J. DIFFERENTIAL GEOMETRY
4 (1970) 5-12

A NEW METHOD FOR INFINITESIMAL RIGIDITY OF SURFACES WITH $K > 0$

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1. Introduction

Classical methods of proving infinitesimal rigidity of convex surfaces use an auxiliary vector function $\boldsymbol{y}$ and integral identities involving it. In this paper a new method will be given using $\boldsymbol{y}$. The main tool is the Lemma (3.0). The lemma applies directly to surfaces (bounded or not) with spherical image in a hemisphere. Where this assumption does not hold the well-known projective transformation of Darboux is used to obtain a surface and a deformation to which the lemma can be applied.

Infinitesimal rigidity is proved for various standard cases, including surfaces with fixed boundaries, Rembs boundaries (along which the unit surface normal is constant) and closed convex surfaces. If the spherical image does lie in a hemisphere, various results seem to be new, namely, Theorems (4.1), (4.2), and also (4.3), in which rigidity is proved while allowing the deformation (velocity) field to have *different* constant values on different boundary components. What also seems new is that these results hold even if the surface is not convex in the large, e.g., if the surface intersects itself or is many-sheeted.

The differentiability assumptions throughout will be (C'', C'') (that is the surface and the deformation respectively are both of class C'') as contrasted with (C''', C''') for classical methods, e.g., of Blaschke [1, p. 75], and with (C'', C') for the method of Minagawa and Rado [3] (the latter assume C''' for the surface in a neighborhood of a Rembs boundary).

The method has the unifying feature that for all the problems solved here the boundary conditions are expressed in the form $\boldsymbol{y} \cdot \boldsymbol{n} = \text{const.}$; cf. (4.1).

This work was initiated while the author was a Visiting Member of the Courant Institute of Mathematical Sciences.

2. General remarks

We state here definitions and hypotheses, which will hold throughout, as well as some general results of the theory of infinitesimal deformations. For a complete discussion see [1, Chap. VI].

$\boldsymbol{r} = \boldsymbol{r}(u, v)$ is the radius vector of a surface S (two-dimensional differentiable manifold) immersed in euclidean 3-space, joining the origin $(0, 0, 0)$ to a point

$(x(u,v), y(u,v), z(u,v))$ of S. The functions x, y, z are assumed to be C'' functions of u, v. In case S has a boundary (which is not assumed to be part of S) we assume that S plus boundary is contained in a C'' surface on which $r(u,v)$ is C'', and that x, y, z and their derivatives of the first two orders are given the induced values on the boundary of S. S will always be assumed regular, i.e., $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$ on S plus boundary.

A vector field $\boldsymbol{z} = \boldsymbol{z}(u,v) = (X, Y, Z)$ of class C'' called the *deformation field* will be defined on S and extended so as to be C'' on the boundary in the same manner as $\boldsymbol{r}$ was. A family of surfaces deformed from the first is given by $\boldsymbol{r}^* = \boldsymbol{r} + \varepsilon \boldsymbol{z}$, where ε is a real parameter. A necessary and sufficient condition that the arc length of a curve on $\boldsymbol{r}^*$ remain constant to the first order in ε is

$$(2.1) \qquad \boldsymbol{z}_u \cdot \boldsymbol{r}_u = 0 , \qquad \boldsymbol{z}_u \cdot \boldsymbol{r}_v + \boldsymbol{z}_v \cdot \boldsymbol{r}_u = 0 , \qquad \boldsymbol{z}_v \cdot \boldsymbol{r}_v = 0 ,$$

or in differential form $d\boldsymbol{z} \cdot d\boldsymbol{r} = 0$. In this case the deformation $\boldsymbol{z}$ is called an *infinitesimal bending*.

In consequence of (2.1) there exists a unique vector field $\boldsymbol{y} = \boldsymbol{y}(u,v) \in C'$ such that

$$(2.2) \qquad \boldsymbol{y} \times \boldsymbol{r}_u = \boldsymbol{z}_u , \qquad \boldsymbol{y} \times \boldsymbol{r}_v = \boldsymbol{z}_v ,$$

or in differential form $d\boldsymbol{z} = \boldsymbol{y} \times d\boldsymbol{r}$. $\boldsymbol{y}$ is called the *rotation field*.

