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TOTALLY GEODESIC MAPS

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Introduction

A C^∞ map $f: X \to Y$ of finite-dimensional connected C^∞ Riemannian manifolds is defined to be *totally geodesic* if for each geodesic x_t in X the image $f(x_t)$ is a geodesic in Y [1, p. 123]. An equivalent definition is that f is connection-preserving, or *affine*. The global structure of these maps is investigated in this paper.

§ 1 contains preliminary material relating to the fundamental form of a map [1, p. 123]. In § 2 a global factorization theorem is proved, namely, if X is complete, then each totally geodesic map factors into a Riemannian submersion [3], [4], [6] and an immersion, both totally geodesic. In § 3 the submersion part of this factorization is studied. It is shown that the nontrivial totally geodesic Riemannian submersions with X complete can be characterized as fibre bundles with integrable (flat) connection [2], having complete Riemannian metrics on base and fibre, the latter being invariant under the structural group.

The proof of the factorization theorem starts with the fact (shown in § 1) that kernels of affine maps are holonomy-invariant. This implies that totally geodesic maps have constant rank, whence a local factorization follows. The global factorization uses results of Reinhart [8] and of Kobayashi and Nomizu [5]. The proofs in § 3 depend on results of Hermann [4] and O'Neill [6].

The material of § 1 and Theorem 3.3 in § 3 were part of the author's study of the fundamental form in his thesis done under the guidance of Professor James Eells, whom the author wishes to thank in this respect.

It is assumed without further mention that everything below is C^∞, and manifolds are connected. "Totally geodesic" is also abbreviated as "t.g.".

1. The fundamental form of a map

The fundamental form of a map expresses the manner in which connections on two manifolds are related by a map of these manifolds. It vanishes iff the map is affine, and also furnishes a good means of establishing the equivalence of the above two definitions of totally geodesic maps.

The definition of the fundamental form is conveniently given in the following

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general setting. Let E and $\hat{E}$ be vector bundles over X with linear connections D and $\hat{D}$, and $\varphi\colon E \to \hat{E}$ a linear map. φ is then a section of the bundle $L(E, \hat{E})$, which has a connection defined by D and $\hat{D}$. The covariant derivative of this section φ is called the *fundamental form of* φ, $\beta(\varphi)$, [1]. It is the section of $L^2(E, TX; \hat{E})$ defined by $\beta(\varphi)(A, v) = \hat{D}_v\varphi A - \varphi D_v A$, where A and v are (germs of) sections of E and TX respectively.

If one starts with a map $f\colon X \to Y$ and a linear map $\varphi\colon E \to F$ over f, with linear connections D and $\bar{D}$ on E and F, respectively, then the above situation is recovered by setting $\hat{E} = f^{-1}F$, identifying φ with the unique linear map $E \to \hat{E}$ defined by it, and putting on $\hat{E}$ the pull-back connection $\hat{D}$ induced by $\bar{D}$. $\beta(\varphi)$ will then be a section of $L^2(E, TX; f^{-1}F)$.

The pull-back connection $\hat{D}$ on $f^{-1}F \to X$ is defined locally by

$$(\hat{D}_v A)_x = A'_x v_x + \bar{\Gamma}_{fx}(A_x, f'_x v_x) ,$$

where $A_x \in (f^{-1}F)_x = F_{fx}$, primes denote derivatives, and $\bar{\Gamma}$ denotes the local Christoffel symbol for $\bar{D}$.

Lemma 1.1. *$\hat{D}$-parallel translation along curves in X coincides with $\bar{D}$-parallel translation along curves in $f(X) \subset Y$.*

Proof. A section A_t of $f^{-1}F \to X$ along a curve x_t in X is parallel iff $\hat{D}_t A_t = 0$ for all t, iff $A'_t + \bar{\Gamma}_{fx_t}(A_t, f'_x x'_t) = 0$ for all t, iff the section A_t of $F \to Y$ along the curve $y_t = fx_t$ is parallel. q.e.d.

