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GAUSSIAN CURVATURE AND CONFORMAL MAPPING

DIMITRI KOUTROUFIOTIS

1. Introduction and preliminaries

An ovaloid in E^3 is a closed convex surface of class C^2 with positive Gaussian curvature. We shall denote the unit sphere in E^3 by Σ.

L. Nirenberg has posed this question: given a C^∞ positive function K on Σ, do there exist an ovaloid S and a conformal mapping φ of S onto Σ so that the Gaussian curvature of S at P is equal to $K(\varphi(P))$ for all points P on S? In this paper we shall prove that the answer is yes in the special case where the given function K is even and sufficiently close to 1 pointwise. In fact, the ovaloid S will also turn out to be "almost spherical" in a natural sense. Some related questions will be discussed briefly in the last section of this paper.

We shall deal with the problem from the point of view of partial differential equations. A priori estimates for solutions and their derivatives with respect to local parameters on Σ will be needed. We shall therefore introduce now coodinate systems and function spaces appropriate to our problem. The unit sphere can be described by two overlapping coodinate patches. Say, we choose two overlapping open regions on Σ, one containing the north pole and bounded by a southern parallel of latitude, and one containing the south pole and bounded by a northern parallel of latitude. These two regions are mapped into the equator plane (x, y) through stereographic projection, the first one from the south pole and the second one from the north pole. Thus the parameter domains for the two spherical regions are two (coincident) open discs G_1 and G_2 in the equator plane. For the sake of definiteness, from now on we shall consistently use this parametrization of Σ, although our final results will be independent of the smooth parameters used to describe it.

The norm of a function of class C^k on Σ is defined as

$$\|f\|_k = \sum_{|\alpha| \leq k} \sup_{G_1} |D^\alpha f| + \sum_{|\alpha| \leq k} \sup_{G_2} |D^\alpha f| ,$$

where $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha f = \partial^{|\alpha|} f / \partial x^{\alpha_1} \partial y^{\alpha_2}$.

If, in addition, the kth derivatives of f, in each patch, satisfy a Hölder-condition with exponent α, $0 < \alpha < 1$, we define the norm of f to be

$$\|f\|_{k+\alpha} = \|f\|_k + h ,$$

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where h is the smallest possible coefficient in the Hölder-conditions for the kth derivatives in G_i, $i = 1, 2$. We have thus defined the normed linear spaces C^m for every nonnegative number m. They are in fact real Banach spaces.

Next we consider the set of functions f on Σ with

$$\int_{G_1} \sum_{|\alpha| \leq k} |D^\alpha f|^r dG_1 + \int_{G_2} \sum_{|\alpha| \leq k} |D^\alpha f|^r dG_2 < \infty ,$$

for some r, $1 < r < \infty$, and some integer k. Banach spaces W_r^k are constructed by completion of these spaces with respect to the norm

$$\|f\|_{k,r} = \Big(\int_{G_1} \sum_{|\alpha| \leq k} |D^\alpha f|^r dG_1\Big)^{1/r} + \Big(\int_{G_2} \sum_{|\alpha| \leq k} |D^\alpha f|^r dG_2\Big)^{1/r} .$$

Note that $W_r^0 = L_r$.

We shall have to solve an equation of the form

$$\tfrac{1}{2}\Delta_2 v + v = f$$

for the unknown function v on Σ. Here Δ_2 denotes the Laplace-Beltrami operator on the unit sphere in E^3. It is well-known that this equation is solvable if and only if f is orthogonal to all the solutions of the corresponding homogeneous equation, i.e., to all linear functions on Σ:

$$\psi = a_1 x + a_2 y + a_3 z , \quad x^2 + y^2 + z^2 = 1 , \quad \int_\Sigma f\psi d\omega = 0 .$$

The solution we shall construct will define a new positive-definite metric on Σ, to be realized as an ovaloid in E^3. Some version of Weyl's realization theorem will be needed. Before formulating the version which is appropriate to our situation, we introduce Banach spaces of quadratic forms on Σ. If

$$ds^2 = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2 , \tag{1.1}$$

we define $\|ds^2\|_{k+\alpha} = \|E\|_{k+\alpha} + \|F\|_{k+\alpha} + \|G\|_{k+\alpha}$, where k is a nonnegative integer and $0 < \alpha < 1$.