It follows from $\boldsymbol{z}_{uv} = \boldsymbol{z}_{vu}$ that there exist real scalars α, β, γ such that

$$(2.3) \qquad \boldsymbol{y}_u = \alpha \boldsymbol{r}_u - \beta \boldsymbol{r}_v , \qquad \boldsymbol{y}_v = \gamma \boldsymbol{r}_u - \alpha \boldsymbol{r}_v , \qquad \text{so } \boldsymbol{n} \cdot d\boldsymbol{y} = 0 .$$

If we regard $\boldsymbol{z}$ and $\boldsymbol{y}$ as position vectors, they describe two surfaces (which may be degenerate) called the *deformation surface* and the *rotation surface* respectively.

A deformation $\boldsymbol{z}$ is said to be *trivial* if $\boldsymbol{z} = \boldsymbol{c} \times \boldsymbol{r} + \boldsymbol{d}$, where $\boldsymbol{c}, \boldsymbol{d}$ are constant vectors. A trivial deformation coincides with the velocity field of a rigid motion. Two deformations which differ by a trivial deformation are called *equivalent* (clearly an equivalence relation on the set of all C'' deformations).

Obviously if one of two equivalent deformations is trivial, so is the other. If a surface admits only trivial bendings, it is said to be *infinitesimally rigid*. If there are restrictions on $\boldsymbol{z}$ on the boundary the surface is said to be infinitesimally rigid with respect to those restrictions.

3. The main lemma

We shall consider surfaces which are immersions in E^3 of parameter domains of the following topological types:

1. An open disk with a finite number (possibly zero) of closed disks removed.

2. A sphere.

(3.0) Lemma. *Let S be a surface (possibly unbounded) with parameter domain D of type one such that*

1. $K > 0$,
2. *the spherical image of S lies in an open hemisphere.*

Then $\boldsymbol{y}\cdot\boldsymbol{n} \neq 0$ on the boundary of D implies $\boldsymbol{y}\cdot\boldsymbol{n} \neq 0$ in the interior of D.

Remark. $\boldsymbol{y}\cdot\boldsymbol{n} = 0$ at a point of D if and only if the point corresponds to a singular point of the deformation surface. Liebmann considered these points in his early work on infinitesimal rigidity [2], however his method is otherwise dissimilar.

Proof. Suppose, by way of contradiction, that $\boldsymbol{y}\cdot\boldsymbol{n} \neq 0$ on the boundary of D, but $\boldsymbol{y}\cdot\boldsymbol{n} = 0$ at some point inside D. Then the set of zeros of $\boldsymbol{y}\cdot\boldsymbol{n}$ is contained in the interior of a compact subset D' of D. We choose the xyz cartesian coordinate system so that the spherical image of S lies in the open "upper" half of the unit sphere centered at $(0, 0, 0)$.

Claim: There exists a constant vector $\boldsymbol{e}$ such that

$$\boldsymbol{y} - \boldsymbol{e} \neq 0 \quad \text{on } D' , \tag{3.1}$$

$$(\boldsymbol{y} - \boldsymbol{e})\cdot\boldsymbol{n} \neq 0 \quad \text{on the frontier of } D' , \tag{3.2}$$

$$(\boldsymbol{y} - \boldsymbol{e})\cdot\boldsymbol{n} = 0 \quad \text{at a point in } D' . \tag{3.3}$$

Proof of Claim: $\boldsymbol{y}\cdot\boldsymbol{n} \not\equiv 0$ on D'. Suppose $\boldsymbol{y}\cdot\boldsymbol{n} > 0$ (<0) at some point of D'. Since the spherical image of D' is a compact subset of the open upper hemisphere, $\boldsymbol{e}\cdot\boldsymbol{n} > 0$ (<0) on D' for any constant vector $\boldsymbol{e}$ whose direction is sufficiently close to that of the positive (negative) z axis. The set of such vectors $\boldsymbol{e}$ form a solid half-cone opening upward (downward) with vertex at $(0, 0, 0)$.

