sqrt{\delta_P}$. Furthermore, if equality holds for some fixed $c \leq 2\sqrt{\delta}$ and each $P \in \partial M$, then ∂M is an $(n-1)$-sphere with radius $1/c$.*

Proof. Since

$$X = \sum_1^{n-1} (X \cdot e_i)e_i + (X \cdot e_n)e_n + \sum_{n+1}^{n+p} (X \cdot e_j)e_j ,$$

we have

$$X \cdot X = \sum_1^{n-1} (X \cdot e_i)^2 + (X \cdot e_n)^2 + \sum_{n+1}^{n+p} (X \cdot e_j)^2 ,$$

so

$$1 = \sum_1^{n-1} (X \cdot e_i)^2/(X \cdot X) + (X \cdot e_n)^2/(X \cdot X) + \sum_{n+1}^{n+p} (X \cdot e_j)^2/(X \cdot X) .$$

Since $\operatorname{grad}|X| = \sum_1^{n-1} e_i(|X|)e_i = \sum_1^{n-1} (X \cdot e_i/|X|)e_i$, we have $|\operatorname{grad}|X||^2 = \sum_1^{n-1} (X \cdot e_i)^2/(X \cdot X)$, so

$$1 = |\operatorname{grad}|X||^2 + (X \cdot e_n)^2/(X \cdot X) + \sum_{n+1}^{n+p} (X \cdot e_j)^2/(X \cdot X) . \tag{8}$$

Now

$$0 \leq \left(X \cdot e_n - \frac{c}{2} X \cdot X\right)^2 = (X \cdot e_n)^2 - c(X \cdot e_n)(X \cdot X) + \frac{c^2}{4}(X \cdot X)^2 ,$$

so

$$(X \cdot e_n)^2/(X \cdot X) - c(X \cdot e_n) = (X \cdot e_n - \frac{c}{2} X \cdot X)^2/(X \cdot X) - \frac{c^2}{4} X \cdot X . \tag{9}$$

Using (8) and (9) the left side of (7) becomes

$$\int_{\partial M} (1 - cX \cdot e_n) d\sigma$$

$$= \int_{\partial M} \left[|\operatorname{grad}|X||^2 - \frac{c^2}{4} X \cdot X + \frac{\left(X \cdot e_n - \frac{c}{2} X \cdot X\right)^2}{X \cdot X} + \frac{\sum_{n+1}^{n+p} (X \cdot e_j)^2}{X \cdot X} \right] d\sigma$$

$$\geq \int_{\partial M} \left(|\operatorname{grad}|X||^2 - \frac{c^2}{4} X \cdot X \right) d\sigma = (d|X|, d|X|) - \frac{c^2}{4}(|X|, |X|) \geq 0 .$$

The last equality holds in view of (4), while the last inequality holds because $c \leq 2\sqrt{\delta_P}$ in view of the definition of δ_P.

Now suppose that equality holds for $c \leq 2\sqrt{\delta}$ and all $P \in \partial M$. Then $X \cdot e_n - \frac{c}{2} X \cdot X = 0$ for all $P \in \partial M$ and $X \in \partial M$. Let $N = e_n$ at some point $P_o \in \partial M$. Let $A = -N/c$, assuming the origin is at P_0. Then $X = P$ and $-X$ is the vector from P to P_0. Hence $(-X) \cdot N - (c/2)(-X) \cdot (-X) = 0$, i.e., $(c/2) X \cdot X = -X \cdot N$. Then

$$|X - A|^2 = X \cdot X - 2X \cdot A + A \cdot A = X \cdot X + 2X \cdot N/c + N \cdot N/c^2 = 1/c^2 .$$

Therefore ∂M is contained on a sphere T with center $A = -N/c$ and radius $1/c$. Furthermore, since equality holds in the inequality, $X \cdot e_j = 0$ for all $X \in \partial M$ and $j \in \{n + 1, \cdots, n + p\}$. For each j let $N_j = e_j$ at P_0. Then $(-X) \cdot N_j = 0$. Thus P lies in a hyperplane normal to N_j and passing through P_0. This is true for each of the mutually orthogonal vectors $N_{n+1}, \cdots, N_{n+p}$, so ∂M lies in an n-dimensional subspace F of E^{n+p}. Then $\partial M \subseteq F \cap T$, and $F \cap T$ is an $(n - 1)$-sphere. Since ∂M is $(n - 1)$-dimensional and $\partial(\partial M) = \emptyset$, it follows that $\partial M = F \cap T$. The radius of $F \cap T$ is $1/c$ since $A \in F$. q.e.d.