Nirenberg proved in [1, Theorem 2, p. 351] that if the given metric (1.1) is close enough to the natural metric of Σ in the $C^{2+\alpha}$-norm, then it can be realized by an ovaloid which is close to Σ in $C^{2+\alpha}$. However, we shall succeed in constructing a metric which is only $C^{1+\alpha}$-close to that of Σ. Nevertheless, making use of the Schauder estimates in [2] for equations in variational form, it is a straightforward matter to modify Nirenberg's proof so as to give the following stronger theorem, to be used subsequently:

Let ds_0^2 be a positive-definite quadratic form of class $C^{4+\alpha}$ on Σ, with positive curvature, which can be realized by an ovaloid $\vec{X}_0(x, y)$ in E^3. There exists a positive number ε such that any quadratic form ds^2 of class $C^{2+\alpha}$ on Σ can be realized by an ovaloid $\vec{X}(x, y)$ of class $C^{2+\alpha}$ in E^3 with $\|\vec{X} - \vec{X}_0\|_{1+\alpha}$ arbitrarily small if only ds^2 is sufficiently close to ds_0^2 in $C^{1+\alpha}$.

The author takes pleasure in thanking L. Nirenberg for having suggested this problem, but also for his interest and assistance.

2. The main result

Theorem. *Given a C^∞ symmetric function K on the unit sphere Σ, sufficiently close to* 1 *pointwise, there exist a centrally-symmetric ovaloid S in E^3 and a conformal homeomorphism φ of S onto Σ so that S has Gaussian curvature $K(\varphi(P))$ at the point P for all $P \in S$. Furthermore, the position of S and parameter systems covering it can be chosen in such a way that, for the position vectors $\vec{X}$ and $\vec{X}_0$ describing S and Σ respectively, $\vec{X}(P) - \vec{X}_0(\varphi(P))$ and its first derivatives are arbitrarily small in absolute value if K is sufficiently close to* 1.

Proof. The parametrization (x, y) of Σ through stereographic projection introduced above is isothermic, i.e., the line element ds_0^2 of Σ assumes the form

$$ds_0^2 = e^{u_0(x,y)}(dx^2 + dy^2) .$$

We shall construct a new line element

$$ds^2 = e^{u(x,y)}(dx^2 + dy^2)$$

on Σ, which has the given function K as Gaussian curvature. If such a metric exists, the Theorema Egregium in these parameters states that

$$K = -\tfrac{1}{2}e^{-u}\Delta u , \tag{2.1}$$

where Δ signifies the ordinary Laplacian: $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We also have

$$1 = -\tfrac{1}{2}e^{-u_0}\Delta u_0 . \tag{2.2}$$

Setting $v = u - u_0$, (2.1) may be written as

$$K = -\tfrac{1}{2}e^{-u_0-v}\Delta(u_0 + v) ,$$

or, in view of (2.2) and the linearity of Δ,

$$\tfrac{1}{2}e^{-u_0}\Delta v + Ke^v = 1 . \tag{2.3}$$

Note that v is a well-defined invariant function on Σ, since $e^v = ds^2/ds_0^2$. Furthermore, the operator $e^{-u_0}\Delta$ is the Laplace-Beltrami operator Δ_2 on Σ, expressed in isothermic parameters. Thus (2.3) is an invariant equation on Σ.