The rotation surface $\boldsymbol{y}$ cannot contain an open subset of E^3, hence the intersection of any ball with center $(0, 0, 0)$ and the half-cone always contains a point not on the surface $\boldsymbol{y}$. A vector $\boldsymbol{e}$ from $(0, 0, 0)$ to such a point satisfies (3.1).

On the frontier of D', which is compact, there is an m such that $|\boldsymbol{y}\cdot\boldsymbol{n}| > m > 0$. Hence (3.2) is satisfied if $|\boldsymbol{e}| < m$.

$\boldsymbol{y}\cdot\boldsymbol{n}$ takes, on D', a maximum (minimum) M. If $|\boldsymbol{e}| < |M|$ then, at the maximum (minimum) point $\boldsymbol{y}\cdot\boldsymbol{n} > \boldsymbol{e}\cdot\boldsymbol{n} > 0$ $(\boldsymbol{y}\cdot\boldsymbol{n} < \boldsymbol{e}\cdot\boldsymbol{n} < 0)$. Since $(\boldsymbol{y} - \boldsymbol{e})\cdot\boldsymbol{n} < 0$ (>0) at a zero of $\boldsymbol{y}\cdot\boldsymbol{n}$ it changes sign and hence vanishes at some point of D'.

Thus a vector $\boldsymbol{e}$ exists satisfying the claim. For each such $\boldsymbol{e}$ we define a new deformation $\boldsymbol{z}_e = \boldsymbol{z} - (\boldsymbol{e} \times \boldsymbol{r})$. $\boldsymbol{z}_e$ is equivalent to $\boldsymbol{z}$ and has the rotation vector $\boldsymbol{y}_e = \boldsymbol{y} - \boldsymbol{e}$. By (3.3) there is a point P_0 in D' at which $\boldsymbol{y}_e\cdot\boldsymbol{n} = 0$. Not both $(\boldsymbol{y}_e\cdot\boldsymbol{n})_u$ and $(\boldsymbol{y}_e\cdot\boldsymbol{n})_v$ vanish at P_0, for then $\boldsymbol{y}_e\cdot\boldsymbol{n}_u = 0$ and $\boldsymbol{y}_e\cdot\boldsymbol{n}_v = 0$ by (2.3) while $\boldsymbol{n}_u$, $\boldsymbol{n}_v$ and $\boldsymbol{n}$ are linearly independent. This contradicts (3.1).

By the Implicit Function theorem there exists a neighborhood of P_0 in which the set of points for which $\boldsymbol{y}_e \cdot \boldsymbol{n} = 0$ is a smooth curve segment which, by (3.2), is a subset of the interior of D'. The endpoints of the segment lie also in D' and we may, by repetition of the argument, extend it indefinitely in both directions to a curve σ lying in the interior of D'. (It is not hard to show that σ must be a closed curve, but this is not explicitly needed here.)

Let σ be represented by $\boldsymbol{r} = \boldsymbol{r}(s)$, $-\infty < s < \infty$. On σ, $\boldsymbol{y}_e \cdot \boldsymbol{n}' = 0$ (the prime indicates differentiation with respect to s). Since the second fundamental form is definite, $\boldsymbol{r}' \cdot \boldsymbol{n}' \neq 0$.

We can conclude that $\boldsymbol{y}_e \times \boldsymbol{r}' \neq 0$ on σ, for the angle between $\boldsymbol{r}'$ and $\boldsymbol{n}'$ is never a right angle on σ while the angle between $\boldsymbol{y}_e$ and $\boldsymbol{n}'$ is a right angle.

