J. DIFFERENTIAL GEOMETRY
4 (1969) 121-131

SUBMANIFOLDS WITH A REGULAR PRINCIPAL NORMAL VECTOR FIELD IN A SPHERE

TOMINOSUKE OTSUKI

Introduction

In [10], the author defined a principal normal vector for a submanifold M in a Riemannian manifold $\bar{M}$. This concept is a generalization of the principal normal vector for a curve and the principal curvature for a hypersurface. In fact, if M is a hypersurface, let $\Phi(X, Y)$ be the value of the 2nd fundamental form for any tangent vector fields X and Y of M. Then, we have

$$\begin{aligned}\Phi(X, Y)e &= -\langle \bar{\nabla}_X e, Y\rangle e \\ &= \text{normal part of } \bar{\nabla}_X Y \equiv T_X Y \ ,\end{aligned}$$

where e is the normal unit vector field and $\bar{\nabla}$ is the covariant differentiation of $\bar{M}$. If λ is a principal curvature at a point x of M and X is a principal tangent vector at x corresponding to λ, then we have

$$T_X Y = \langle X, Y\rangle \lambda e \qquad \text{at } x \ .$$

If we consider λe as the principal normal vector at x of M, then the above concepts for curves and hypersurfaces are in the same category.

In [10], the author investigated the properties of the integral submanifolds in M for the distribution corresponding to a regular princial normal vector field of M in an $\bar{M}$ of constant curvature. In the present paper, the properties of M will be investigated for admitting a regular principal normal vector field, and then the results will be applied to the case in which $\bar{M}$ is a sphere and M is minimal and has two principal normal vector fields such that the corresponding principal tangent spaces span the tangent space of M. Theorem 4 in this paper is a generalization of Theorems 3 and 4 in [9].

1. Preliminaries

We will use the notation in [10]. Let $\bar{M} = \bar{M}^{n+p}$ be an $(n+p)$-dimensional C^∞ Riemannian manifold of constant curvature $\bar{c}$, and $M = M^n$ an n-dimensional C^∞ submanifold immersed in $\bar{M}$ by an immersion $\psi \colon M \to \bar{M}$ which has

Received May 29, 1969. Work done under partial support from State University of New York Funds.

the naturally induced Riemannian metric by ϕ. Let $P\colon \phi^*T(\bar{M}) \to T(M)$ be the projection defined by the orthogonal decomposition:

$$T_{\phi(x)}(\bar{M}) = \phi_*(T_x(M)) + N_x\,, \qquad x \in M\,,$$

and put $P^\perp = 1 - P$. Let $N(M, \bar{M})$ denote the normal vector bundle of M in $\bar{M}$ by the immersion ϕ. Then we have

$$\phi^*T(\bar{M}) = T(M) \oplus N(M, \bar{M})\,.$$

In the following, we denote the sets of C^∞ cross sections of $T(M)$ and $N(M, \bar{M})$ by $\mathfrak{X}(M)$ and $\mathfrak{X}^\perp(M)$, and the covariant differentiations for $\bar{M}$ and M by $\bar{\nabla}$ and ∇, respectively. For the vector bundle $N(M, \bar{M})$, we have the naturally induced metric connection from $\bar{M}$ and denote the corresponding covariant differentiation by $\nabla^\perp$. Then for any $X \in \mathfrak{X}(M)$, we have

$$\bar{\nabla}_X = \nabla_X + T_X \quad \text{on} \quad \mathfrak{X}(M) \tag{1.1}$$

with $\nabla_X = P\bar{\nabla}_X$ and $T_X = P^\perp\bar{\nabla}_X$, and

$$\bar{\nabla}_X = T_X + \nabla^\perp_X \quad \text{on} \quad \mathfrak{X}^\perp(M) \tag{1.2}$$

with $T_X = P\bar{\nabla}_X$ and $\nabla^\perp_X = P^\perp\bar{\nabla}_X$.