It is a quasilinear elliptic partial differential equation for v, linear in the highest-order derivatives. We rewrite it as

$$\frac{1}{2}\Delta_2 v + v = (1 - K)e^v + 1 + v - e^v \ . \tag{2.4}$$

Note that the right-hand side of (2.4) is of order ε, if $K - 1$ and v are of order ε, and so it is plausible to try to construct a "small" solution of (2.4) by iteration, starting with $v_0 \equiv 0$ and defining v_n as the suitably normalized solution of the linear equation

$$\frac{1}{2}\Delta_2 v_n + v_n = (1 - K)e^{v_{n-1}} + 1 + v_{n-1} - e^{v_{n-1}} \ , \qquad n = 1, 2, 3, \cdots \ . \tag{2.5}$$

We must make sure that, at each step of the iteration, the right-hand side is orthogonal to all the linear functions on Σ. Since K is a symmetric-, i.e., even-function on Σ, assuming v_{n-1} is symmetric, the right-hand side of (2.5) is symmetric. Starting with $v_0 \equiv 0$, we choose at each step as v_n the uniquely determined symmetric solution of (2.5). This choice is possible since, if $v(\vec{\nu})$, $|\vec{\nu}| = 1$, is a solution of

$$\frac{1}{2}\Delta_2 \varphi + \varphi = f \ , \tag{2.6}$$

$f(\vec{\nu})$ being a given symmetric function on Σ, then $\bar{v}(\vec{\nu}) = v(-\vec{\nu})$ is also a solution of (2.6) and so is the function $v_s = \frac{1}{2}(v + \bar{v})$. Now v_s is a symmetric function on Σ and, in fact, is the sole symmetric solution of (2.6), since any other solution is obtained from v_s by adding to it an arbitrary linear function and hence is not symmetric any more. With this choice of v_n the right-hand side of (2.5) remains constantly symmetric, therefore always orthogonal to the linear functions on Σ, and the sequence of iterates $\{v_n\}$ is well-defined. Note that they are all C^∞ functions, since K is C^∞.

Since we have no information about the Hölder norms of K and about its derivatives in our fixed coodinate covering of Σ, we shall work in the space W_r^2. We shall now prove in standard fashion that if $\varepsilon > 0$ is sufficiently small, and K is close enough to one pointwise, then the sequence $\{v_n\}$ converges in W_r^2 to a smooth solution v of (2.3) satisfying $\|v\|_{2,r} \leq \varepsilon$. By virtue of its choice, v_n satisfies the inequality

$$\begin{aligned} \|v_n\|_{2,r} &\leq c_1\|(1 - K)e^{v_{n-1}} + 1 + v_{n-1} - e^{v_{n-1}}\|_{0,r} \\ &\leq c_1\left[\max_{\Sigma}|1 - K|\cdot\|e^{v_{n-1}}\|_{0,r} + \|1 + v_{n-1} - e^{v_{n-1}}\|_{0,r}\right] , \\ & \qquad n = 1, 2, \cdots \end{aligned} \tag{2.7}$$

where c_1 is some constant depending only on r. If we assume that $\|v_{n-1}\|_{2,r} < \varepsilon$, from Sobolev's inequalities we obtain

$$(2.8) \qquad \max_{\Sigma} |v_{n-1}| \leq \|v_{n-1}\|_{\alpha} \leq c_2 \|v_{n-1}\|_{2,r} < c_2\varepsilon ,$$

where $r > 2$, $0 < \alpha < 1 - 2/r$ and $c_2 = c_2(r)$. It follows that

$$(2.9) \qquad \|e^{v_{n-1}}\|_{0,r} < e^{c_2\varepsilon}\|1\|_{0,r} = c_3 e^{c_2\varepsilon} , \qquad c_3 = c_3(r) .$$