Thus we can represent $\boldsymbol{y}_e \times \boldsymbol{r}' = \lambda \boldsymbol{n}$ where λ is a scalar bounded away from zero. Let $\boldsymbol{f}$ be a constant vector parallel to the z axis. Then $(\boldsymbol{f} \cdot \boldsymbol{z}_e)' = \boldsymbol{f} \cdot \boldsymbol{y}_e \times \boldsymbol{r}' = \lambda \boldsymbol{f} \cdot \boldsymbol{n}$ is bounded away from zero; hence $\boldsymbol{f} \cdot \boldsymbol{z}_e$ is unbounded, which is a contradiction.

4. Surfaces with spherical image in a hemisphere

(4.1) Theorem. *Let S be a surface with parameter domain of type* 1 *with a finite number* (≥ 1) *of piecewise smooth boundary curves $B_0, \cdots, B_n$. Suppose*

1. $K > 0$ *on S, except on a denumerable set,*
2. *the spherical image of S plus boundary lies in an open hemisphere*, i.e., *there exists a constant vector $\boldsymbol{e}$ such that $\boldsymbol{e} \cdot \boldsymbol{n} > 0$ on S* plus boundary,
3. $\boldsymbol{y} \cdot \boldsymbol{n} \equiv 0$ *on B_0 and on each of the remaining B_i, $\boldsymbol{y} \cdot \boldsymbol{n}$ either vanishes identically or not at all. Then $y \equiv 0$ on S.*

Note. We do not assume $K > 0$ on B_i, the immersion of D in E^3 need not be an imbedding, and S, in the large, need not be a piece of a convex surface.

Throughout this paper the hypothesis 1 of (4.1) is assumed. It is an open question whether the method can be extended to the case $K > 0$ on a dense set (as in [3]).

Proof. Choose the xyz coordinate system such that the spherical image of S plus boundary lies in the open upper half of the unitsphere with center at $(0, 0, 0)$. Then $(0, 0, c) \cdot \boldsymbol{n} > 0$ (<0) on S plus boundary if $c > 0$ (<0). We can choose the values of c from an arbitrary deleted neighborhood of zero on the real line so that $\boldsymbol{y} \cdot \boldsymbol{n} - (0, 0, c) \cdot \boldsymbol{n} \neq 0$ on the union of the boundary of S and the set of zeros of K on S.

By (3.0), $(\boldsymbol{y} - (0, 0, c)) \cdot \boldsymbol{n} \neq 0$ on S. If $c > 0$ (<0) then $(\boldsymbol{y} - (0, 0, c)) \cdot \boldsymbol{n} < 0$ (>0) on B_0 and hence on S. Since c is arbitrarily small, this means that $\boldsymbol{y} \cdot \boldsymbol{n} \leq 0$ (≥ 0) on S and hence that $\boldsymbol{y} \cdot \boldsymbol{n} \equiv 0$. Then $\boldsymbol{y} \cdot \boldsymbol{n}_u = \boldsymbol{y} \cdot \boldsymbol{n}_v \equiv 0$, which implies, since $\boldsymbol{n}, \boldsymbol{n}_u, \boldsymbol{n}_v$ are linearly independent, that $\boldsymbol{y} \equiv 0$.

(4.2) Corollary. *Let the hypotheses be as in Theorem* (4.1) *except that hypothesis* 3 *is replaced by*: $\boldsymbol{n}$ *is constant to first order in the deformation* *parameter on the boundary of S. Then $y \equiv 0$.*

Proof. Since $\boldsymbol{n} \to \boldsymbol{n} + \varepsilon(\boldsymbol{y} \times \boldsymbol{n})$ under a deformation with parameter ε, the hypothesis is equivalent to $\boldsymbol{y} \times \boldsymbol{n} = 0$ on the boundary. Hence $\boldsymbol{y} = (\boldsymbol{y} \cdot \boldsymbol{n})\boldsymbol{n}$. Differentiating along B_i we obtain

$$\boldsymbol{y}' = (\boldsymbol{y} \cdot \boldsymbol{n})\boldsymbol{n}' + (\boldsymbol{y} \cdot \boldsymbol{n})'\boldsymbol{n} \ .$$