Before defining an affine map, it is convenient to note

Lemma 1.2. *$\beta(\varphi) = 0$ iff φ preserves parallel translation.*

Proof. Let e_t be a curve in E over curve x_t in X. Then $\beta(\varphi)(e_t, x'_t) = \hat{D}_t\varphi e_t - \varphi D_t e_t = \hat{D}_t\varphi e_t$ if e_t is parallel (i.e., if $D_t e_t = 0$). Then clearly $\beta(\varphi)(e_t, x'_t) = 0$ iff φe_t is parallel along x_t. Thus $\beta(\varphi) = 0$ iff φ preserves parallel translation for all curves in E over all curves in X. q.e.d.

A linear map $\varphi\colon E \to \hat{E}$ is called an *affine map* if any of the following equivalent conditions holds:

(i) $\beta(\varphi) = 0$,

(ii) φ preserves parallel translation,

(iii) $\hat{D}_v\varphi A = \varphi D_v A$ for all v and A.

The following proposition is the basis of all the results below.

Proposition 1.3. *If φ is affine then* $\ker\varphi$ *is a holonomy-invariant subbundle of E.*

Proof. Along a given curve, parallel translation is an isomorphism of the fibres at corresponding points and thus sends zero into zero. Also translation along the reverse curve is the inverse of translation along the curve. Hence $\ker\varphi$ is invariant along curves. But X is connected, so $\ker\varphi$ is invariant under parallel translation and therefore under the holonomy group. If $\ker\varphi$ is zero at one point, it is zero everywhere.

Ker φ is a subbundle of E because the above implies φ is a linear map of constant rank, and a standard argument then applies. q.e.d.

For a map $f\colon X \to Y$ of manifolds with linear connections, the *fundamental form of* f, $\beta(f)$, is defined to be $\beta(f_*)$ (where $f_*\colon TX \to TY$ denotes the tangent map of f); also f affine is defined to mean f_* affine.

In the case of symmetric connections, a map is affine if and only if it is totally geodesic. This is true because a symmetric connection is uniquely determined by its geodesics, as is well known. A short proof of this will nevertheless be given, using the fundamental form, as follows.

$\beta(f)$ is a bilinear function. Each bilinear function $B(u, v)$ can be written as the sum of its symmetric and antisymmetric parts $B_s(u, v) = \frac{1}{2}[B(u, v) + B(v, u)]$ and $B_a(u, v) = \frac{1}{2}[B(u, v) - B(v, u)]$, respectively. Note that B restricted to the diagonal determines B_s:

$$B_s(u, v) = B(u + v, u + v) - B(u - v, u - v) \ .$$

Therefore $B_s = 0$ iff $B|$ diagonal $= 0$.

Lemma 1.4. (i) *$\beta(f)|$ diagonal $= 0$ iff f is totally geodesic.*

(ii) *$\beta(f)_a(u, v) = \bar{\mathscr{J}}(f_*u, f_*v) - f_*\mathscr{J}(u, v)$, where $\mathscr{J}$ is torsion.*

Proof. $\beta(f)(x_t', x_t') = \hat{D}_t f_* x_t' - f_* D_t x_t'$, but $\hat{D}_t f_* x_t' = \bar{D}_t y_t'$ where $y_t = f x_t$. Now a curve is a geodesic iff its tangent vector is parallel; and through each point in each direction goes a geodesic. Reasoning analogous to that in Lemma 1.2 then proves (i). (ii) is proved by local calculations. Namely,

$$\mathscr{J}(u, v) = \tfrac{1}{2}(D_v u - D_u v - [v, u]) \overset{\text{loc.}}{=} \tfrac{1}{2}(\Gamma(u, v) - \Gamma(v, u)) \ ,$$

because $D_v u \overset{\text{loc.}}{=} u'v + \Gamma(u, v)$ and $[u, v] \overset{\text{loc.}}{=} u'v - v'u$. Now

$$\begin{aligned}\hat{D}_v f_* u &\overset{\text{loc.}}{=} (f'u)'v + \bar{\Gamma}(f'u, f'v) \\ &= f''(u, v) + f'u'v + \bar{\Gamma}(f'u, f'v) \ ,\end{aligned}$$

whence

$$\begin{aligned}\hat{D}_v f'u - \hat{D}_u f'v - f'[v, u] &\overset{\text{loc.}}{=} \tfrac{1}{2}(\bar{\Gamma}(f'u, f'v) - \bar{\Gamma}(f'v, f'u)) \\ &= \bar{\mathscr{J}}(f_*u, f_*v) \ .\end{aligned}$$

Substitution into the equation $\beta(f)(u, v) = \hat{D}_v f_* u - f_* D_v u$ and an easy calculation yield the conclusion.