From Taylor's theorem, (2.8) and (2.9), we have

$$(2.10) \qquad \|e^{v_{n-1}} - (1 + v_{n-1})\|_{0,r} = \|\tfrac{1}{2}v_{n-1}^2 e^{\theta v_{n-1}}\|_{0,r} < \tfrac{1}{2}c_2^2 c_3 \varepsilon^2 e^{c_2\varepsilon} ,$$

where θ lies between 0 and 1. Thus we obtain the following estimate for v_n:

$$(2.11) \qquad \|v_n\|_{2,r} \leq c_1\left[\max_{\Sigma} |1 - K| c_3 e^{c_2\varepsilon} + \tfrac{1}{2}c_2^2 c_3 \varepsilon^2 e^{c_2\varepsilon}\right] .$$

Now pick an $\varepsilon > 0$, and assume $\varepsilon < \min\{1, (c_1 c_3 c_2^2 e^{c_2})^{-1}\}$ and $\max_{\Sigma} |1 - K| < \frac{1}{2}c_2^2\varepsilon^2$. Then using (2.7), (2.11) and $v_0 \equiv 0$, by induction it follows immediately that $\|v_n\|_{2,r} < \varepsilon$ for all n. We proceed to prove convergence:

$$\|v_n - v_{n-1}\|_{2,r} \leq c_1\| - K(e^{v_{n-1}} - e^{v_{n-2}}) + v_{n-1} - v_{n-2}\|_{0,r} ,$$

since $v_n - v_{n-1}$ is orthogonal to all the linear functions on Σ and satisfies the equation

$$\tfrac{1}{2}\Delta_2\varphi + \varphi = -K(e^{v_{n-1}} - e^{v_{n-2}}) + v_{n-1} - v_{n-2} .$$

We may write

$$e^{v_{n-1}} - e^{v_{n-2}} = e^{v_{n-1}^*}(v_{n-1} - v_{n-2}) ,$$

v_{n-1}^* being between v_{n-1} and v_{n-2}, and we obtain

$$(2.12) \qquad \|v_n - v_{n-1}\|_{2,r} \leq c_1 \max_{\Sigma} |1 - Ke^{v_{n-1}^*}| \cdot \|v_{n-1} - v_{n-2}\|_{2,r} .$$

Since $-c_2\varepsilon < v_n < c_2\varepsilon$ for all n,

$$1 - Ke^{c_2\varepsilon} < 1 - Ke^{v_{n-1}^*} < 1 - Ke^{-c_2\varepsilon} .$$

Using $1 - \frac{1}{2}c_2^2\varepsilon^2 < K < 1 + \frac{1}{2}c_2^2\varepsilon^2$, we have that

$$(2.13) \qquad c_1 \max_{\Sigma} |1 - Ke^{v_{n-1}^*}| < \mu < 1$$

for a fixed constant μ and all n, provided that we restrict ε further so that

$$-1 < c_1[1 - (1 + \tfrac{1}{2}c_2^2\varepsilon^2)e^{c_2\varepsilon}] \quad \text{and} \quad c_1[1 - (1 - \tfrac{1}{2}c_2^2\varepsilon^2)e^{-c_2\varepsilon}] < 1 .$$

From (2.12) and (2.13) we now obtain

$$\|v_n - v_{n-1}\|_{2,r} < \mu^{n-1}\|v_1\|_{2,r} .$$

This shows the sequence $\{v_n\}$ to be a Cauchy sequence in W_r^2 which therefore converges to a function v. Clearly, v is a weak, symmetric solution of (2.3) satisfying $\|v\|_{2,r} \leq \varepsilon$, provided $r > 2$, ε is sufficiently small and $\max_{\Sigma}|1 - K| < \bar{c}(r)\varepsilon^2$. A fortiori, v is in $C^{1+\alpha}$ for $0 < \alpha < 1 - 2/r$, and satisfies the inequalities

$$\|v\|_{1+\alpha} \leq c(r)\|v\|_{2,r} \leq c(r)\varepsilon .$$

Equation (2.3) is quasilinear of the type investigated in [3]; since all its coefficients are C^∞, the bounded weak solution v must itself be C^∞ [3, Theorem 7, p. 493]. We now set $u = u_0 + v$, and consider on Σ the new C^∞ metric

