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EXTENDIBILITY AND TRANSVERSALITY

STEPHEN J. GREENFIELD & MICHAEL MENN

1. Introduction

In [1] Errett Bishop wrote: "It is thought that a manifold $M^{n+1} \subset C^n$ has, in general, the property that holomorphic functions in a neighborhood of M extend to be holomorphic in some fixed open set." In this paper we analyze Bishop's statement and discover an interpretation for "in general".

We say a subset K of C^n is extendible to a connected subset K' of C^n (with $K \subsetneqq K'$) if every function holomorphic about K extends to a holomorphic function defined in a neighborhood of K'.

In [5] conditions were obtained for a real $(n+k)$-dimensional submanifold M of C^n to be extendible to a set containing an open subset of C^n. These conditions were stated in terms of holomorphic and antiholomorphic vector fields on M and their Lie brackets.

But from the point of view of [8] the conditions mentioned above can be interpreted as restrictions on the $(n+k)$-jet of the map $i: M \to C^n$, where i is the inclusion of M in C^n. Careful examination of the restrictions on the jet of i reveals that "most" $(n+k)$-jets satisfy these restrictions; so, therefore, do "most" maps in C^m topology, for m large enough (verifying Bishop's remark). More precise statements of this are made in § 4, where a corollary on function algebras is also deduced.

In § 2 the notation and some of the main ideas of [8] are reviewed with special attention to the situation considered here. Computations comparing jets of maps and Lie brackets are done in § 3.

2. Singularities of maps of real manifolds into complex manifolds

If $\phi: X \to Y$ is a map of topological spaces and $x \in X$, then ϕ_x will denote the germ of ϕ at x. Let $\mathscr{F}(p,q) = \{\phi: R^p \to R^q \mid \phi \text{ is } C^\infty \text{ and } \phi(0) = 0\}$ and $J(p,q) = \{\phi_0 \mid \phi \in \mathscr{F}(p,q)\}$. If $\phi \in \mathscr{F}(p,q)$ or $\phi \in J(p,q)$, then $[\phi]^n$ will denote the set of germs at the origin of elements of $\mathscr{F}(p,q)$ which agree with ϕ up to and including order n. Let $J^n(p,q) = \{[\phi]^n \mid \phi \in J(p,q)\}$. $J^n(p,q)$ is a real finite dimensional vector space. $[\phi]^n$ will occasionally be abbreviated to ϕ.

Whenever m is an integer, $\mathscr{L}_m$ will denote the group of invertible germs in $J(m,m)$. There is a group action of $\mathscr{L}_p \times \mathscr{L}_q$ on $J^n(p,q)$; $(\alpha,\beta)([\phi]^n =$

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$[\beta\phi\alpha^{-1}]^n$. Similar definitions can be made in the complex case. Let $C\mathscr{F}(p, q) = \{\phi\colon C^p \to C^q \mid \phi$ is holomorphic and $\phi(0) = 0\}$, $CJ(p, q) = \{\phi_0 \mid \phi \in C\mathscr{F}(p, q)\}$, $CJ^n(p, q) = \{[\phi]^n \mid \phi \in CJ(p, q)\}$, and $C\mathscr{L}_m$ be the group of invertible germs in $CJ(m, m)$. $C\mathscr{L}_p \times C\mathscr{L}_q$ acts on $CJ^n(p, q)$.

By manifold we mean real C^∞ paracompact Hausdorff manifold. All maps of manifolds are C^∞. By complex manifold we mean complex analytic paracompact Hausdorff manifold. Maps of complex manifolds are holomorphic.

Let $U \subset R^p(U \subset C^p)$ be open and let $\phi\colon U \to R^q(\phi\colon U \to C^q)$. Define $t_\phi\colon U \to J(p, q)(t_\phi\colon U \to CJ(p, q))$ by $t_\phi(x)$ to be the germ at the origin of $y \to \phi(x + y) - \phi(x)$. The projection of t_ϕ onto $J^n(p, q)(CJ^n(p, q))$ will also be written t_ϕ.

Let $\tilde{\mathscr{L}}_m(C\tilde{\mathscr{L}}_m)$ be a subgroup of $\mathscr{L}_m(C\mathscr{L}_m)$. Suppose M is an m-dimensional (complex) manifold and Q is an atlas of coordinate functions for M. The pair (M, Q) will be called a (complex) manifold of type $\tilde{\mathscr{L}}_m(C\tilde{\mathscr{L}}_m)$ if $t_{\alpha_2\alpha_1^{-1}}(\alpha_1(x)) \in \tilde{\mathscr{L}}_m(C\tilde{\mathscr{L}}_m)$ for all $x \in M$ and coordinate functions $\alpha_1, \alpha_2 \in Q$ whose domain contains x. The atlas Q will be suppressed from the notation.

If X is a (complex) p-manifold and Y is a (complex) q-manifold, then $J^n(X, Y)(CJ^n(X, Y))$ will denote the fiber bundle with base $X \times Y$, fiber $J^n(p, q)(CJ^n(p, q))$ and group $\mathscr{L}_p \times \mathscr{L}_q(C\mathscr{L}_p \times C\mathscr{L}_q)$. If X is a (complex) manifold of type $\tilde{\mathscr{L}}_p(C\tilde{\mathscr{L}}_p)$ and Y is a (complex) manifold of type $\tilde{\mathscr{L}}_q(C\tilde{\mathscr{L}}_q)$, then the group of $J^n(X, Y)(CJ^n(X, Y))$ is reducible to $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q(C\tilde{\mathscr{L}}_p \times C\tilde{\mathscr{L}}_q)$.

Let X and Y be manifolds of type $\tilde{\mathscr{L}}_p$ and $\tilde{\mathscr{L}}_q$, respectively. If $A \subset J^n(p, q)$ and is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$, then A determines a subbundle $J^n(X, Y; A)$ of $J^n(X, Y)$. If A is a submanifold of $J^n(p, q)$, then $J^n(X, Y; A)$ is a submanifold of $J^n(X, Y)$. Furthermore, the codimension of $J^n(X, Y; A)$ on $J^n(X, Y)$ is the codimension of A in $J^n(p, q)$.