Now, for a fixed point $x \in M$, a normal vector $v \in N_x$ is called a *principal normal vector* of M at x if there exists a nonzero vector $u \in M_x = T_x(M)$ such that

$$T_u z = \langle u, z\rangle v \qquad \text{for any } z \in M_x\,, \tag{1.3}$$

and the vector u is called a *principal tangent vector* for v. The set of all principal tangent vectors for v and the zero vector form a linear subspace of M_x, which is called the *principal tangent vector space* for v and is denoted by $E(x, v)$.

A normal vector field $V \in \mathfrak{X}^\perp(M)$ is called a *regular principal normal vector field* of M, if $V(x)$ is a principal normal vector and $\dim E(x, V(x)), x \in M$, is constant.

In the following, we suppose that V is a regular principal normal vector field of M. By Lemma 2 in [10], $E(x, V(x)), x \in M$, form a C^∞ distribution of M, which we denote by $E(M, V)$. By Theorem 1 in [10], $E(M, V)$ is completely integrable. Now, we decompose M_x in the following orthogonal sum:

$$M_x = E(\mathrm{x}, V(x)) + N(x, V(x))\,,$$

and denote the distribution of $N(x, V(x)), x \in M$, by $N(M, V)$. Then

$$T(M) = E(M, V) \oplus N(M, V)\,.$$

Let $Q\colon T(M) \to E(M, V)$ and $Q^{\perp}\colon T(M) \to N(M, V)$ be the natural projections by this decomposition· $E(M, V)$ and $N(M, V)$ have the naturally defined metric connections induced from the one of M as vector bundles over M.

By means of Theorem 2 in [10], if the dimension m of the distribution $E(M, V)$ is greater than 1 and $V \neq 0$ everywhere, then there exists a uniquely determined cross section U of $N(M, V)$ such that for any integral submanifold M^m of $E(M, V)$, $U | M^m$ is a principal normal vector field of M^m in M^n, and M^m is totally umbilic in M^n.

2. The integrability condition of $N(M, V)$

In this section, we consider the case stated in the last paragraph in the 1st section. For any $y \in E(x, V(x))$, we define a linear mapping $\Phi_y\colon N(x, V(x)) \to N(x, V(x))$ by

$$\Phi_y(z) = Q^{\perp}(\nabla_z Y) , \tag{2.1}$$

where Y is a C^∞ local cross section of $E(M, V)$ at x with $Y(x) = y$.

Lemma 1. *Φ_y is well defined.*

Proof. Let B_1 be the set of frames $b = (x, e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p})$ such that $e_1, \cdots, e_m \in E(x, V(x))$ and

$$V(x) = \lambda(x) e_{n+1} , \qquad \lambda(x) > 0 . \tag{2.2}$$

Then, we have[1]

$$\omega_{ar} = \rho_r \omega_a + \sum_{t=m+1}^{n} \Gamma_{art} \omega_t , \qquad a = 1, \cdots, m, r = m+1, \cdots, n ; \tag{2.3}$$

$$U = \sum_{r=m+1}^{n} \rho_r e_r . \tag{2.4}$$

Now, we put $Y = \sum_{n=1}^{m} f_a e_a$ about x and $z = \sum_{r=m+1}^{n} z_r e_r$ at x. Then by (2.3)

$$\begin{aligned} Q^{\perp}(\nabla_z Y) &= Q^{\perp}\Big(\sum_r z_r \nabla_{e_r}\Big(\sum_a f_a e_a\Big)\Big) \\ &= \sum_{a=1}^{m} \sum_{r,t=m+1}^{n} z_r f_a \omega_{at}(e_r) e_t = \sum_{a,r,t} f_a z_r \Gamma_{atr} e_t . \end{aligned}$$

The right hand side of the above equation does not depend on the choice of frame $b \in B_1$ at x and the extension Y of y, since Γ_{atr} are the components of a cross section of $E^*(M, V) \otimes N(M, V) \otimes N^*(M, V)$ where $E^*(M, V)$ and $N^*(M, V)$ are the dual vector bundles over M of $M(M, V)$ and $N(M, V)$ respectively.

As in [10], we denote the set of all C^∞ cross sections for any vector bundle

[1] see the proof of Theorem 2 in [10].