$$ds^2 = e^u(dx^2 + dy^2) . \tag{2.14}$$

It defines an abstract two-dimensional Riemannian manifold M homeomorphic to the sphere $\Sigma \subset E^3$. The underlying topological space of M is Σ itself, and the homeomorphism is the identity on Σ. The same fixed parameters (x, y) are used to describe corresponding points of M and Σ. Since, with respect to these parameters, both the metrics ds^2 and $ds_0^2 = e^{u_0}(dx^2 + dy^2)$ assume isothermic form, the homeomorphism in question is, in fact, a conformal mapping of M onto Σ. Note that, since v is a symmetric function and u_0 defines the natural metric on Σ, the metric ds^2 is symmetric on Σ in the sense that a curve on Σ and its reflection on the center have the same length with respect to ds^2. Furthermore, the curvature of this new metric is our given K, since, using (2.3) and (2.2), we have

$$-\tfrac{1}{2}e^{-u}\Delta u = (-\tfrac{1}{2}e^{-u_0}\Delta u_0)e^{-v} + (-\tfrac{1}{2}e^{-u_0}\Delta v)e^{-v} = K .$$

The metric ds_0^2 is analytic, the metric ds^2 is C^∞ and, since $C^{1+\alpha}$ is a Banach algebra, we have the estimates

$$\begin{aligned}\|ds^2 - ds_0^2\|_{1+\alpha} &= 2\|e^u - e^{u_0}\|_{1+\alpha} = 2\|e^{u_0}(e^v - 1)\|_{1+\alpha} \\ &\leq 2k_1\|e^v - 1\|_{1+\alpha} \leq 2k_1k_2\|v\|_{1+\alpha} \leq \text{const.}\ \varepsilon .\end{aligned}$$

We may thus apply the Realization Theorem stated in § 1 to obtain an ovaloid $S \subset E^3$ and a global isometry $M \to S$. This ovaloid is given by a vector $\vec{X}(x, y)$ in terms of the isothermic parameters (x, y), and $\vec{X}(x, y)$ differs by little in the $C^{1+\alpha}$-norm from the vector $\vec{X}_0(x, y)$ representing Σ. Note that, since the same parameters (x, y) are used on S and Σ, we have a conformal mapping of S onto Σ whereby points with the same parameter values correspond. Furthermore, a point on S with parameter values (x, y) has Gaussian curvature equal to the value of K at the point of Σ with the same parameter values (x, y). In addition, the ovaloid S is centrally symmetric: the metric ds^2 is symmetric; if we choose as origin of coodinates the center of Σ, all the known quantities in the differential equation for $\vec{X}$ are symmetric, considered as functions on Σ, and the solution vector is symmetric too. Also, the conformal mapping $S \to \Sigma$ carries antipodes into antipodes. Hence the theorem is proven.

3· Related questions

It is not known to the author whether the ovaloid S obtained as above is unique up to congruence. More generally, we may ask: if two centrally-symmetric ovaloids are such that there exists a conformal mapping between them preserving antipodes and Gaussian curvature, are they congruent? An affirmative answer to this question would yield the following result: if the centrally-symmetric C^∞ ovaloid S_0 imbedded in E^3 has curvature K sufficiently close to 1 pointwise, then it is almost spherical in the sense of the theorem of the previous section. Namely, as we shall show below, S_0 can be mapped conformally onto Σ so that K becomes an even function on Σ. We can then apply the theorem of the previous section to construct an almost-spherical centrally-symmetric ovaloid S and a conformal mapping of S onto Σ preserving antipodes. The induced conformal mapping of S onto S_0 would therefore preserve curvature and antipodes, and hence S_0 would be almost spherical.

The possibility of mapping S_0 conformally onto Σ so that K, considered as a function on Σ, is symmetric will be clearly guaranteed, if we can prove that there exists a conformal mapping of S_0 onto Σ carrying antipodes on S_0 into antipodes on Σ. The proof which follows makes use of the group-theoretic properties of conformal mappings.