It follows from Theorem 4 that $\lambda \leq \delta_P$ for each $P \in \partial M$, hence $\lambda \leq \delta$. For each $P \in \partial M$ Theorem 5 gives two specific versions of the isoperimetric inequality, one for $c = 2\sqrt{\lambda}$ and one for $c = 2\sqrt{\delta_P}$. If $c = 2\sqrt{\lambda}$, then Theorem 5 can be strengthened.

Corollary 1. *If $c = 2\sqrt{\lambda}$ and equality holds in either* (6) *or* (7) *for each $P \in \partial M$, then not only is ∂M an $(n - 1)$-spere of radius $1/c$ but also $n = 2$.*

Proof. Equality holds in (6) and (7) only if equality holds in Theorem 4 where $f = |X|$, i.e., only when $|X|$ is an eigenfunction for problem (5). Suppose P is a point on the unit sphere, and φ is the angle with vertex at the center of the sphere subtending X. Then a calculation shows that

$$|X| = 2 \sin (\varphi/2) , \qquad \Delta |X| = (n - 2) \cot \varphi \cos (\varphi/2) - \tfrac{1}{2} \sin (\varphi/2) .$$

Clearly, $|X|$ is an eigenfunction if and only if $n - 2 = 0$. q.e.d.

Finally, Theorem 5 gives information about upper bounds for the first eigenvalue for problem (5).

Corollary 2. *Let $P \in \partial M$ be fixed. If N is any n-dimensional orientable compact Riemannian manifold with boundary ∂N such that $\partial N = \partial M$ and H_N is the mean curvature vector on N, then*

$$\lambda \leq \sigma(\partial M)^2 \Big/ \left[4n^2 \left(\nu(N) + \int_N X \cdot H_N d\nu \right)^2 \right] .$$

Proof. This inequality is obtained by solving (6) where $c = 2\sqrt{\lambda}$.

6. Further versions

If $n = 2$, and ∂M is diffeomorphic to a circle, then $2\sqrt{\lambda} = 1/\rho$. This together with Theorem 4 is one way of stating Wirtinger's inequality (see, for example, [4, p. 185]). Thus Theorem 5 gives an isoperimetric inequality when $c = 1/\rho = 2\sqrt{\lambda}$ and $n = 2$. This is essentially the inequality proved by Hsiung.

Let W be the class of those manifolds M for which $1/\rho \leq 2\sqrt{\delta}$. Then for each $M \in W$ Theorem 5 gives an isoperimetric inequality where $c = 1/\rho$. As mentioned in the last section if $p = 0$, M is a solid body, and $c = 1/\rho$, then Schmidt has proved that M belongs to W.

Suppose M is the n-ball of radius ρ. If $P \in \partial M$, then a calculation shows that $\delta_P = 1/(4\rho^2)$. Consequently, if N is any compact n-dimensional orientable Riemannian C^∞ manifold such that $\partial N = \partial M$ while N is tangent to M everywhere on ∂M, then, with respect to N, $1/\rho = 2\sqrt{\delta_P} = \sqrt{\delta}$, and N belongs to W.

In addition to the manifolds with spherical boundary just discussed there are other manifolds which belong to W.

Theorem 6. *Suppose ∂M lies in an n-dimensional subspace of E^{n+p}. Suppose N, a convex compact orientable n-dimensional Riemannian manifold with boundary ∂N, lies in the subspace containing ∂M and $\partial N = \partial M$. Then* (6) *and* (7) *give an isoperimetric inequality for M when $c = 1/\rho$, and M belongs to W.*

Proof. N is flat, so $H_N = 0$. Then Theorem 1 gives

$$\int_{\partial M} X \cdot f d\sigma = n\nu(N) \geq 0 ,$$

where f is the outward unit vector tangent to N and normal to ∂M. According to the isoperimetric inequality of Schmidt,

$$0 \leq \sigma(\partial M) - \frac{1}{\rho} n\nu(N) = \int_{\partial M} \left(1 - \frac{1}{\rho} X \cdot f\right) d\sigma .$$

Suppose that the origin is at $P \in \partial N$. Then $X \cdot f \geq 0$ for all $X \in \partial M$. Since e_n is normal to ∂M while e_n and f are both unit vectors, it follows that $X \cdot e_n \leq X \cdot f$ everywhere on ∂M. Therefore

$$\int_{\partial M} \left(1 - \frac{1}{\rho} X \cdot e_n\right) d\sigma \geq \int_{\partial M} \left(1 - \frac{1}{\rho} X \cdot f\right) d\sigma \geq 0 ,$$

so $M \in W$. q.e.d.