$E \to M$ by $\Gamma(E, M)$. Then, by Lemma 1, for any $Y \in \Gamma(E(M, V))$, we can define a mapping $\Phi_Y: \Gamma(N(M, V)) \to \Gamma(N(M, V))$ in a natural way.

Theorem 1. *Let M be an immersed submanifold of a Riemannian manifold $\bar{M}$ of constant curvature, and V a nonzero regular principal normal vector field of M in $\bar{M}$ such that the dimension of the distribution $E(M, V) > 1$. Then the distribution $N(M, V)$ is completely integrable if and only if Φ_Y for any $Y \in \Gamma(E(M, V))$ is self-adjoint on $\Gamma(N(M, V))$.*

Proof. The completely integrability of the distribution $N(M, V)$ is equivalent to the following condition:

$$d\omega_a \equiv 0 \quad (\mathrm{mod}\, \omega_1, \cdots, \omega_m)\,, \quad \text{on} \quad B_1\,, \quad a = 1, \cdots, m\,.$$

From the structure equations and (2.3), we obtain

$$\begin{aligned} d\omega_a &= \sum_b \omega_b \wedge \omega_{ba} - \sum_r \omega_r \wedge \Big(\rho_r \omega_a + \sum_t \Gamma_{art}\omega_t\Big) \\ &\equiv -\sum_{r,t} \Gamma_{art}\omega_r \wedge \omega_t \qquad (\mathrm{mod}\, \omega_1, \cdots, \omega_m)\,. \end{aligned}$$

Therefore, $N(M, V)$ is completely integrable if and only if $\Gamma_{art} = \Gamma_{atr}$, which is clearly equivalent to that for any $Y \in \Gamma(E(M, V))$, and $Z, W \in \Gamma(N(M, V))$, we have

$$\langle \Phi_Y(Z), W \rangle = \langle \Phi_Y(W), Z \rangle\,.$$

3. Properties of Φ_X and F

On B_1, we have

$$\begin{aligned} &\omega_{an+1} = \lambda\omega_a\,, \qquad \omega_{a\beta} = 0, \\ &a = 1, \cdots, m\,, \qquad \beta = n+2, \cdots, n+p\,. \end{aligned} \tag{3.1}$$

From (2.3), (3.1) and the structure equations it follows that[2]

$$\begin{aligned} d\omega_{ar} &= \sum_{B=1}^{n+p} \omega_{aB} \wedge \omega_{Br} - \bar{c}\omega_a \wedge \omega_r \\ &= \rho_r \sum_b \omega_{ab} \wedge \omega_b + \sum_{b,s} \Gamma_{brs}\omega_{ab} \wedge \omega_s + \sum_s \rho_s\omega_a \wedge \omega_{sr} \\ &\quad + \sum_{s,t} \Gamma_{ast}\,\omega_t \wedge \omega_{sr} - \lambda \sum_s A_{n+1,rs}\omega_a \wedge \omega_s - \bar{c}\omega_a \wedge \omega_r\,, \end{aligned}$$

[2] In the following, the ranges of indices are:

$$a, b, c, \cdots = 1, \cdots, m;\ r, s, t, \cdots = m+1, \cdots, n\,;$$
$$i, j, k, \cdots = 1, \cdots, n\,.$$

$$\begin{aligned} d\Big(\rho_r\omega_a + \sum_s \Gamma_{ars}\omega_s\Big) &= d\rho_r \wedge \omega_a + \rho_r \sum_{i=1}^n \omega_i \wedge \omega_{ia} \\ &\quad + \sum_s d\Gamma_{ars} \wedge \omega_s + \sum_s \Gamma_{ars} \sum_{i=1}^n \omega_i \wedge \omega_{is} \\ &= d\rho_r \wedge \omega_a + \rho_r \sum_b \omega_b \wedge \omega_{ba} + \rho_r \sum_s \rho_s\omega_a \wedge \omega_s \\ &\quad + \rho_r \sum_{s,t} \Gamma_{ast}\omega_t \wedge \omega_s + \sum_s d\Gamma_{ars} \wedge \omega_s \\ &\quad + \sum_{t,s,b} \Gamma_{art}\Gamma_{bts}\omega_b \wedge \omega_s + \sum_{t,s} \Gamma_{art}\omega_{ts} \wedge \omega_s \,, \end{aligned}$$