From (2.3) it follows that $(\boldsymbol{y} \cdot \boldsymbol{n})' = 0$ and thus $\boldsymbol{y} \cdot \boldsymbol{n} = \text{const.}$ on B_i. If $\boldsymbol{y} \cdot \boldsymbol{n} = 0$ on some B_i, (4.1) gives the result. If $\boldsymbol{y} \cdot \boldsymbol{n} \neq 0$ on any B_i, then by (3.0), $\boldsymbol{y} \cdot \boldsymbol{n} \neq 0$ on S. But the boundary integral of the identity

$$\oint_{\partial S} \boldsymbol{n} \cdot d\boldsymbol{z} = 2 \iint_S H(\boldsymbol{y} \cdot \boldsymbol{n}) dA, \qquad (H = \text{mean curvature}) \ ,$$

vanishes by (2.2), and we have a contradiction for, since $H > 0$, it follows that $\boldsymbol{y} \cdot \boldsymbol{n} \equiv 0$. To prove the identity let d denote the exterior differential operator. Then

$$d(\boldsymbol{n} \cdot d\boldsymbol{z}) = d\boldsymbol{n} \wedge d\boldsymbol{z} = d\boldsymbol{n} \wedge \boldsymbol{y} \times d\boldsymbol{r} = -\boldsymbol{y} \cdot d\boldsymbol{r} \mathbin{\hat{\times}} d\boldsymbol{n} = 2H(\boldsymbol{y} \cdot \boldsymbol{n}) dA \ .$$

Question. Does the result hold if the condition on the spherical image is dropped?

(4.3) Theorem. *Let S be a surface of type 1 with a finite number (≥ 1) of piecewise smooth boundary curves $B_0, \cdots, B_n$. Then $\boldsymbol{y} \equiv 0$, if*

1. *$K > 0$ on S, except on a denumerable set,*
2. *the spherical image of S plus boundary lies in an open hemisphere,*
3. *$z = c_i = \text{const.}$ on $B_i, i = 0, \cdots, m$, with $0 \leq m \leq n$,*
4. *$\boldsymbol{n} = \boldsymbol{d}_i = \text{const.}$ on B_i for $i = m + 1, \cdots, n$ (i.e., B_i is a Rembs boundary).*

Proof. Along B_i, $i = 0, \cdots, m$, $d\boldsymbol{z} = 0$ implies $\boldsymbol{y}$ is parallel to $d\boldsymbol{r}$, hence $\boldsymbol{y} \cdot \boldsymbol{n} = 0$. Along B_i, $i = m + 1, \cdots, n$, $\boldsymbol{y} \cdot \boldsymbol{n} = \text{const.}$ Hence the theorem follows from (4.1).

5. Rigidity for convex surfaces

We consider some surfaces without the restriction on the spherical image.

(5.0) Theorem. *Let S be a surface defined on a parameter domain of type 1 with piecewise smooth boundary curves $B_0, \cdots, B_n$. Then $\boldsymbol{z} \equiv 0$, if*

1. *$K > 0$ on S, except on a denumerable set,*
2. *S may be completed to a closed convex surface of class C'' by the addition of closed caps, each homeomorphic to a closed disk and having $K > 0$ at some point P_0 of one of the caps (say the one bounded by B_0),*
3. *$\boldsymbol{z} = 0$ on B_i, $i = 0, \cdots, M$, where $0 \leq M \leq N$,*

4. *$\boldsymbol{n}$ is constant on each B_i, $i = M + 1, \cdots, N$.*

Proof. Place the completed surface tangent to the xy plane so that $P_0 = (0, 0, 0)$ and so that S lies in the half-space $z > 0$. Now the projective transformation (of Darboux, cf. [1, § 25, § 95] and [3, § 1.2.3])

$$x' = x/z, \qquad y' = y/z, \qquad z' = 1/z \tag{5.1}$$

transforms S into a surface S' with a representation in the form $z' = z'(x', y')$. Let the images of the B_i be B_i' (B_0 becomes the outer boundary). The parameters x', y' vary in a region of type 1.