Do all manifolds of the type considered in this paper belong to W? The following example shows that the answer must be negative.

Let $n = 2$, $p = 1$, and let M be a surface whose boundary consists of two circles C_1 and C_2. P lies on C_2, and $|X| = a$ for each $X \in C_1$. M is such that $e_2 = X/|X|$ on C_1, while $e_2 \cdot X = 0$ on C_2. Let $L_1 = \sigma(C_1)$ and $L_2 = \sigma(C_2)$. Then

$$\int_{\partial M} \left(1 - \frac{1}{\rho} X \cdot e_2\right) d\sigma = L_1 + L_2 - \frac{a}{\rho} L_1 .$$

This expression is negative provided $a < [(L_1 + L_2)/L_1]\rho$ where $\rho = (L_1 + L_2)/(2\pi)$. For such a, $M \notin W$.

7. Minimal manifolds

A manifold M is minimal provided $H \equiv 0$. The results obtained thus far can be interpreted for this case.

Theorem 7. *Let M be a minimal manifold, and P any point on ∂M. Then*

$$\sigma(\partial M) - cn\nu(M) \geq 0 , \tag{10}$$

where $c \leq 2\sqrt{\delta_P}$. If M is an n-ball and $c = 2\sqrt{\delta_P}$, then $c = 1/\rho$ and equality holds. If equality holds and $c \leq 2\sqrt{\delta}$, then M is an n-ball and $c = 1/\rho = 2\sqrt{\delta}$.

Proof. The inequality is obtained from Theorem 5 by letting $H = 0$ in (6). The calculation in the previous section shows that if M is an n-ball, then $2\sqrt{\delta_P} = 1/\rho$ and equality holds for this value of c. Now suppose that equality holds. Then it holds for each $P \in \partial M$ since neither P nor X appears in (10). Then from Theorem 5 it follows that ∂M is an $(n-1)$-sphere with radius $\rho = 1/c$. It remains to be shown that M is the n-ball determined by ∂M. The proof given by Hsiung [6] may be generalized. Since $e_i(X) = e_i$, we have

$$e_i(e_i(X)) = D_{e_i} e_i = \sum_{j=1}^{n+p} \omega_i{}^j(e_i) e_j .$$

Substitution into the expression for Δ gives

$$\begin{aligned} \Delta X &= \sum_{i=1}^{n} \left[\sum_{j=1}^{n+p} \omega_i{}^j(e_i) e_j + \sum_{j=1}^{n} \omega_j{}^i(e_i) e_j \right] \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \omega_i{}^j(e_i) + \omega_j{}^i(e_i) \right) e_j + \sum_{j=n+1}^{n+p} \sum_{i=1}^{n} \omega_i{}^j(e_i) e_j = nH , \end{aligned}$$

since $\omega_i{}^j(e_i) + \omega_j{}^i(e_i) = 0$. Since $H = 0$, $\Delta X = 0$. Now let $X = (x_1, \cdots, x_{n+p})$ where the functions x_i are the components of X relative to some fixed orthonormal frame in E^{n+p}. Since ∂M lies in an n-dimensional subspace F of E^{n+p}, it may be supposed that $x_{n+1} = \cdots = x_{n+p} = 0$ on M. Since $\Delta X = 0$, we have $\Delta x_{n+1} = \cdots = \Delta x_{n+p} = 0$ on M. Thus $x_{n+1}, \cdots, x_{n+p}$ are functions which are harmonic on M and vanish on ∂M. They must then vanish on all of M. Consequently M lies in F, and so M must be the n-ball bounded by ∂M. Finally, the calculation in the previous section shows that $1/\rho = 2\sqrt{\delta}$. q.e.d.

This result was obtained by Carleman for the case $n = 2$ and $p = 1$, and was extended by Hsiung for the case $n = 2$ and p arbitrary.

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Eastern Washington State College