and hence

$$\begin{aligned} &\Big(d\rho_r + \sum_s \rho_s\omega_{sr} - \rho_r \sum_s \rho_s\omega_s\Big) \wedge \omega_a \\ &+ \sum_s \Big(d\Gamma_{ars} + \sum_b \Gamma_{brs}\omega_{ba} + \sum_t \Gamma_{ats}\omega_{tr} + \sum_t \Gamma_{art}\omega_{ts} \\ &+ \sum_{t,b} \Gamma_{art}\Gamma_{bts}\omega_b + \rho_r \sum_t \Gamma_{ast}\omega_t + \bar{c}\delta_{rs}\omega_a + \lambda A_{n+1,rs}\omega_a\Big) \wedge \omega_s = 0 \,. \end{aligned}$$

Since $m > 1$, from the above equations we have

$$d\rho_r + \sum_s \rho_s\omega_{sr} - \rho_r \sum_s \rho_s\omega_s = \sum_t F_{rt}\omega_t \,, \tag{3.2}$$

$$\begin{aligned} &d\Gamma_{ars} + \sum_b \Gamma_{brs}\omega_{ba} + \sum_t \Gamma_{ats}\omega_{tr} + \sum_t \Gamma_{art}\omega_{ts} \\ &\quad + \sum_{t,b} \Gamma_{art}\Gamma_{bts}\omega_b + \rho_r \sum_t \Gamma_{ast}\omega_t + (\bar{c}\delta_{rs} + \lambda A_{n+1,rs})\omega_a \\ &= F_{rs}\omega_a + \sum_t B_{arst}\omega_t \,, \end{aligned} \tag{3.3}$$

where F_{rt} and B_{arst} are functions on B_1, and components of a tensor of type $(1,1)$ of $N(M,V)$ and a tensor of type $(0,1) \otimes (1,2)$ of $E(M,V) \otimes N(M,V)$ respectively, and

$$B_{arst} = B_{arts} \,. \tag{3.4}$$

Now, let F and B_{XW}, for $X \in \Gamma(E(M,V))$ and $W \in \Gamma(N(M,V))$, be the endomorphisms on $N(M,V)$ defined by

$$F(e_t) = \sum_r F_{rt}e_r \,,$$

$$B_{XW}(e_t) = \sum_{a,r,s} B_{arts}X_aW_se_r \,,$$

where $X = \sum_a X_ae_a$ and $W = \sum_r W_re_r$. We denote the covariant differentiation of the tensor product bundles of $E(M,V)$ and $N(M,V)$ by D, Then, (3.2) and (3.3) can be written as

$$D_Z U = \langle Z, U\rangle U + F(Q^{\perp}(Z)) \,, \tag{3.5}$$

$$\begin{aligned} &D_Z(\Phi_X(W)) - \Phi_{D_Z X}(W) - \Phi_X(D_Z W) + \Phi_X(\Phi_{Q(Z)}(W)) \\ &\quad + \langle \Phi_X(Q^{\perp}(Z)), W\rangle U + \langle Z, X\rangle\{\bar{c}W - T_W(V)\} \\ &= \langle Z, X\rangle F(W) + B_{XW}(Q^{\perp}(Z)) \,, \end{aligned} \tag{3.6}$$

where $Z \in \mathfrak{X}(M)$, $X \in \Gamma(E(M, V))$, $W \in \Gamma(N(M, V))$, and the 2nd term on the right hand side of (3.6) is expressed, by means of (3.4), as

$$B_{XW}(Y) = B_{XY}(W) \,, \qquad Y \in \Gamma(N(M, V)) \,. \tag{3.7}$$

From (3.5) follows easily

Lemma 2. *Under the conditions of Theorem* 1, $U \in \Gamma(N(M, V))$ *is parallel along any integral submanifold of the distribution* $E(M, V)$.