An infinitesimal bending field $\boldsymbol{z}'$ of class C'' is defined on S' by $\boldsymbol{z}' = (X', Y', Z')$ where

$$X' = X/z\ , \qquad Y' = Y/z\ , \qquad Z' = -(\boldsymbol{r}\cdot\boldsymbol{z})/z\ . \tag{5.2}$$

This implies immediately that the conditions $\boldsymbol{z}' = 0$ hold on B_i', $i = 0, \cdots, M$. n' is constant on each B_i', $i = M + 1, \cdots, N$, because n constant on a boundary curve implies that S is in contact with a plane along the boundary curve, and a projective transformation preserves planes and tangency. Also, as is well known, a projective transformation preserves the sign of the gauss curvature. Now by (4.1), the rotation vector for $\boldsymbol{z}'$ vanishes identically; hence, by (2.2) and the boundary condition $\boldsymbol{z}' \equiv 0$ on S', $\boldsymbol{z} \equiv 0$ on S follows from (5.2).

(5.3) Theorem. *A closed convex surface with $K > 0$ (except on a denumerable set) is infinitesimally rigid.*

Proof. Choose a point P of the surface S at which the gauss curvature is positive. Place S tangent to the xy plane so that $P = (0, 0, 0)$ and S lies in the half-space $z \geq 0$. Then $z > 0$ except at P.

We may assume without loss of generality that $\boldsymbol{z} = 0$ and $\boldsymbol{y} = 0$ at P. For, if $\boldsymbol{z}^*$ is any bending field with rotation field $\boldsymbol{y}^*$, then $\boldsymbol{z} = \boldsymbol{z}^* - (\boldsymbol{y}^*(P) \times \boldsymbol{r} + \boldsymbol{z}^*(P))$ is an equivalent deformation.

We apply the transformation (5.1) to obtain an unbounded convex surface S' of class C'' with gauss curvature positive, except on a denumerable set, which can be represented in the form $z' = z'(x', y')$ on the entire x', y' plane. P, of course, is mapped to infinity (cf. [3, § 1.2.3]).

Elementary calculation using (5.1) gives (with numerical subscripts denoting components of vectors on the x, y, z and $x'y'z'$ axes):

$$\begin{aligned}(\boldsymbol{r}_u' \times \boldsymbol{r}_v')_i &= -(\boldsymbol{r}_u \times \boldsymbol{r}_v)_i/z^3, \qquad i = 1, 2\ , \\ (\boldsymbol{r}_u' \times \boldsymbol{r}_v')_3 &= |\boldsymbol{r}_u \times \boldsymbol{r}_v|\,(\boldsymbol{r}\cdot\boldsymbol{n})/z^3\ .\end{aligned} \tag{5.4}$$

Therefore the unit "outer" normal $\boldsymbol{n}$ of S is mapped into the unit "inner" normal $\boldsymbol{n}'$ of S', and the spherical image of S' lies in the open "upper" hemisphere of the unit sphere.

As in the previous proof, (5.2) is used to obtain a bending field on S'. The rotation field y' corresponding to z' is given by

$$y_1' = s_2, \quad y_2' = -s_1, \quad y_3' = -y_3, \quad s_1' = -y_2, \quad s_2' = y_1, \quad s_3' = s_3, \tag{5.5}$$

where $s = z - y \times r$ and $s' = z' - y' \times r'$. (5.5) is a special case of the transformation formula [1, § 25, p. 60, (11)].

From (5.4) and (5.5) it follows easily that

$$y' \cdot n' = [(s \times n)_3 - y_3(r \cdot n)]/D\ , \tag{5.6}$$

where $D = (n_1^2 + n_2^2 + (r \cdot n)^2)^{1/2}$.

Define $z_c = z - (0, 0, c) \times r$, where c is a real number. z_c is a bending equivalent to z with rotation vector $y_c = y - (0, 0, c)$. If we define $s_c = (z_c - y_c \times r)$ and substitute in (5.6) we obtain

$$\begin{aligned} D(y_c' \cdot n') &= [(z_c - y_c \times r) \times n]_3 - (y_c)_3(r \cdot n) \\ &= (z \times n)_3 - (y \cdot n)z + c(r \cdot n). \end{aligned} \tag{5.7}$$

Now S can be represented in a neighborhood of the origin in the xy plane in the form $z = z(x, y)$. Expansion by Taylor's theorem to terms of second order using the conditions (2.2) and (2.3) at P shows that the first two terms of the righthand side of (5.7) are $o(x^2 + y^2)$ at P. The third has the sign of c, since $r \cdot n \geq 0$, and furthermore it dominates the first two terms in a neighborhood of P, for it can be represented to terms of second order as

$$r \cdot n = -\tfrac{1}{2}(Lx^2 + 2Mxy + Ny^2) \neq o(x^2 + y^2),$$

where L, M, N are evaluated at P.

It follows that $y_c' \cdot n' > 0$ (<0) if $c > 0$ (<0), in some neighborhood of infinity on S'. By (5.7) we may choose c arbitrarily close to zero so that $y_c' \cdot n' \neq 0$ on the denumerable set of points of S' where $K' = 0$, i.e., if Q is such a point we avoid the value c such that

$$0 = (z \times n)_3 - (y \cdot n)z + c(r \cdot n)$$

at Q.

Now by (3.0) $y_c' \cdot n' > 0$ (<0) on S' if $c > 0$ (<0). Since the limit of $y_c' \cdot n'$ as c approaches zero is $y' \cdot n'$, it follows that $y' \cdot n' \equiv 0$. As before $y' \equiv 0$ on S'. From (5.5) (cf. also [1, p. 60, (14) et seq.]) it follows that $y =$ const. on S. Hence $y \equiv 0$ on S since $y(P) = 0$.

Theorem. *A convex surface with a finite number of piecewise smooth boundary curves on each of which* n *is const. is infinitesimally rigid.*

The proof is virtually identical to the previous one. We note that the condition $n =$ const. on a curve is preserved under a projective transformation, as mentioned above in the proof of (5.0). Therefore $y'_c \cdot n'$ is const. on each boundary curve of S'. At the stage of the proof corresponding to the next to last paragraph of the previous proof we avoid the finite number of values of c for which $y'_c \cdot n' = 0$ on the boundary curves of S'.

6. Remarks

We conclude with some remarks on unbounded complete surfaces with $K > 0$. Stoker [4] showed that the spherical image of such a surface lies on an open hemisphere. From this it follows that there exists a unit vector $\boldsymbol{e}$ such that $\boldsymbol{e} \cdot \boldsymbol{n} > 0$ for all points P on S.

Theorem. *Let $\boldsymbol{y}$ be the rotation vector of a deformation $\boldsymbol{z}$ such that $|\boldsymbol{y} \cdot \boldsymbol{n}| = o(\boldsymbol{e} \cdot \boldsymbol{n})$ as $P \to \infty$. Then $\boldsymbol{y} \equiv 0$ on S.*

Proof. Let c be a real number. Then there exists a neighborhood $N(c)$ of ∞ such that $(\boldsymbol{y} - c\boldsymbol{e}) \cdot \boldsymbol{n} < 0$ (>0) if $c > 0$ (<0) when $P \in N(c)$. By (3.0) the inequalities hold everywhere on S. Hence $\boldsymbol{y} \cdot \boldsymbol{n} \equiv 0$ which implies $\boldsymbol{y} \equiv 0$.

If the spherical image of S lies in a compact subset of a hemisphere, then the hypothesis can be weakened to $\lim \boldsymbol{y} \cdot \boldsymbol{n} = 0$, for then $\boldsymbol{e}$ may be chosen so that $\boldsymbol{e} \cdot \boldsymbol{n}$ is positive and bounded from zero.

Stoker in [5] proved infinitesimal rigidity for complete unbounded convex surfaces under the (c''', c''') hypothesis and the assumption that the deformation $\boldsymbol{z}$ is bounded. It is not clear what the relation between this and the present assumption on $\boldsymbol{y} \cdot \boldsymbol{n}$ is.

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