$J^n(X, Y)$ may be looked at as the set of n-equivalence classes of germs of maps of X into Y where two germs are n-equivalent if they agree to order n. If $f\colon X \to Y$ and $x \in X$, let $f^n(x)$ be the n-equivalence class containing the germ of f at x. Thus a map $f\colon X \to Y$ induces a commutative triangle:

$$\begin{tikzcd} & J^n(X, Y) \arrow[d] \\ X \arrow[ur, "f^n"] \arrow[r, "(id, f)"'] & X \times Y \end{tikzcd}$$

Let $A(f)$, the singular set of f of type A, be defined by $A(f) = (f^n)^{-1}J^n(X, Y; A)$. If f is such that f^n is transversal to $J^n(X, Y; A)$, then f will be called A-transversal. If f is A-transversal, then $A(f)$ is a submanifold of X with codimension equal to that of A in $J^n(p, q)$. Similar definitions and statements may be made in the complex case.

If $f\colon X \to Y$, let $Tf\colon TX \to TY$ be the induced map of tangent bundles.

If $(a_1, \cdots, a_m)$ is a tuple of integers with $0 \leq a_m \leq \cdots \leq a_1$, define $P(a_1, \cdots, a_m)$ to be the dimension of the symmetric product $\boldsymbol{R}^{a_m} \circ \cdots \circ \boldsymbol{R}^{a_1}$ (see [8, § 6] for a definition of the symmetric product).

Theorem 2.1. *Let p and q be positive integers. It is possible to assign to each tuple $(a_1, \cdots, a_n)$ of nonnegative integers, with $a_1 \geq p - q$ and $a_1 \geq \cdots \geq a_n$, a submanifold $Z(a_1, \cdots, a_n)$ of $J^n(p, q)$ in such a way that*

i) *each $Z(a_1, \cdots, a_n)$ is invariant undr $\mathscr{L}_p \times \mathscr{L}_q$,*

ii) *if $f: X \to Y$ is a map of a p-manifold into a q-manifold, then $Z(a)(f) = \{x \in X \mid \text{dimension kernel } Tf_x = a\}$,*

iii) *if $f: X \to Y$ is a $Z(a_1, \cdots, a_m)$-transversal map of a p-manifold into a q-manifold (so $Z(a_1, \cdots, a_m)(f)$ is a manifold), then $Z(a_1, \cdots, a_m, a_{m+1})(f) = \{x \in Z(a_1, \cdots, a_m)(f) \mid \text{dimension } (\text{kernel } Tf_x \cap TZ(a_1, \cdots, a_m)(f)_x) = a_{m+1}\}$,*

iv) *if $f: X \to Y$ is $Z(a)$-transversal, then the codimension of $Z(a)(f)$ in X is $a(q - p + a)$. If $m \geq 2$ and f is $Z(a_1, \cdots, a_{m-1})$-transversal and $Z(a_1, \cdots, a_m)$-transversal, then the codimension of $Z(a_1, \cdots, a_m)(f)$ in $Z(a_1, \cdots, a_{m-1})(f)$ is $P(a_1, \cdots, a_m)(q - p + a_1) - \sum_{i=2}^{m} P(a_i, \cdots, a_m)(a_{i-1} - a_i)$.*

For a proof, see [2] or [8].

It is possible to define complex submanifolds $\boldsymbol{C}Z(a_1, \cdots, a_n)$ of $\boldsymbol{C}J^n(p, q)$ which are invariant under $\boldsymbol{C}\mathscr{L}_p \times \boldsymbol{C}\mathscr{L}_q$ behaving analogously to the $Z(a_1, \cdots, a_n)$ with respect to holomorphic maps of complex manifolds. The proof is formally identical to that of Theorem 2.1.

If X and Y are manifolds, let $C^m(X, Y)$ denote the set of C^∞ maps of X into Y, provided with the topology of compact convergence of all partials of order less than or equal to n.

Let B be a submanifold of $J^n(X, Y)$. Then, according to the Thom transversality theorem, $\{f: X \to Y \mid f^n \text{ is transversal to } B\}$ is dense (in fact, a Baire set) in $C^{n+1}(X, Y)$. If X is compact, this set is open as well as dense in $C^{n+1}(X, Y)$. See [7] for a proof of the transversality theorem.

If $f: X \to \boldsymbol{R}^q$ (or $f: X \to \boldsymbol{C}^q$), then f_j will denote the jth coordinate function of f. If $\phi: \boldsymbol{R}^{2p} \to \boldsymbol{R}^{2q}$, define $\hat{\phi}: \boldsymbol{C}^p \to \boldsymbol{C}^q$ by $\hat{\phi}_j(x_1^1 + ix_2^1, \cdots, x_1^p + ix_2^p) = \phi_j(x_1^1, \cdots, x_1^p, x_2^1, \cdots, x_2^p) + i\phi_{q+j}(x_1^1, \cdots, x_1^p, x_2^1, \cdots, x_2^p)$. (Note that $\hat{\phi}$ is not necessarily holomorphic.) If $S \subset \boldsymbol{C}J(p, q)$, let $\check{S} = \{\phi \in J(2p, 2q) \mid \hat{\phi} \in S\}$. A real $2q$-manifold Y is a complex q-manifold if and only if Y is a manifold of type $(\boldsymbol{C}\mathscr{L}_q)^\vee$.

If $P: \boldsymbol{R}^p \to \boldsymbol{R}^{2q}$ is a polynomial with $P_j(x_1, \cdots, x_p) = \sum_{j_1, \cdots, j_p} a^j_{j_1, \ldots, j_p} x_1^{j_1} \cdots x_p^{j_p}$, define $\rho(P): \boldsymbol{C}^p \to \boldsymbol{C}^q$ by

$$(\rho P)_j(z_1, \cdots, z_p) = \textstyle\sum_{j_1, \ldots, j_p} (a^j_{j_1, \ldots, j_p} + ia^{q+j}_{j_1, \ldots, j_p}) z_1^{j_1} \cdots z_p^{j_p} .$$

The function ρ induces a map $J^n(p, 2q) \to \boldsymbol{C}J^n(p, q)$ also denoted by ρ. This map is an isomorphism of real vector spaces. If A is a submanifold of $\boldsymbol{C}J^n(p, q)$ then, since ρ is an isomorphism, $\rho^{-1}(A)$ is a submanifold of $J^n(p, 2q)$. It is easy to show that if A is invariant under $C\mathscr{L}_p \times C\tilde{\mathscr{L}}_q$, then $\rho^{-1}(A)$ is invariant under $\mathscr{L}_p \times (C\tilde{\mathscr{L}}_q)^\vee$.

Thus if X is a p-manifold, Y is a complex q-manifold, $a_1 \geq p - q$ and $a_1 \geq \cdots \geq a_n \geq 0$, then $J^n(X, Y; \rho^{-1}CZ(a_1, \cdots, a_n))$ is a submanifold of $J^n(X, Y)$.