Proposition 1.5. *f is affine iff f is totally geodesic and $\bar{\mathscr{J}} f_* = f_* \mathscr{J}$.*

Proof. The bilinear function $\beta(f)$ is zero iff its symmetric and antisymmetric parts vanish. The conclusion follows from Lemma 1.2.

Corollary 1.6. *Suppose connections are symmetric.*

(i) *Then $\beta(f)$ is symmetric.*

(ii) *f is affine iff f is totally geodesic.*

2. The factorization theorem

The theorem will be deduced from Proposition 1.3 via the theory of foliations (=integrable subbundles of TX; see Palais [7] for terminology). Let $f\colon X \to Y$ be, as above, a map of manifolds with linear connections.

Lemma 2.1. *If f is affine, then either f is an immersion, or* $\ker f_*$ *is a regular foliation on X.*

Proof. By Proposition 1.3 either f is an immersion or f_* has constant positive rank k. A version of the implicit function theorem then implies that locally there are coordinates on X and Y such that $f(x^1, \cdots, x^n) = (x^1, \cdots, x^k, 0, \cdots, 0)$. But then it is clear that $\ker f_*$ is a regular foliation, because each slice (= set of points on X defined by fixing $x^1, \cdots, x^k$) clearly lies on a different $f^{-1}(y)$. q.e.d.

To factor f through a submerison, all that needs to be done is to collapse the leaves of the foliation, and make sure the quotient space is a manifold. Without additional hypotheses, however, Hausdorfness fails (see the example in [7, p. 20]). Reinhart's results [8] say that one such hypothesis is completeness of a Riemann metric on X, which is bundle-like for the foliation. This happens to work for the foliation at hand.

Henceforth let X and Y be Riemannian manifolds. Since their connections are symmetric, a map $f\colon X \to Y$ is affine iff it is totally geodesic (Corollary 1.6). f is a *Riemannian submersion* [6] if f_* restricted to $(\ker f_*)^\perp$ is an isometry onto TY. The identity map and, more generally, immersions, which are Riemannian coverings, are trivial cases of this definition.

Theorem 2.2. *If X is complete, then each totally geodesic map f factors into a totally geodesic Riemannian submersion followed by a totally geodesic immersion.*

Proof. If f is an immersion, then a trivial factorization works; otherwise, $\ker f_*$ is nontrivial and holonomy-invariant (Proposition 2.1) so X is reducible. But then [5, Chap. IV, § 5] shows X is locally isometric to a product of open neighborhoods $V_1 \times V_2$ lying on a leaf of $\ker f_*$ and of $(\ker f_*)^\perp$, respectively. This means that the metric is bundle-like for $\ker f_*$, a fortiori [8, § 2]. Hence by [8, Cor. 3], collapsing the leaves of $\ker f_*$ produces a manifold B. The projection map $\pi\colon X \to B$ is a submersion [7], whence B inherits a metric from X such that π is a Riemannian submersion. It is also easy to see that the induced map $g\colon B \to Y$ is an immersion. The local diagram

$$\begin{tikzcd} V_1 \times V_2 \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & V_2 \arrow[ur, "g"'] & \end{tikzcd}$$

shows that π and g are t.g., since π is projection onto V_2 and f is t.g.

Remark. The leaves of $\ker f_*$ are totally geodesic submanifolds. This is also true without a metric [5, p. 181].

The following fact, contained in the proof of the theorem, is stated here for emphasis.

Proposition 2.3. *If f is affine and not an immersion, then X is reducible.*

Corollary 2.4. *If X is irreducible, then it admits only immersions as totally geodesic maps, and these are homothetic ($=$ isometric up to constant).*

Proof. See the argument in [5, p. 242].

3. Riemannian submersions

The structure theorem for totally geodesic Riemannian submersions which are nontrivial (i.e., not immersions) follows from a theorem of Hermann [4, Th. 1] and its (partial) converse, via a preliminary characterization depending on results of Hermann [4] and O'Neill [6].

Let $\pi\colon X \to B$ be a Riemannian submersion, and set $V = \ker \pi_*$ and $H = (\ker \pi_*)^\perp$; also assume that V is nontrivial. V and H are subbundles of TX, and V is integrable with the components of the fibres as leaves. In terms of the orthogonal decomposition $TX = H \oplus V$, $\beta(\pi)$ has matrix form

$$\begin{bmatrix} \beta(\pi)\,|\,H \times H & \beta(\pi)\,|\,H \times V \\ \beta(\pi)\,|\,V \times H & \beta(\pi)\,|\,V \times V \end{bmatrix},$$

which is symmetric, since $\beta(\pi)$ is symmetric by Corollary 1.6.

Lemma 3.1 [4, *Prop.* 3.1]. $\beta(\pi)|H \times H = 0$.

Lemma 3.2 [6].

(i) $\beta(\pi)\,|\,V \times V = 0$ *iff V has t.g. leaves.*

(ii) $\beta(\pi)\,|\,V \times H = 0$ *iff H is integrable.*

Proof. In [8, § 2] O'Neill defines for Riemannian submersions two "fundamental tensors" T and A. It follows from definitions in § 1 and [6] that

$$(\pi_*\,|\,H)^{-1}\beta(\pi)(Vu, Vv) = -H(D_{Vv}Vu) = -T_{Vv}Vu\,,$$
$$(\pi_*\,|\,H)^{-1}\beta(\pi)(Vu, Hv) = -H(D_{Hv}Vu) = -A_{Hv}Vu\,.$$

Since T is the second fundamental form of the fibres, it vanishes iff the fibres are t.g. submanifolds, which proves (i). Now A_{Hv} is skew-symmetric for the metric of X, and switches H and V. Hence $A_{Hv}Hw = 0$ for all v, w iff $A_{Hv}Vu = 0$ for all u, v, whence [6, Lemma 2] gives (ii). q.e.d.

The lemmas, together with the symmetry of $\beta(\pi)$, yield

Theorem 3.3. *A (nontrivial) Riemannian submersion is totally geodesic iff the fibres are totally geodesic submanifolds and the horizontal subbundle is integrable.*

Theorem 3.4 [6, *Th.* 1]. *If X is complete and $\pi\colon X \to B$ is a Riemannian submersion with t.g. fibres, then π is a fibre bundle with connection and the group of isometries of a fibre as structural group.*

Let $\pi\colon X \to B$ be a (G,F) bundle with connection [2], and assume B and F have Riemannian metrics with the one on F being G-invariant.

Theorem 3.5. *With the above hypothesis, there exists on X a natural Riemannian metric such that π is a Riemannian submersion with t.g. fibres.*

Proof. By definition of a connection [2], there is a horizontal subbundle H such that TX is a direct sum of H and $V = \ker \pi_*$. The inner product on TB is transferred to H by $(\pi_* | H)^{-1}$. The metric on F is G-invariant, which allows it to be transferred to the fibres in a well-defined way by local maps of the bundle structure. Since the fibres are integral manifolds of V, this gives an inner product on V. Then the inner product on TX is defined as a direct sum, and π is a Riemannian submersion.

The definition of a connection also specifies that there is unique C^∞ H-lift of curves in B which gives G-isomorphism of fibres (called parallel translation). Since the metric on F is G-invariant, the above definition of the metric on fibres shows parallel translation to be an isometry of the fibres.

Now Hermann's argument for proving [6, Th. 1] can be reversed in the following manner to show the fibres to be totally geodesic: Let $\alpha(t)$ be a curve on the fibre X_b parametrized proportionally to the arc length, with $0 \leq t \leq 1$. Let $v(0)$ be a given horizontal vector at $\alpha(0)$, and define $\sigma(t, s)$ to be the parallel translation of $\alpha(t)$ along some curve in B with intial vector $\pi_* v(0)$. Let $v(t) = \dfrac{\partial}{\partial s}\sigma(t, 0)$, a horizontal vector field along $\alpha(t)$, and let $f(s)$ denote the arc length of the curve $\sigma(s, t)$, with s fixed, from $t = 0$ to $t = 1$. But since parallel translation is isometric, $f(s)$ is constant, i.e., $f'(0) = 0$. But then the well-known first variation of the arc length formula (see [6, (2.2)]) gives $\int_0^1 \frac{1}{f(0)} < D_t\alpha'(t), v(t) > dt = 0$, so $\langle D_t\alpha'(t_1), v(t_1)\rangle = 0$ for some $0 \leq t_1 \leq 1$. Next let $\alpha_1(t) = \alpha(t/2)$, $0 \leq t \leq 1$, and apply the same procedure. Since $D_t\alpha_1'(t)$ and $D_t\alpha'(t)$ are proportional, one gets $\langle D_t\alpha'(t_2), v(t_2)\rangle = 0$ for some $0 \leq t_2 \leq \frac{1}{2}$. This procedure gives a sequence $t_i \to 0$ on which $\langle D_t\alpha'(t), v(t)\rangle$ vanishes. Thus $\langle D_t\alpha'(0), v(0)\rangle = 0$.

Since $v(0)$ was an arbitrary horizontal vector, this shows $D_t\alpha'(0)$ is vertical, which means it equals $D_t^F\alpha'(0)$, where D^F refers to the connection on the fibre. But then the fundamental form β of the inclusion $X_b \subset X$ vanishes, since $\beta(\alpha'(0), \alpha'(0)) = D_t\alpha'(0) - D_t^F\alpha'(0)$, and $\alpha(t)$ was an arbitrary curve on X_b.

Theorem 3.6. *$\pi\colon X \to B$ is a (nontrivial) totally geodesic Riemannian submersion with complete X iff it is a fibre bundle with flat connection, having complete metrics on the base and fibre, the latter being invariant under the structural group.*

Proof. This theorem, apart from the completeness statement, follows from Theorems 3.3, 3.4, and 3.6, since a flat connection implies that H is integrable. For the completeness part, if X is complete, then the fibres, being totally geodesic submanifolds, are complete. B is complete since each geodesic in it is the image of a horizontal geodesic (Lemma 3.1). On the other hand, it was noted in the proof of Theorem 2.2 that π being totally geodesic implies the metric on X is locally a product of vertical and horizontal neighborhoods. Hence the base and fibre being complete implies that X is complete.

Corollary 3.7. *If $\pi\colon X \to B$ is a totally geodesic (nontrivial) Riemannian submersion with complete and simply connected X, then X is a Riemannian product and π is projection.*

Proof. Theorem 3.6, Proposition 2.3, and the well-known de Rham reducibility theorem give this result.

The following results deal with the question of existence of totally geodesic nontrivial Riemannian submersions for a given Riemannian manifold X.

Lemma 3.8. *A Riemannian submersion $\pi\colon X \to B$ is totally geodesic iff* $\ker \pi_*$ *is holonomy-invariant.*

Proof. The "only if" part follows from Proposition 1.3. For the "if" part, observe that if $\ker \pi_*$ is holonomy-invariant, then [5, Ch. IV, § 5] shows $\ker \pi_*$ and its complement are integrable with t.g. leaves. Thus π is t.g. by Theorem 3.3. q.e.d.

For a given reducible Riemannian manifold X, let $T_xX = A_1 \oplus \cdots \oplus A_k$ be a complete decomposition of a tangent space into holonomy-invariant mutually orthogonal subspaces. Note the only holonomy-invariant subspaces are those of form $A_{i_1} \oplus \cdots \oplus A_{i_p}$.

Theorem 3.9. *The set of totally geodesic Riemannian submersions on a reducible complete Riemannian manifold X is in a* 1–1 *correspondence with the set of holonomy-invariant subspaces of the tangent space at a point.*

Proof. By Lemma 3.8 each t.g. Riemannian submersion corresponds to one of these subspaces. Conversely, given such a subspace, the proof of Theorem 2.2 shows that this subspace generates a foliation, whose leaves collapsed produce a space B such that $\pi\colon X \to B$ is the required submersion.

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