Proof. For any $X \in \Gamma(E(M, V))$, we have

$$D_X U = 0 \,. \tag{3.5$'$}$$

Lemma 3. *Under the conditions of Theorem* 1, F *can be defined by the equation*

$$F(W) \equiv D_W U - \langle W, U\rangle U \,. \tag{3.5$''$}$$

It is clear that (3.5) is equivalent to (3.5′) and (3.5″). Substituting (3.5″) into (3.6), we get

$$\begin{aligned} B_{XW}(Q^{\perp}(Z)) = {} & D_Z(\Phi_X(W)) - \Phi_{D_Z X}(W) - \Phi_X(D_Z(W)) + \Phi_X(\Phi_{Q(Z)}(W)) \\ & + \{\langle \Phi_X(Q^{\perp}(Z)), W\rangle + \langle X, Z\rangle\langle W, U\rangle\}U \\ & + \langle X, Z\rangle\{\bar{c}W - T_W(V) - D_W U\} \,. \end{aligned}$$

In particular, for $Z = Y \in \Gamma(E(M, V))$,

$$\begin{aligned} &D_Y(\Phi_X(W)) - \Phi_{D_Y X}(W) - \Phi_X(D_Y(W)) + \Phi_X(\Phi_Y(W)) \\ &\quad + \langle X, Y\rangle\{\langle W, U\rangle U + \bar{c}W - T_W(V) - D_W U\} = 0 \,, \end{aligned} \tag{3.6$'$}$$

and, for $Z \in \Gamma(N(M, V))$,

$$\begin{aligned} B_{XW}(Z) = {} & D_Z(\Phi_X(W)) - \Phi_{D_Z X}(W) \\ & - \Phi_X(D_Z W) + \langle \Phi_X(Z), W\rangle U \,, \end{aligned} \tag{3.6$''$}$$

which may be considered as the formula of definition of B_{XW}.

Now, for any $X, Y \in \Gamma(E(M, V))$, we have

$$D_X Y - D_Y X = Q(\nabla_X Y - \nabla_Y X) = Q([X, Y]) = [X, Y] \,,$$

since $E(M, V)$ is completely integrable. Therefore, from (3.6′) follows

$$(3.8)\quad D_Y\cdot\Phi_X - D_X\cdot\Phi_Y + \Phi_{[X,Y]} - \Phi_X\cdot D_Y + \Phi_Y\cdot D_X + [\Phi_X, \Phi_Y] = 0\ .$$

Lemma 4. *For any $X, Y \in \Gamma(E(M, V))$, by defining θ_X: $\Gamma(N(M, V)) \to \Gamma(N(M, V))$ by*

$$\theta_X = D_X - \Phi_X\ , \tag{3.9}$$

we have

$$\theta_X\cdot\theta_Y - \theta_Y\cdot\theta_X = \theta_{[X,Y]} + R^{\perp}_{XY}\ ,$$

where $R^{\perp}$ denotes the curvature tensor of $N(M, V)$.

Proof. By means of (3.8), we obtain

$$\begin{aligned}\theta_X\cdot\theta_Y - \theta_Y\cdot\theta_X &= (D_X - \Phi_X)(D_Y - \Phi_Y) - (D_Y - \Phi_Y)(D_X - \Phi_X)\\ &= D_XD_Y - D_YD_X + [\Phi_X, \Phi_Y] - D_X\Phi_Y\\ &\quad - \Phi_XD_Y + D_Y\Phi_X + \Phi_YD_X\\ &= R^{\perp}_{XY} + D_{[X,Y]} - \Phi_{[X,Y]}\\ &= R^{\perp}_{XY} + \theta_{[X,Y]}\ .\end{aligned}$$

From Lemma 4 follows easily

Theorem 2. *Under the conditions of Theorem 1, if $N(M, V)$ is flat along any integral submanifold of the distribution $E(M, V)$, then θ is a representation of the Lie algebra $\Gamma(E(M, V))$ on the space of endomorphisms of $N(M, V)$.*

Formula (3.6)′ implies immediately

Lemma 5. *For any $X \in \Gamma(E(M, V))$, with $\|X\| = 1$, and $W \in \Gamma(N(M, V))$,*

$$\begin{aligned}D_X(\Phi_X(W)) - \Phi_X(D_X(W)) - \Phi_{D_XX}(W) + \Phi^2_X(W)\\ = D_WU + T_W(V) - \langle W, U\rangle U - \bar{c}W\ .\end{aligned}$$

4. Case $\bar{M}^{n+p} = S^{n+p}$

In this section, we suppose furthermore that $\bar{M}^{n+p}$ is an $(n + p)$-dimensional unit sphere S^{n+p} in Euclidean space R^{n+p+1}. We may consider the frame $b = (x, e_1, \cdots, e_{n+p})$ of $\bar{M}$ to be Euclidean in R^{n+p+1} and define a vector field on M by

$$\xi = U + V - e_{n+p+1} = \sum_r \rho_r e_r + \lambda e_{n+1} - e_{n+p+1}\ , \tag{4.1}$$

where $e_{n+p+1} = x \in M$. ξ is clearly orthogonal to $E(x, V(x))$. Then, by (2.3), (3.1) and $\omega_{i,n+p+1} = -\omega_i$, we have

$$\begin{aligned}de_a &= \sum_{B=1}^{n+p} \omega_{aB}e_B + \omega_{a,n+p+1}e_{n+p+1}\\ &= \sum_b \omega_{ab}e_b + \omega_a\xi + \sum_{r,s} \Gamma_{ars}\omega_s e_r\ .\end{aligned} \tag{4.2}$$

Next, we also have

$$
\begin{aligned}
d\xi &= \sum_r d\rho_r e_r + d\lambda e_{n+1} + \sum_r \rho_r \left(\sum_{B=1}^{n+p} \omega_{rB} e_B - \omega_r e_{n+p+1} \right) \\
&\quad + \lambda \sum_{B=1}^{n+p} \omega_{n+1,B} e_B - \sum_i \omega_i e_i \\
&\equiv \sum_r \left(d\rho_r + \sum_t \rho_t \omega_{tr} - \lambda \sum_t A_{n+1,rt} \omega_t - \omega_r \right) e_r \\
&\quad + \left(d\lambda + \sum_{t,r} A_{n+1,tr} \rho_t \omega_r \right) e_{n+1} \\
&\quad + \sum_{\beta > n+1} \left(\lambda \omega_{n+1,\beta} + \sum_{t,r} A_{\beta tr} \rho_t \omega_r \right) e_\beta \\
&\quad - \sum_r \rho_r \omega_r e_{n+p+1} \qquad (\mathrm{mod}\ e_1, \cdots, e_m) ,
\end{aligned}
$$

where $\omega_{i\alpha} = \sum_i A_{\alpha ij} \omega_j$. On the other hand, using (3.3) and (3.4) in [10]:

$$
d\lambda = \sum_r B_{n+1,r} \omega_r , \qquad \lambda \omega_{n+1,\beta} = \sum_r B_{\beta r} \omega_r , \tag{4.3}
$$

exterior differentiation of (3.1) gives

$$
\begin{aligned}
&\sum_t \omega_{at} (A_{n+1,tr} - \lambda \delta_{tr}) + B_{n+1,r} \omega_a \equiv 0 , \\
&\sum_t \omega_{at} A_{\beta tr} + B_{\beta r} \omega_a \equiv 0 , \qquad (\mathrm{mod}\ \omega_{m+1}, \cdots, \omega_n) .
\end{aligned}
$$

Substituting (2.3) into the above equations, we get

$$
\begin{aligned}
&B_{n+1,r} + \sum_t \rho_t A_{n+1,tr} = \lambda \rho_r , \\
&B_{\beta r} + \sum_t \rho_t A_{\beta tr} = 0 , \qquad \beta > n + 1 .
\end{aligned} \tag{4.4}
$$

Making use of (4.3) and (4.4), we have

$$
\begin{aligned}
d\xi &\equiv \sum_r \left(d\rho_r + \sum_t \rho_t \omega_{tr} - \lambda \sum_t A_{n+1,rt} \omega_t - \omega_r \right) e_r \\
&\quad + \lambda \sum_r \rho_r \omega_r e_{n+1} - \sum_r \rho_r \omega_r e_{n+p+1} , \qquad (\mathrm{mod}\ e_1, \cdots, e_m) .
\end{aligned} \tag{4.5}
$$

Now, we consider the following Euclidean $(m + 1)$-vector in R^{n+p+1},

$$
\pi = e_1 \wedge \cdots \wedge e_m \wedge \xi . \tag{4.6}
$$

By means of (4.2) and (4.5), we obtain

$$
\begin{aligned}
d\pi = & \sum_{r=m+1}^{n} \rho_r \omega_r \pi \\
& + \sum_{a+1}^{m} e_1 \wedge \cdots \wedge e_{a-1} \wedge \sum_{r,s} \Gamma_{ars} \omega_s e_r \wedge e_{a+1} \wedge \cdots \wedge e_m \wedge \xi \\
& + \sum_{r=m+1}^{n} \left(d\rho_r + \sum_t \rho_t \omega_{tr} - \lambda \sum_t A_{n+1,rt} \omega_t - \omega_r \right) e_1 \wedge \cdots \wedge e_m \wedge e_r \\
& - \sum_r \rho_r \omega_r e_1 \wedge \cdots \wedge e_m \wedge \sum_t \rho_t e_t ,
\end{aligned} \tag{4.7}
$$

which is equivalent to the following equation:

$$
\begin{aligned}
d_Z \pi = & \langle U, Z \rangle \pi + e_1 \wedge \cdots \wedge e_m \wedge (D_Z U - \langle U, Z \rangle U + T_Z(V) - Z) \\
& + \sum_{a=1}^{m} e_1 \wedge \cdots \wedge e_{a-1} \wedge \Phi_{e_a}(Q^{\perp}(Z)) \wedge e_{a+1} \wedge \cdots \wedge e_m \wedge \xi ,
\end{aligned} \tag{4.8}
$$

for $Z \in \mathfrak{X}(M)$. In particular, we have

$$
d_X \pi = 0 , \qquad \text{for } X \in \Gamma(E(M, V)) . \tag{4.9}
$$

Hence, we can easily reach

Theorem 3. *Let V be a nonzero regular principal normal vector field of M in $S^{n+p} \subset R^{n+p+1}$ such that the dimension m of the distribution $E(M, V) > 1$. Then for any maximal integral submanifold of $E(M, V)$ there exists an $(m+1)$-dimensional linear subspace E^{m+1} such that it is contained in the m-dimensional sphere $E^{m+1} \cap S^{n+p}$. Furthermore, the condition for all the E^{m+1} to be parallel to a fixed one is*

$$
D_Z U - \langle U, Z \rangle U + T_Z(V) - Z = 0 \qquad \textit{for any } Z \in \Gamma(N(M, V)) \tag{4.10}
$$

and

$$
\Phi_X = 0 \qquad \textit{for any } X \in \Gamma(E(M, V)) . \tag{4.11}
$$

Remark. If M is a minimal hypersurface in S^{n+1} and $m = n-1$, then we have (see [10, § 3])

$$
\omega_{an} = (\log \lambda^{1/n})' \omega_a ,
$$

where $\lambda = \| V \|$ (principal curvature of multiplicity $n-1$), and λ is a function of arc length v of an orthogonal trajectory of the family of the integral submanifolds. Thus $\Gamma_{ann} = 0$ and $U = (\log \lambda^{1/n})' e_n$. Hence (4.11) is trivially true and (4.10) becomes

$$
(\log \lambda^{1/n})'' - \{(\log \lambda^{1/n})'\}^2 + ((n-1)\lambda^2 - 1 = 0 .
$$

Theorem 4. *Let $M^n (n \geq 3)$ be a minimal submanifold in $S^{n+p} \subset R^{n+p+1}$*

with two regular principal normal vector fields V and W such that

$$E(M, V) \oplus E(M, W) = T(M) .$$

Then there exists a linear subspace E^{n+2} through the origin of R^{n+p+1} such that $M^n \subset E^{n+2} \cap S^{n+p}$.

Proof. We may suppose the dimension m of the distribution $E(M, V) > 1$. Since $V \neq W$ at each point, $E(M, V)$ and $E(M, W)$ are orthogonal by Lemma 1 in [10]. We use frames $b = (x, e_1, \cdots, e_{n+p})$ such that $e_1, \cdots, e_m \in E(M, V)$ and $e_{m+1}, \cdots, e_n \in E(M, W) = N(M, V)$. By putting $V = \sum_{\alpha>n} \lambda_\alpha e_\alpha$ and $W = \sum_{\alpha>n} \mu_\alpha e_\alpha$, we obtain

$$\begin{aligned} A_{\alpha a j} &= \lambda_\alpha \delta_{a j} , & A_{\alpha r j} &= \mu_\alpha \delta_{r j} , \\ \alpha &= n+1, \cdots, n+p ; & a &= 1, \cdots, m ; \\ r &= m+1, \cdots, n ; & j &= 1, \cdots, n . \end{aligned}$$

Since M^n is minimal, it follows that

$$0 = \sum_i A_{\alpha i i} = m\lambda_\alpha + (n-m)\mu_\alpha = 0 ,$$

that is,

$$mV + (n-m)W = 0 .$$

Since $V \neq W$, we see that $V \neq 0$ and $W \neq 0$. Therefore we may put $V = \lambda e_{n+1} (\lambda > 0)$, $W = \mu e_{n+1}$, and then have

$$\omega_{a n+1} = \lambda \omega_a , \quad \omega_{r n+1} = \mu \omega_r , \quad \omega_{i\beta} = 0 \quad (\beta = n+2, \cdots, n+p) .$$

Hence M-index of M^n in S^{n+p} is 1 everywhere. By Theorem 1 in [9], there exists an $(n+1)$-dimensional totally geodesic submanifold of S^{n+p} containing M^n as a minimal hypersurface, which is the intersection of a linear subspace E^{n+2} through the origin of R^{n+p+1} and S^{n+p}.

References

[1] S. S. Chern, M. Do Carmo & S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length,* to appear in Marshall Stone Jubilee Volume.
[2] R. Maltz, *Isometric immersion into spaces of constant curvature,* to appear.
[3] B. O'Neill, *Isometric immersions which preserve curvature operators,* Proc. Amer. Math. Soc. **13** (1962) 759–763.
[4] ——, *Isometric immersion of flat Riemannian manifolds in Euclidean space,* Michigan Math. J. **9** (1962) 199–205.
[5] ——, *Umbilics of constant curvature immersions,* Duke Math. J. **32** (1965) 149–160.
[6] B. O'Neill & E. Stiel, *Isometric immersions of constant curvature manifolds,* Michigan Math. J. **10** (1963) 335–339.

[7] T. Otsuki, *A theory of Riemannian submanifolds*, Kōdai Math. Sem. Rep. **20** (1968) 282–295.

[8] ——, *Pseudo-umbilical submanifolds with* M-*index* $\leq$ 1 *in Euclidean spaces*, Kōdai Math. Sem. Rep. **20** (1968) 296–304.

[9] ——, *Minimal hypersurfaces in a Riemannian manifold of constant curvature*, Amer. J. Math. **92** (1970).

[10] ——, *On principal normal vector fields of submanifolds in a Riemannian manifold of constant curvature*, J. Math. Soc. Japan **22** (1970) 35–46.

[11] E. Stiel, *Isometric immersions of manifolds of nonnegative constant sectional curvature*, Pacific J. Math. **15** (1965) 1415–1419.

STATE UNIVERSITY OF NEW YORK, STONY BROOK
TOKYO INSTITUTE OF TECHNOLOGY