Let X and Y be as above and let $f\colon X \to Y$ be C^∞ (as a map of real manifolds). It is immediate that $\rho^{-1}CZ(a_1)(f) = \{x \in X \mid \text{the complex span of } Tf(TX_x)$ is a $(p - a_1)$-dimensional complex subspace of $TY_{f(x)}\}$. Suppose $p \leq 2q$ so that it is possible for $Z(0)(f)$ to be nonempty. From the fact that $Z(0)(f)$ is open in X, it follows that if f is $\rho^{-1}CZ(a_1)(f)$-transversal, then $Z(0)(f) \cap \rho^{-1}CZ(a_1)(f)$ is a submanifold of X with codimension $2a_1(q - p + a_1)$. Define a vector subbundle K of TX over $Z(0)(f) \cap \rho^{-1}CZ(a_1)(f)$ by $K = \{v \mid v \in TX_x$ for some $x \in Z(0)(f) \cap \rho^{-1}CZ(a_1)(f)$ and $iTf(v) \in Tf(TX_x)\}$. The fiber of K is $2a_1$-dimensional. Define $\alpha\colon K \to K$ by $Tf(\alpha(v)) = iTf(v)$.

R^{2q} will be identified with C^q by associating the tuple $(a_1 + ib_1, \cdots, a_q + ib_q)$ with the tuple $(a_1, \cdots, a_q, b_1, \cdots, b_q)$. We will need the following computational facts about ρ: Let $f \in \mathscr{F}(p, 2q)$ be a polynomial and let $v, w \in TR_0^p$. Let $\rho\colon J^n(p, 2q) \to CJ^n(p, q)$ be as above. Then it is simple to show:

i) $T(\rho f)(v + iw) = Tf(v) + iTf(w)$,

ii) $Tt_{\rho f}(v + iw) = T_\rho Tt_f(v) + iT_\rho Tt_f(w)$.

Proposition 2.2. *Let X be a real p-manifold, Y be a complex q-manifold, and $F\colon X \to Y$ be $\rho^{-1}CZ(a_1, \cdots, a_m)$-transversal. If $x \in Z(0)(f) \cap \rho^{-1}CZ(a_1, \cdots, a_m)(f)$, let $W_x = \{v \in K_x \mid v$ and $\alpha(v)$ both are elements of $T\rho^{-1}CZ(a_1, \cdots, a_m)(f)\}$. Let $V = \{x \in Z(0)(f) \cap \rho^{-1}CZ(a_1, \cdots, a_m)(f) \mid \text{dimension } W_x = 2a_{m+1}\}$. Then $V \subset \bigcup_{b \geq a_{m+1}} \rho^{-1}CZ(a_1, \cdots, a_m, b)(f)$.*

Proof. This is a local question. Suppose $X = R^p$, $Y = C^q = R^{2q}$, $f\colon R^p \to C^q$ is a $\rho^{-1}CZ(a_1, \cdots, a_m)$-transversal polynomial, and $0 \in V$. Let $v_1, \cdots, v_{a_{m+1}} \in TR_0^p$ be such that $W_0 = \text{span } \{v_1, \cdots, v_{a_{m+1}}, \alpha(v_1), \cdots, \alpha(v_{a_{m+1}})\}$. It follows from i) that for $j = 1, \cdots, a_{m+1}$, $T(\rho f)(v_j + i\alpha(v_j)) = Tf(v_j) + iTf(\alpha(v_j)) = 0$.

We will show that $v_j + i\alpha(v_j) \in \text{kernel } T(\rho f)_0 \cap TCZ(a_1, \cdots, a_m)(\rho f)_0$ for each j so that the complex dimension of kernel $T(\rho f)_0 \cap TCZ(a_1, \cdots, a_m)(\rho f)_0$ is at least a_{m+1}. If we also show that ρf is $CZ(a_1, \cdots, a_m)$-transversal at 0, then the result will follow from the complex analogue of Theorem 2.1.

$J^m(R^p, R^{2q}) = R^p \times R^{2q} \times J^m(p, 2q)$, and t_f is the projection of f^m onto $J^m(p, 2q)$. Thus $\rho^{-1}CZ(a_1, \cdots, a_m)(f) = t_f^{-1}(\rho^{-1}CZ(a_1, \cdots, a_m))$, and t_f is transversal to $\rho^{-1}(CZ(a_1, \cdots, a_m))$. If $v, w \in TR_0^p$, then $Tt_{\rho f}(v + iw) = T_\rho Tt_f(v) + iT_\rho Tt_f(w)$. That $t_{\rho f}$ is transversal to $CZ(a_1, \cdots, a_m)$ at 0 follows from the fact that t_f is transversal to $\rho^{-1}CZ(a_1, \cdots, a_m)$. Thus $v + iw \in TCZ(a_1, \cdots, a_m)(\rho f)$ if and only if $Tt_{\rho f}(v + iw) \in TCZ(a_1, \cdots, a_m)$. But for $j = 1, \cdots, m$, $Tt_{\rho f}(v_j + i\alpha(v_j)) = T_\rho Tt_f(v_j) + iT_\rho Tt_f(\alpha(v_j))$. Since v_j and $\alpha(v_j)$ both are elements of $T\rho^{-1}CZ(a_1, \cdots, a_m)(f)$, $Tt_f(v_j)$ and $Tt_f(a(v_j))$ are elements of $T\rho^{-1}CZ(a_1, \cdots, a_m)$. Thus $Tt_{\rho f}(v_j + i\alpha(v_j)) \in TCZ(a_1, \cdots, a_m)$, and $v_j +$

$i\alpha(v_j) \in TCZ(a_1, \cdots, a_m)(\rho f)$. Hence the proposition is proved.

Example 2.3. Let $f\colon \boldsymbol{R}^2 \to \boldsymbol{C}^2$ be defined by $f(x, y) = (x + iy, i(x^2 + y^2))$. f is $\rho^{-1}CZ(1)$-transversal. Furthermore, $0 \in Z(0)(f) \cap \rho^{-1}CZ(1, 1)(f)$, but $W_0 \cap TZ(0)(f) = \{0\}$ since $TZ(0)(f) = \{0\}$. It follows that the inclusion $V \subset \bigcup_{b > a_{m+1}} \rho^{-1}CZ(a_1, \cdots, a_m, b)(f)$ of Proposition 2.2 cannot be replaced by $V \subset \rho^{-1}CZ(a_1, \cdots, a_{m+1})$.