Lemma. *A centrally-symmetric surface S in E^3, of genus zero and class C^3, can be mapped onto the sphere Σ conformally and preserving antipodes.*

Proof. Since the metric on S is of class C^2, the isothermic local coodinate systems define a Riemann surface structure on S. The possibility of mapping S conformally onto Σ is therefore guaranteed by the Uniformization Theorem. Now pick such a conformal mapping $\varphi\colon S \to \Sigma$, and let $\sigma\colon S \to S$ denote the anticonformal mapping which assigns to every point on S its antipode with respect to the center. Then the mapping $h = \varphi\sigma\varphi^{-1}\colon \Sigma \to \Sigma$ is an anticonformal involution: $h^2 = \varphi\sigma\varphi^{-1}\varphi\sigma\varphi^{-1} =$ identity. Let $r\colon \Sigma \to \Sigma$ denote the reflection of Σ in its center. The mapping $rh\colon \Sigma \to \Sigma$, being the product of two anticonformal mappings, is a conformal mapping of the sphere onto itself, i.e., a Möbius transformation of Σ. It follows that we may set $h = g_1 r$, where g_1 denotes a certain Möbius transformation of Σ. Since h is an involution,

$$g_1 r g_1 r = \text{identity} .$$

This last relation imposes restrictions on the form which g_1 may assume. We seek to determine the admissible forms of g_1. We shall represent conformal and anticonformal mappings of the sphere Σ onto itself, in the usual way, through stereographic projection from the north pole onto the equator plane, as transformations of the extended complex z-plane. The totality of Möbius transformations is given by

$$(3.1)\qquad z \to (az+b)/(cz+d)\ , \qquad ad - bc \neq 0\ .$$

The reflection r is represented by the transformation $z \to -1/\bar{z}$.

If g_1 is given by (3.1), an elementary direct calculation easily shows that $g_1 r g_1 r =$ identity implies

$$(3.2)\qquad |b|^2 - a\bar{d} = |c|^2 - \bar{a}d \neq 0\ ,$$

$$(3.3)\qquad a\bar{c} - \bar{a}b = 0\ ,$$

$$(3.4)\qquad d\bar{b} - \bar{d}c = 0\ .$$

The mapping $\varphi\sigma\varphi^{-1} = g_1 r\colon \Sigma \to \Sigma$ clearly has no fixed points. This implies $a \neq 0$, since otherwise 0 would be a fixed point. Also $d \neq 0$, since otherwise ∞ would be a fixed point. Thus we may assume that $d = 1$. From (3.2) it follows then that a is real. Furthermore, from (3.4) we have either $b = c = 0$ or $b = \rho e^{i\varphi_1}$, $c = \rho e^{i\varphi_2}$ with $\varphi_1 + \varphi_2 = 2k\pi$, k an integer.

Since the mapping $g_1 r$ has no fixed points, the equation

$$(3.5)\qquad (az+b)/(cz+1) = -1/\bar{z}$$

has no solution z. From this fact we shall derive necessary conditions for the numbers a, b, c. Setting $z = |z|e^{i\alpha}$, we consider the equivalent equation

$$(3.6)\qquad a|z|^2 + 1 = -|z|\rho(e^{i(\varphi_1-\alpha)} + e^{i(\varphi_2+\alpha)})\ .$$