It is possible, despite Example 2.3, to interpret the sets $\rho^{-1}CZ(a_1, \cdots, a_{m+1})(f)$ (for suitably transversal f) in a more precise fashion than Proposition 2.2. This would, however, take space. The point we are trying to make here is that the singular types constructed in [8] give rise to singular types of maps of real manifolds into complex manifolds.

3. Lie brackets

If U is an open subset of $\boldsymbol{R}^p$, then $\phi\colon U \to \boldsymbol{R}^q$ and $x \in U$ define $D\phi_x\colon \boldsymbol{R}^p \to \boldsymbol{R}^q$ by $T\phi(v_x) = (D\phi_x(v))_{\phi(x)}$. $D\phi$ will abbreviate $D\phi_0$. Let $\Sigma \subset J^n(p, q)$ be open, and E_1, E_2, B be vector subbundles of $\Sigma \times \boldsymbol{R}^p$. Define F by the exactness of $0 \to B \to \Sigma \times \boldsymbol{R}^p \to F \to 0$. Let $\pi\colon J^{n+1}(p, q) \to J^n(p, q)$ be the projection.

If s and t are nonnegative integers, let $M(s, t)$ denote the set of linear maps from $\boldsymbol{R}^s$ to $\boldsymbol{R}^t$. Give $M(s, t)$ the usual structure as a real vector space, so we may identify $M(s, t)$ with $\boldsymbol{R}^{st}$.

Suppose that the fiber dimension of E_i is $e(i)$. Let $\phi \in \mathscr{F}(p, q)$ be such that $[\phi]^n \in \Sigma$, and U be a neighborhood of $[\phi]^n$ in Σ such that E_1 and E_2 are both trivial over U. Then there are bundle equivalences $\delta_i\colon U \times \boldsymbol{R}^{e(i)} \to E_i/U$. Define C^∞ maps $C_i\colon U \to M(e(i), p)$ by $\delta_i([\psi]^n, v) = ([\psi]^n, C_i([\psi]^n)(v))$. $C_i([\psi]^n)$ has rank $e(i)$ and its image is $\{w \in \boldsymbol{R}^p \mid ([\psi]^n, w) \in E_i\}$. Straightforward linear algebra shows that there are an integer N and smooth functions $A_i\colon U \to M(p, N)$ such that $([\psi]^n, v) \in E_i$ if and only if $A_i([\psi]^n)(v) = 0$.

Let $v_i\colon U \to E_i$ be sections for $i = 1, 2$. Recall that since $\phi \in \mathscr{F}(p, q)$ there is a map $t_\phi\colon \boldsymbol{R}^p \to J^n(p, q)$. The sections v_i are pulled back to sections $t_\phi^* v_i$ of $t_\phi^* E_i$ over $t_\phi^{-1}(U)$. Note that the bundles $t_\phi^* E_i$ and $t_\phi^* B$ are equivalent to subbundles of $T\boldsymbol{R}^p$ over $t_\phi^{-1}(U)$. Furthermore, there is an exact sequence $0 \to t_\phi^* B \to T\boldsymbol{R}^p \xrightarrow{\varepsilon} t_\phi^* F \to 0$ over $t_\phi^{-1}(U)$.

Define $\bar{v}_i\colon t_\phi^{-1}(U) \to \boldsymbol{R}^p$ by: $t_\phi^* v_i(x) = (\bar{v}_i(x))_x$. $A_i(t_\phi(x)) \cdot \bar{v}_i(x)$ is zero for each $x \in t_\phi^{-1}(U)$. Consequently all directional derivatives of $A_i(t_\phi(\cdot))\bar{v}_i(\cdot)$ are 0. Thus $(D(A_1 \circ t_\phi)(\bar{v}_2(0))) \cdot \bar{v}_1(0) + A_1([\phi]^n) \cdot D\bar{v}_1(\bar{v}_2(0)) = 0$ and $(D(A_2 \circ t_\phi)(\bar{v}_1(0))) \cdot \bar{v}_2(0) + A_2([\phi]^n) \cdot D\bar{v}_2(\bar{v}_1(0)) = 0$. Since $D(A_i \circ t_\phi)$ is determined by $[\phi]^{n+1}$ and the kernel of $A_i([\phi]^n)$ is $\{v \mid v_0 \in (t_\phi^* E_i)_0\}$, it follows that the Lie bracket $[t_\phi^* v_1, t_\phi^* v_2](0)$ is determined up to $(t_\phi^* E_1 + t_\phi^* E_2)_0$ by $[\phi]^{n+1}$ and the $v_i([\phi]^n)$.

If we suppose that $E_i \subset B$ for $i = 1, 2$, then $\varepsilon([t_\phi^* v_1, t_\phi^* v_2](0))$ is determined by $[\phi]^{n+1}$ and $v_i([\phi]^n)$. $E_1^* \otimes E_2^* \otimes F = \{([\psi]^n, L) \mid [\psi]^n \in \Sigma$ and $L\colon (E_1)_{[\psi]^n} \times$

$(E_2)_{[\psi]^n} \to F_{[\psi]^n}$ is bilinear}. Thus, if each $E_i \subset B$, then Lie bracketing induces a morphism $\gamma\colon \pi^{-1}\Sigma \to E_1^* \otimes E_2^* \otimes F$ of fiber bundles over Σ. If a is less than or equal to the fiber dimension of F, define $\Sigma(\gamma, a)$ to be the set of points ψ in $\pi^{-1}\Sigma$ such that the linear map $(E_1)_{[\psi]^n} \otimes (E_2)_{[\psi]^n} \to F_{[\psi]^n}$ corresponding to $\gamma(\psi)$ has rank a.

A function $f\colon J^n(p, q) \to \boldsymbol{R}$ will be called a polynomial if, given some choice of vector space basis for $J^n(p, q)$, f is a polynomial in the coordinate functions of $J^n(p, q)$. A function $g\colon J^n(p, q) \to \boldsymbol{R}^s$ will be called a polynomial if each coordinate projection of g is a polynomial.

Suppose Σ is such that there is a polynomial $g\colon J^n(p, q) \to \boldsymbol{R}^N$ such that $\Sigma = \{[\phi]^n \mid g([\phi]^n) \neq 0\}$. Let U be a vector subbundle of $\Sigma \times \boldsymbol{R}^p$. We will say that U is polynomially determined if there are an integer K and a polynomial function $G\colon J^n(p, q) \to M(p, K)$ such that for $[\psi]^n \in \Sigma$, then $([\psi]^n, v) \in U$ if and only if $G([\psi]^n)\cdot v = 0$. It is apparent that if the bundles E_1, E_2 and B are polynomially determined, each $\Sigma(\gamma, a)$ is determined by polynomial equalities and inequalities. If a is maximal with respect to the property that $\Sigma(\gamma, a) \neq \phi$, then there is a polynomial h on $J^{n+1}_{(p,q)}$ such that $[\psi]^{n+1} \in \Sigma(\gamma, a)$ if and only if $h([\psi]^{n+1}) \neq 0$. Consequently, $\Sigma(\gamma, a)$ is open.

Now suppose that $\tilde{\mathscr{L}}_p \subset \mathscr{L}_p$ and $\tilde{\mathscr{L}}_q \subset \mathscr{L}_q$ are subgroups, and that Σ is invariant under the action of $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Define an action of $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$ on $\Sigma \times \boldsymbol{R}^p$ by $(\alpha, \beta)([\phi]^n, v) = ([\beta\phi\alpha^{-1}]^n, D\alpha(v))$, and suppose that E_1, E_2 and B are invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. The actions of $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$ on $\Sigma \times \boldsymbol{R}^p$ and B determine an action on F. The actions on E_1, E_2 and F determine an action on $E_1^* \otimes E_2^* \otimes F$ as follows: an element of $E_1^* \otimes E_2^* \otimes F$ is a pair $([\phi]^n, L)$ where $[\phi]^n \in \Sigma$ and $L\colon (E_1)_{[\phi]^n} \times (E_2)_{[\phi]^n} \to F_{[\phi]^n}$ is bilinear. Define $(\alpha, \beta)([\phi]^n, L) = ([\beta\phi\alpha^{-1}]^n, (\alpha, \beta)L)$ where $(\alpha, \beta)L$ is defined by $((\alpha, \beta)L)(([\beta\phi\alpha^{-1}]^n, D\alpha v), ([\beta\phi\alpha^{-1}]^n, D\alpha w))(\alpha, \beta)(L(([\phi]^n, v), ([\phi]^n, w)))$. We now show that γ is equivariant thereby showing that $\Sigma(\gamma, a)$ is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$.

Let U, open in Σ, be such that E_1 and E_2 are trivial over U, and let $v_i\colon U \to E_i$ be sections. If $(\alpha, \beta) \in \tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$ then, for $i = 1, 2$, $(\alpha, \beta)v_i$ is a section of E_i over $(\alpha, \beta)U$. Since $(t^*_{\beta\phi\alpha^{-1}}(\alpha, \beta)v_i)(\alpha(x)) = T\alpha(t^*_\phi v_i)(x))$, it follows that

$$[t^*_{\beta\phi\alpha^{-1}}(\alpha, \beta)v_1, t^*_{\beta\phi\alpha^{-1}}(\alpha, \beta)v_2](0) = T\alpha[t^*_\phi v_1, t^*_\phi v_2](0) .$$

The equivariance of γ is now immediate.

Since $\Sigma(\gamma, a)$ is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$ and is determined by polynomial equalities and inequalities, it may (see [3]) be written as a finite union of disjoint manifolds each of which is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$.

Let X be a manifold of type $\tilde{\mathscr{L}}_p$, and Y a manifold of type $\tilde{\mathscr{L}}_q$. Then $J^{n+1}(X, Y; \Sigma(\gamma, a))$ is a finite union of disjoint manifolds. If a is maximal with respect to the property that $\Sigma(\gamma, a) \neq \phi$ then $\cup_{b<a} J^{n+1}(X, Y; \Sigma(\gamma, b))$ is a finite union of disjoint manifolds, each of which has positive codimension in $J^{n+1}(X, Y)$. Thus, if $f\colon X \to Y$ is such that f^{n+1} is transversal to each of these

manifolds, then $X \sim \Sigma(\gamma, a)(f)$ is a finite union of manifolds of dimension less than p.

Let $A_1(A_2)$ be a maximal atlas of coordinate functions for $X(Y)$ such that if $\alpha_1, \alpha_2 \in A_1(A_2)$ and x belongs to the domain of both α_1 and α_2, then $t_{\alpha_2\alpha_1^{-1}}(\alpha_1(x)) \in \tilde{\mathscr{L}}_p(\tilde{\mathscr{L}}_q)$. Let $p_1: X \times Y \to X$ and $n: J^n(X, Y) \to X \times Y$ be the projections. We will define for $i = 1, 2$ a vector subbundle $E_i(X, Y)$ of $n^*p_1{}^*TX$ over $J^n(X, Y; \Sigma)$, which corresponds to E_i. An element of $n^*p_1{}^*TX$ over Σ is a pair (ϕ, v) where $\phi \in J^n(X, Y; \Sigma)$ and $v \in TXp_{1n(\phi)}$. Let $n(\phi) = (x, y)$, $\alpha \in A_1$ be such that $\alpha(x) = 0$, and $\beta \in A_2$ be such that $\beta(y) = 0$. Then $\beta\phi\alpha^{-1} \in \Sigma$. Let $T\alpha(v) = w(v, \alpha)_0$, and define $E_i(X, Y) = \{(\phi, y) \in n^*p_1{}^*TX \mid (\beta\phi\alpha^{-1}, w(v, \alpha)) \in E_i\}$. This definition is independent of the choices of α and β. We may, in a similar fashion, define a vector subbundle $B(X, Y)$ and a factor bundle $F(X, Y)$ of $n^*p_1{}^*TX$ over $J^n(X, Y; \Sigma)$, which correspond respectively to B and F.

The equivariance of γ ensures that γ induces a morphism of fiber bundles, $J^{n+1}(X, Y; \pi^{-1}\Sigma) \to E_1(X, Y)^* \otimes E_2(X, Y)^* \otimes F(X, Y)$, which will also be denoted γ. If $f: X \to Y$, then $E_i(f)$ (respectively $B(f)$, $F(f)$) will denote $f^{n*}E_i(X, Y)$ (respectively $f^{n*}B(X, Y)$, $f^{n*}F(X, Y)$) over $\Sigma(f)$. γ induces a section $\sigma(f): \Sigma(f) \to E_1(f)^* \otimes E_2(f)^* \otimes F(f)$ defined by $f^{n+1*}\sigma(f)(x) = \gamma(f^{n+1}(x))$. $\sigma(f)$ is induced by Lie-bracketing vector fields in $E_1(f)$ with vector fields in $E_2(f)$ and projecting onto $F(f)$, i.e., if $v_i: \Sigma(f) \to E_i(f)$ are sections, then $\sigma(f)(x)(v_1(x) \otimes v_2(x))$ is the projection of $[v_1, v_2](x)$ on $F(f)$. If $x \in \Sigma(f)$, let $L_x(f) = \{[v_1, v_2](x) \mid v_i$ is a section of $E_i(f)\}$. Then $\Sigma(\gamma, b)(f) = \{x \in \Sigma(f) \mid \dim(L_x + B(f)_x) = b + \dim B(f)_x\}$. If a is maximal with respect to the property that $\Sigma(\gamma, a) \neq \phi$, then $J^{n+1}(p, q) \sim \Sigma(\gamma, a)$ may be written as $\bigcup_{i=1}^r M_i$ where each M_i is a manifold invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. If f is M_i-transversal for each i, then $X \sim \Sigma(\gamma, a)(f)$ is a finite union of disjoint manifolds of dimension less than p.

We now summarize.

Theorem 3.1. *Let $g: J^n(p, q) \to \boldsymbol{R}^N$ be a polynomial, and let $\Sigma = \{[\phi]^n \mid g([\phi]^n) \neq 0\}$. Let $\tilde{\mathscr{L}}_p \subset \mathscr{L}_p$ and $\tilde{\mathscr{L}}_q \subset \mathscr{L}_q$ be subgroups. Suppose that Σ is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$, and further that E_1, E_2 and B are polynomially determined vector subbundles of $\Sigma \times \boldsymbol{R}^p$, which are invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Define F by the exactness of $0 \to B \to \Sigma \times \boldsymbol{R}^p \to F \to 0$. Let $\pi: J^{n+1}(p, q) \to J^n(p, q)$ be the projection, and assume that $E_1 + E_2 \subset B$. Then Lie-bracketing of vector fields in E_1 with vector fields in E_2 induces a map $\gamma: \pi^{-1}\Sigma \to E_1^* \otimes E_2^* \otimes F$, i.e., γ assigns to each $[\phi]^{n+1} \in \pi^{-1}\Sigma$ a linear map $\gamma([\phi]^{n+1}): (E_1 \otimes E_2)_{[\phi]^n} \to F_{[\phi]^n}$. γ is equivariant. If b is a nonnegative integer, let $\Sigma(\gamma, b) = \{[\phi]^{n+1} \in \pi^{-1}\Sigma \mid$ image $\gamma([\phi]^{n+1})$ has rank $b\}$. Each $\Sigma(\gamma, b)$ is a union of a finite number of submanifolds of $J^{n+1}(p, q)$ each of which is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Define $\tilde{B}$, a bundle over $\Sigma(\gamma, b)$, by $\tilde{B} = \{([\phi]^{n+1}, v + w) \mid ([\phi]^n, v) \in B$, and the projection of $([\phi]^n, w)$ on F is an element of the image of $\gamma([\phi]^{n+1})\}$. $\tilde{B}$ is polynomially determined and is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Let a be maximal with respect to the property that $\Sigma(\gamma, a) \neq \phi$.*

There is a polynomial h on $J^{n+1}(p, q)$ *such that* $\Sigma(\gamma, a) = \{[\phi]^{n+1} \mid h([\phi]^{n+1}) \neq 0\}$.

Let X be a manifold of type $\tilde{\mathscr{L}}_p$, *and Y a manifold of type* $\tilde{\mathscr{L}}_q$. *The bundles* E_i *and B induce bundles* $E_i(X, Y)$ *and* $B(X, Y)$ *over* $J^n(X, Y; \Sigma)$ *and hence induce bundles* $E_i(f)$ *and* $B(f)$ *over* $\Sigma(f)$ *for* $f\colon X \to Y$. *If* $x \in \Sigma(f)$, *let* $L_x(f) = \{[v_1, v_2](x) \mid v_i$ *is a section of* $E_i(f)\}$. *Then* $\Sigma(\gamma, b)(f) = \{x \in \Sigma(f) \mid$ *dimension* $(L_x + B(f)_x) = b +$ *fiber dimension* $B\}$. $J^{n+1}(X, Y) \sim J^{n+1}(X, Y; \Sigma(\gamma, a))$ *may be written as a finite union of manifolds of positive codimension in* $J^{n+1}(X, Y)$. *If* $f\colon X \to Y$ *is such that* f^{n+1} *is transversal to each of these manifolds, then* $\{x \in X \mid x \notin \Sigma(f)$ *or* $\dim(L_x + B(f)_x) \neq a +$ *fiber dimension* $B\}$ *is a finite union of manifolds of dimension less than p.*

The set of functions obeying the above transversality conditions is a Baire set in $C^{n+2}(X, Y)$, and is open and dense if X is compact.

Corollary 3.2. *Let* $p > q$, *X be a real p-manifold, and Y be a complex q-manifold. If* $f\colon X \to Y$ *and* $x \in X$, *let* $E_x(f) = \{v \in TX_x \mid iTf(v) \in Tf(TX_x)\}$ *and* $E(f) = \cup\{E_x(f) \mid x \in X\}$. *Let* $L(f)$ *be the Lie algebra of vector fields generated by vector fields in* $E(f)$. *If* $x \in X$, *let* $L_x(f) = \{v(x) \mid v \in L(f)\}$. *Let* $S(f) = \{x \in X \mid L_x(f) \neq TX_x\}$. *Then there are an integer m and a Baire set* $\mathscr{F}$ *(open and dense if X is compact) in* $C^m(X, Y)$ *such that if* $f \in \mathscr{F}$ *then* $S(f)$ *is contained in a finite union of manifolds of dimension less than p.*

Proof. Case 1, $p \geq 2q$: Let $\Sigma = \{[\phi]^{-1} \in J^1(p, 2q) \mid T\phi_0$ has rank $2q\}$. Straightforward linear algebra shows that if $f\colon X \to Y$ and $x \in \Sigma(f)$, then $E_x(f) = TX_x$. Let $\mathscr{F} = \{f\colon X \to Y \mid f$ is $Z(a)$-transversal for all $a\}$.

Case 2, $p < 2q$: Identify $\boldsymbol{R}^{2q}$ with $\boldsymbol{C}^q$, and let $\Sigma^1 = \{[\phi]^1 \in J^1(p, 2q) \mid T\phi_0$ has rank p and $T\phi(T\boldsymbol{R}^p_0) + iT\phi(T\boldsymbol{R}^p_0) = T\boldsymbol{C}^q_0\}$. There is a polynomial g^1 on $J^1(p, 2q)$ such that $[\phi]^1 \in \Sigma^1$ if and only if $g^1([\phi]^1) \neq 0$. Let $E^1 = \{([\phi]^1, v) \mid [\phi]^1 \in \Sigma^1$ and $T\phi(v_0) \in iT\phi(T\boldsymbol{R}^p_0)\}$. Now suppose that g^k is a polynomial on $J^k(p, 2q)$, $\Sigma^k = \{[\phi]^k \mid g^k([\phi]^k) \neq 0\}$, and E^k is a polynomially determined vector subbundle of $\Sigma^k \times \boldsymbol{R}^p$. Define F^k by the exactness of $0 \to E^k \to \Sigma^k \times \boldsymbol{R}^p \to F \to 0$, let $\pi^{k+1}\colon J^{k+1}_{(p,2q)} \to J^k_{(p,2q)}$ be the projection, and $\gamma^k\colon (\pi^{k+1})^{-1}\Sigma^k \to E^{k*} \otimes E^{k*} \otimes F^k$ be the map induced by Lie-bracketing. Let a^k be maximal with respect to the property that $\Sigma^k(\gamma^k, a^k) \neq \phi$. Define $\Sigma^{k+1} = \Sigma^k(\gamma^k, a^k)$, and let g^{k+1} be a polynomial on $J^{k+1}_{(p,2q)}$ such that $[\phi]^{k+1} \in \Sigma^{k+1}$ if and only if $g^{k+1}([\phi]^{k+1}) \neq 0$. Complete the inductive definition by defining $E^{k+1} = \{([\phi]^{k+1}, v + w) \in \Sigma^{k+1} \times \boldsymbol{R}^p \mid ([\phi]^k, v) \in E^k$ and the projection of $([\phi]^k, w)$ on F^k is in the image of $\gamma^k([\varphi]^{k+1})\}$. The proof will be complete if we can show that there is a k such that $E^k = \Sigma^k \times \boldsymbol{R}^p$ (for then we can choose $m = k + 1$). To show this it suffices to show that if $E^j \neq \Sigma^j \times \boldsymbol{R}^p$ then $a^j \neq 0$.

But suppose $E^j \neq \Sigma^j \times \boldsymbol{R}^p$ and $\phi\colon \boldsymbol{R}^p \to \boldsymbol{C}^q$ is such that $[\phi]^j \in \Sigma^j$. We may assume that $D\phi_0$ is given by

$$\left(\begin{array}{ccc|c} 1i & & 0 & \\ & \ddots & & 0 \\ 0 & & 1i & \\ \hline & 0 & & I_{2q-p} \end{array}\right)$$

where I_{2q-p} denotes the $(2q-p)\times(2q-p)$ identity matrix, and the matrix in the upper left hand corner has 1 for each $(k, 2k-1)$-entry and i for each $(k, 2k)$-entry. Let U be a small open neighborhood of the origin in $\boldsymbol{R}^p$. If $u\colon U \to \boldsymbol{R}^p$ defines a section $\tilde{u}\colon U \to T\boldsymbol{R}^p$ by $\tilde{u}(x) = u(x)_x$.

We may find functions $v, w\colon U \to \boldsymbol{R}^p$ such that

i) $v(0) = (1, 0, \cdots, 0)$,

ii) if $x \in U$, then $v_1(x) = 1$; and if $2 \leq k \leq 2p - 2q$, then $v_k(x) = 0$,

iii) if $x \in U$, then $iD\phi_x v(x) = D\phi_x w(x)$.

Define functions f and g from U to $\boldsymbol{R}^q$ by $\phi(x) = f(x) + ig(x)$. If $x \in U$, let $A(x)$ be the matrix consisting of the last $2q - p$ columns of Df_x, and $B(x)$ be the matrix consisting of the last $2q - p$ columns of Dg_x. Let $M(x)$ be the $(2q) \times (2q)$ matrix $\begin{pmatrix} B(X) & Df_x \\ A(X) & -Dg_x \end{pmatrix}$, and let $N(x)$ be the first column of $\begin{pmatrix} Dg_x \\ Df_x \end{pmatrix}$. If v, w obey i)–iii), then $M(x)\begin{pmatrix} v_{2p-2q+1}(x) \\ \vdots \\ v_p(x) \\ w_1(x) \\ \vdots \\ w_p(x) \end{pmatrix} + N(x) = 0$ for all $x \in U$.

Repeated differentiation of this matrix equation enables us to compute the derivatives of v and w in terms of the derivatives of f and g. In particular, if n is an integer, the nth order derivatives of v and w at the origin are determined by the $(n+1)$-jets of f and g at the origin. Also if $2p - 2q + 1 \leq k \leq p$, there are real numbers R_k and S_k depending only on $[\phi]^j$ such that

$$\frac{\partial^j w_k}{\partial x_1^j}(0) = -\frac{\partial^{j+1} f_k}{\partial x_1^j \partial x_2}(0) + \frac{\partial^{j+1} g_k}{\partial x_1^{j+1}}(0) + R_k\,,$$

$$\frac{\partial^j v_k}{\partial x_1^{j-1}\partial x_2}(0) = \frac{\partial^{j+1} g_k}{\partial x_1^{j-1}\partial x_2^2}(0) - \frac{\partial^{j+1} f_k}{\partial x_1^j \partial x_2}(0) + S_k\,.$$

Define a vector field L_2 by $L_2 = [\tilde{v}, \tilde{w}]$, and define $L_{r+1} = [\tilde{v}, L_r]$ if L_r is defined. A direct computation shows that the kth component of $L_{j+1}(0)$ is $(\partial^j w_k/\partial x_1^j)(0) - (\partial^j v_k/\partial x_1^{j-1}\partial x_2)(0) + T_k$ where T_k depends only on the derivatives of v and w at the origin of order less than j. It follows that if $2p - 2q + 1 \leq k \leq p$, then the kth component of $L_{j+1}(0)$ is $-((\partial^{j+1} g_k/\partial x_1^{j+1})(0) + (\partial^{j+1} g_k/\partial x_1^{j-1}\partial x_2^2)(0)) + U_k$ where U_k depends only on $[\phi]^j$. Thus given $[\phi]^j \in \Sigma^j$ one can choose $[\phi]^{j+1} \in (\pi^{j+1})^{-1}([\phi]^j)$ in such a way that $\gamma^j([\phi]^{j+1}) \neq 0$, so $a_j \neq 0$ and the result follows.

4. Results on extendibility

We briefly review the terminology and principal result of [5].

If V is a real vector bundle, $V \otimes \boldsymbol{C}$ has a natural automorphism "$\overline{}$" obtained by extending complex conjugation from $\boldsymbol{C}$. There is a natural linear map $re\colon V \otimes \boldsymbol{C} \to V$, which is just "taking real parts".

The *holomorphic tangent bundle* $H(\boldsymbol{C}^n)$ of $\boldsymbol{C}^n$ is the complex subbundle of $T(\boldsymbol{C}^n) \otimes \boldsymbol{C}$ generated (at $p \in \boldsymbol{C}^n$) by tangent vectors of the form $\sum a_j(\partial/\partial z_j)_p$. Let W be a real differentiable submanifold of $\boldsymbol{C}^n$. $H(W)$, the holomorphic tangent bundle of W, is just $H(\boldsymbol{C}^n) \cap (T(W) \otimes \boldsymbol{C})$ over W. $\mathscr{L}(W)$ (called the *Levi algebra* of W in [5]) is the Lie algebra of vector fields generated by sections of $H(W)$ and $\overline{H(W)}$.

Then $VA3$ of [5] gives:

Theorem 4.1. *Suppose W is a real $(n + k)$-dimensional differentiable submanifold of an n-dimensional complex manifold Y, and that fiber* $\dim_{\boldsymbol{C}} H(W) = k$ *($H(W)$ can be defined locally as above). Then W is extendible to a subset of Y containing a real submanifold N with* $\dim N = n + e$ *where $e =$* sup *fiber* $\dim_{\boldsymbol{C}} \mathscr{L}(W)$.

It is easy to connect the work of § 3 with this theorem. If $f\colon X \to Y$ is as in Corollary 3.2, then take $W = f(X)$. The bundle $E_x(f)$ of Corollary 3.2 is just $re(H(W) + \overline{H(W)})$. The integer e of Theorem 4.1 above can be obtained as sup fiber $\dim_{\boldsymbol{R}} L(f)$ ($L(f)$ as in Corollary 3.2). This is true, since $\mathscr{L}(W) = \overline{\mathscr{L}(W)}$ are $re\mathscr{L}(W) = L(f)$.

We say that a subset S of a complex manifold Y is *locally extendible* to an open set if and only if every relatively open subset of S is extendible to a set containing an open subset of Y. Clearly, a set which is locally extendible to an open set is extendible to a set containing an open subset of Y. Then the remarks at the end of Corollary 3.2 translate as:

Theorem 4.2. *Let X be an $(n + k)$-dimensional real differentiable manifold, and Y an n-dimensional complex manifold. Let $\mathscr{M}$ be a set of maps from X to Y, equipped with the C^m topology (m sufficiently large).*

a) *If X is compact, then there is an open and dense subset $\mathscr{O}$ of $\mathscr{M}$, such that if $f \in \mathscr{O}$, then $f(X)$ is locally extendible (and hence extendible) to an open subset of Y.*

b) *If X is not compact, then there is a Baire subset of $\mathscr{M}$ with the same properties as $\mathscr{O}$ in a).*

Proof. We prove a). Take for $\mathscr{O}$ the set of functions described in Corollary 3.2, and suppose $f \in \mathscr{O}$. Then fiber $\dim_{\boldsymbol{R}} L(f) = n$ except possibly on some lower dimensional manifolds. An open subset of X has, therefore, some point where fiber $\dim_{\boldsymbol{C}} \mathscr{L}(f(X)) = n$. Applying Theorem 4.1 shows that $f(X)$ is locally extendible to an open subset of Y. b) is proven similarly.

Remark. The integer m in the statement of Theorem 4.2 above can be more explicitly obtained by carefully examining the work of § 3. In particular, if $\dim_{\boldsymbol{R}} X = \dim_{\boldsymbol{C}} Y + 1$, then $m = \dim_{\boldsymbol{R}} X$ suffices. (In fact, as $\dim_{\boldsymbol{R}} X$ increases, m can be much less than $\dim_{\boldsymbol{R}} X$.)

Precise results will be given in a forthcoming paper by M. Menn, We can derive a simple corollary about analyticity in maximal ideal spaces of function algebras. (See [4] for background on function algebras.) Suppose K is a compact subset of $\boldsymbol{C}^n$. $\underline{C(K)}$ will denote the algebra of continuous complex-valued functions on K with the uniform norm; $\underline{A(K)}$ is the closure in $C(K)$ of restrictions to K of functions analytic in a neighborhood of K. spec $\underline{A(K)}$ will denote the maximal ideal space of $A(K)$, with the Gelfand topology. We recall that each function $f \in A(K)$ extends to a continuous function $\hat{f}$ on spec $A(K)$.

An important question arises: how can one describe the behavior of $\hat{f}$ on spec $A(K) - K$. (See [4, p. 56].) We can contribute the following:

Theorem 4.3. *Let $\mathscr{H}$ be the collection of compact subsets of $\boldsymbol{C}^n$, topologized with the Hausdorff metric* [6, p. 131]. *There is a dense subset D of $\mathscr{H}$ such that if $K \in D$, then there are an open subset U of $\boldsymbol{C}^n$ and an embedding $h\colon U \to \operatorname{spec} A(K) - K$ such that $\hat{f} \circ h\colon U \to \boldsymbol{C}$ is analytic for every $f \in A(K)$.*

Remarks. 1) We do not know, but suspect, that D is also open in $\mathscr{H}$.

2) Suppose $K \in D$. Put $C = \{x \in \operatorname{spec} A(K) - A(K) \mid x \in \text{image of some embedding } h\}$. Is $\bar{C} = \operatorname{spec} A(K)$? (The appropriate corona problem.)

Proof. The subset D of $\mathscr{H}$ is the collection of images of all $(n+1)$-dimensional compact real manifolds X by maps $f\colon X \to \boldsymbol{C}^n$ which have the properties of Theorem 4.2a). Thus $f(X)$ is extendible to a set containing an open subset U of $\boldsymbol{C}^n$. Since every analytic function defined in a neighborhood of $f(X)$ extends to U (with a sup norm on U dominated by that on $f(X)$), we can see that each element of $A(f(X))$ extends to U hence evaluation at each point of U is a member of spec $A(F(X))$. The Gelfand topology is easily seen to agree with the natural topology on U. So the elements of D have the desired property.

We must show that D is dense in $\mathscr{H}$. If $K \in \mathscr{H}$, consider $K(t) = K + S(t)$ (vector sum), where $S(t)$ is a closed ball of radius t centered at the origin. As $t \to 0$, $K(t) \to K$ in the Hausdorff metric. The sets $K(t)$ have a finite number of arcwise connected components, and it is fairly clear how to approximate them by images of $(n+1)$-dimensional manifolds; then (since C^m approximation is finer than Hausdorff metric approximation) by elements of D, using the density of Theorem 4.2a).

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Massachusetts Institute of Technology
& Rutgers University
Boston College