If $b = c = 0$, i.e., $\rho = 0$, we must have $a > 0$, since otherwise any z with $|z|^2 = -1/a$ would be a solution of (3.5). If $\rho \neq 0$, we have $\varphi_1 + \varphi_2 = 2k\pi$ and we may write, instead of (3.6),

$$(3.7)\qquad (a|z|^2 + 1)/(2\rho|z|) = -\cos(\varphi_1 - \alpha)\ ,$$

from which it follows again that $a > 0$; otherwise, $z = \sqrt{-1/a}\cdot e^{i\alpha}$ and $\cos(\varphi_1 - \alpha) = 0$ would be a solution. Furthermore, we must have

$$|(a|z|^2+1)/(2\rho|z|)| = (a|z|^2+1)/(2\rho|z|) > 1\ ,$$

or

$$(3.8)\qquad a|z|^2 - 2\rho|z| + 1 > 0\ , \qquad \text{for all } z\ ,$$

which is satisfied for all z if and only if

$$(3.9)\qquad a - \rho^2 > 0\ .$$

Thus condition (3.9) must be satisfied by the coefficients of

$$g_1 \colon z \to (az + b)/(cz + 1) , \quad b = \rho e^{i\varphi_1} , \quad c = \rho e^{-i\varphi_1} .$$

We now consider the equation

$$g\varphi\sigma\varphi^{-1}g^{-1} = r \tag{3.10}$$

for the unknown Möbius transformation $g \colon \Sigma \to \Sigma$, and maintain that if we can find a conformal g satisfying (3.10), then the conformal mapping $\psi = g\varphi \colon S \to \Sigma$ maps antipodes on S into antipodes on Σ, as desired. Indeed, from (3.10) we deduce $\psi\sigma = r\psi$, which, in turn, gives

$$\psi(\bar{P}) = \psi\sigma(P) = r\psi(P) = \overline{\psi(P)} ,$$

where P is any point on S, and the bar denotes reflection in the pertinent center.

Thus it remains to show that (3.10) has a conformal solution g. In accordance with our previous considerations, we may write this equation as

$$gg_1rg^{-1} = r . \tag{3.11}$$

We seek a solution g of (3.11) in the form

$$z \to x_1 z + x_2 , \qquad x_1 \neq 0 .$$

The inverse mapping g^{-1} is then given by

$$z \to \frac{1}{x_1} z - \frac{x_2}{x_1} .$$

Substituting for g^{-1}, r, g_1, g in (3.11) their representations, an elementary calculation easily shows that the validity of (3.11) for all z implies

$$c\bar{x}_1 + \bar{x}_2 = 0 , \tag{3.12}$$

$$bx_1 + x_2 = 0 , \tag{3.13}$$

$$a|x_1|^2 + bx_1\bar{x}_2 + c\bar{x}_1x_2 + |x_2|^2 = 1 . \tag{3.14}$$

(3.12) and (3.13) are equivalent, since $\bar{b} = c$. Using (3.12), (3.14) reduces to

$$x_1(a\bar{x}_1 + b\bar{x}_2) = 1 . \tag{3.15}$$

Setting $x_2 = -bx_1$ in (3.14) we obtain $|x_1|^2 = 1/(a - \rho^2) > 0$.

Thus any mapping

$$g \colon z \to x_1 z + x_2$$

with $|x_1|^2 = 1/(a - \rho^2)$, $x_2 = -bx_1$ is a solution of (3.11), and our object has been attained.

Added in proof. The existence part of the theorem in this paper has been recently generalized by J. Moser to the case of an arbitrary smooth symmetric K with $\max_{\Sigma} K > 0$ and will appear under the title "On a nonlinear problem in differential geometry". For recent related results consult J. L. Kazdan and F. W. Warner, *Integrability conditions for $\Delta u = k - Ke^{2u}$ with applications to Riemannian geometry*, Bull. Amer. Math. Soc. **77** (1971) 819–823, and the bibliography therein.

References

[1] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large,* Comm. Pure Appl. Math. **6** (1953) 337–394.

[2] S. Agmon, A. Douglis & L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* I, Comm. Pure Appl. Math. **12** (1959) 623–727.

[3] O. A. Ladyzhenskaia & N. N. Ural'tzeva, *On the smoothness of weak solutions of quasilinear equations in several variables and of variational problems,* Comm. Pure Appl. Math. **14** (1961) 481–495.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA