J. DIFFERENTIAL GEOMETRY
2 (1968) 217-223

THE GAUSS MAP OF IMMERSIONS OF RIEMANNIAN MANIFOLDS IN SPACES OF CONSTANT CURVATURE

MORIO OBATA

Dedicated to Professor H. Hombu on his 60th birthday

0. Introduction

With an immersion x of a Riemannian n-manifold M into a Euclidean N-space E^N there is associated the Gauss map, which assigns to a point p of M the n-plane through the origin of E^N and parallel to the tangent plane of $x(M)$ at $x(p)$, and is a map of M into the Grassmann manifold $G_{n,N} = O(N)/O(n) \times O(N-n)$.

An isometric immersion of M into a Euclidean N-sphere S^N can be viewed as one into a Euclidean $(N+1)$-space E^{N+1}, and therefore the Gauss map associated with such an immersion can be determined in the ordinary sense. However, for the Gauss map to reflect the properties of the immersion into a sphere, instead of into the Euclidean space, it seems desirable to modify the definition of the Gauss map appropriately. To this end we consider the set Q of all the great n-spheres in S^N, which is naturally identified with the Grassmann manifold of $(n+1)$-planes through the center of S^N in E^{N+1}, since such $(n+1)$-planes determine unique great n-spheres and conversely.

In this paper by the Gauss map of an immersion x into S^N is meant a map of M into the Grassmann manifold $G_{n+1,N+1}$ which assigns to each point p of M the great n-sphere tangent to $x(M)$ at $x(p)$, or the $(n+1)$-plane spanned by the tangent space of $x(M)$ at $x(p)$ and the normal to S^N at $x(p)$ in E^{N+1}.

More generally, with an immersion x of M into a simply-connected complete N-space V of constant curvature there is associated a map which assigns to each point p of M the totally geodesic n-subspace tangent to $x(M)$ at $x(p)$. Such a map is called the (generalized) Gauss map. Thus the Gauss map in our generalized sense is a map: $M \to Q$, where Q stands for the space of all the totally geodesic n-subspaces in V.

The purpose of the present paper will be first to obtain a relationship among the Ricci form of the immersed manifold and the second and third fundamental forms of the immersion, and then to give a geometric interpretation of the

Received April 15, 1967 and, in revised form, April 10, 1968. This paper grew out of conversations with Professor J. Eells, Jr. at the University of Cambridge while the author was a Senior Visiting Fellow at the University of Southampton supported by the Science Research Council of the United Kingdom.

third fundamental form in this case by using the notion of the Gauss map. As a result, we will be able to consider the case where these three forms are proportional to the original Riemannian metric.

By a result of Mostow [3] that every compact homogeneous space can be equivariantly immersed into a certain sphere, we can show that if the isotropy subgroup is irreducible the immersion is pseudo-umbilical and the Gauss map is conformal. More strongly such a homogeneous space can be minimally immersed into a certain sphere by results of Hsiang [2] and Takahashi [5], and the Gauss map associated with such minimal immersion is shown to be homothetic, the latter being a generalization of a result of Chern [1] that a minimal surface in a Euclidean space has the anti-holomorphic Gauss map. Indeed, we shall show that the Gauss map associated with a minimal immersion is conformal if and only if the manifold is Einsteinian. If the dimension of the manifold is greater than 2, then "conformal" turns out to be "homothetic".

Obviously one can expect to generalize our method to pseudo-Riemannian manifolds with arbitrary signature of metric.

1. Preliminaries [6]

Throughout this paper let V denote one of the following simply-connected complete Riemannian manifold of dimension N:

(i) An N-sphere S^N of radius a (or of curvature $1/a^2$).

(ii) A Euclidean N-space E^N.

(iii) A hyperbolic N-space H^N of curvature $-1/a^2$.

The bundle $F(V)$ of the orthonormal frames on V can be identified with the group $G(N)$ which is one of the following according as the type of V:

(i) The orthogonal group $O(N+1)$.

(ii) The group $E(N)$ of the Euclidean motions of E^N.

(iii) The group $O(1,N)$ of inhomogeneous Lorentz transformations on H^N.

In fact, fixing a point p^0 in V and an orthonormal frame $b^0 = (p^0, e_1^0, \cdots, e_N^0)$ at p^0, there is one and only one transformation g in $G(N)$ which sends b^0 into a frame $b = (p, e_1, \cdots, e_N)$ at a point p in V, and the correspondence $b \leftrightarrow g$ is the desired identification. The isotropy subgroup at p^0 is $O(N)$ in any case, and obviously V is the homogeneous space $G(N)/O(N)$.

Let θ_{AB} be the Maurer-Cartan forms on $G(N)$, where from now on we agree on the following ranges of indices:

$$0 \leq A, B, C, \cdots \leq N; \qquad 1 \leq \lambda, \mu, \nu, \cdots \leq N .$$

Then θ_{AB} satisfy the following algebraic relations:

$$\theta_{00} = 0, \quad \varepsilon\theta_{0\lambda} + \theta_{\lambda 0} = 0, \quad \theta_{\lambda\mu} + \theta_{\mu\lambda} = 0 ,$$

where from now on ε takes the value:

$$\varepsilon = \begin{cases} 1 \text{ if } G(N) = O(N+1), & V = S^N, \\ 0 \text{ if } G(N) = E(N), & V = E^N, \\ -1 \text{ if } G(N) = O(1, N), & V = H^N. \end{cases}$$

θ_{AB} also satisfy the structure equations:

$$d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB}. \tag{1}$$

On putting

$$\theta_\lambda = a\theta_{0\lambda},$$

the Riemannian metric $d\sigma^2$ on V is given by

$$d\sigma^2 = \sum_\lambda (\theta_\lambda)^2,$$

and (1) becomes

$$\begin{aligned} d\theta_\lambda &= \sum_\mu \theta_\mu \wedge \theta_{\mu\lambda}, \\ d\theta_{\lambda\mu} &= \sum_\nu \theta_{\lambda\nu} \wedge \theta_{\nu\mu} - \frac{\varepsilon}{a^2}\theta_\lambda \wedge \theta_\mu, \end{aligned} \tag{2}$$

which are the structure equations on V. Denoting by $\Theta_{\lambda\mu}$ the curvature forms on V, from (2) we have

$$\Theta_{\lambda\mu} = -\frac{\varepsilon}{a^2}\theta_\lambda \wedge \theta_\mu.$$

Let M be a Riemannian n-manifold isometrically immersed into the space V by a mapping $x: M \to V$, $F(M)$ denote the bundle of frames on M, and B be the set of elements $b = (p, e_1, \cdots, e_N)$ such that $(p, e_1, \cdots, e_n) \in F(M)$ and $(x(p), e_1, \cdots, e_N) \in F(V)$, where e_i, $1 \leq i \leq n$, are identified with $dx(e_i)$. Then $\psi: B \to M$ can be viewed as a principal bundle with the fibre $O(N) \times O(N-n)$, and $\tilde{x}: B \to F(V) = G(N)$ is the natural immersion defined by $\tilde{x}(b) = (x(p), e_1, \cdots, e_N)$.

Let ω_λ, $\omega_{\lambda\mu}$ be the 1-forms on B induced from θ_λ, $\theta_{\lambda\mu}$ by the map $\tilde{x}$. Then we have

$$\omega_r = 0, \tag{3}$$

and the Riemannian metric ds^2 on M is given by

$$ds^2 = \sum_i (\omega_i)^2,$$

where from now on we agree on the following ranges of indices:

$$1 \leq i, j, k, \cdots \leq n; \qquad n+1 \leq r, s, t, \cdots \leq N.$$

Furthermore, from (2) we obtain

$$\omega_{ir} = \sum_j A_{rij}\omega_j, \quad A_{rij} = A_{rji},$$
$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji},$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \sum_r \omega_{ir} \wedge \omega_{jr} - \frac{\varepsilon}{a^2}\omega_i \wedge \omega_j.$$

The curvature forms Ω_{ij} of M can then be written as

$$\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{\varepsilon}{a^2}\omega_i \wedge \omega_j - \sum_r \omega_{ir} \wedge \omega_{jr}.$$

By expressing Ω_{ij} as a 2-form of ω_k:

$$\Omega_{ij} = \frac{1}{2}\sum_{k,l} K_{ijkl}\omega_k \wedge \omega_l,$$

we obtain

$$(4) \qquad K_{ijkl} = -\frac{\varepsilon}{a^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_r (A_{rik}A_{rjl} - A_{ril}A_{rjk}).$$

Obviously K_{ijkl} give the components of the curvature tensor of M.

At a point $b = (p, e_1, \cdots, e_N)$ in B by forming the form

$$II = \sum_{i,r} \omega_{ir}\omega_i e_r = \sum_{r,i,j} A_{rij}\omega_i\omega_j e_r,$$

we know that II is independent of the choice of the point b over p and is a normal-vector-valued quadratic differential form on M. II is called the *second fundamental form* of the immersion x, whose vanishing defines a *totally geodesic* immersion. The normal vector

$$N = \sum_i II(e_i, e_i) = \sum_r A_r e_r,$$

where $A_r = \sum_i A_{rii}$, is independent of the choice of the frame and is called the *mean curvature vector* of the immersion x. If N vanishes identically, then x is said to be *minimal*.

Let $X = \sum_r X_r e_r$ be a normal vector of $x(M)$ at $x(p)$. Then the quadratic differential form defined by

$$II_X = \langle II, X \rangle = \sum_{r,i,j} A_{rij}X_r\omega_i\omega_j,$$

is called the second fundamental form of the immersion x in the direction X. Since N is uniquely determined by the immersion, the form

$$II_N = \sum_{r,i,j} A_r A_{rij} \omega_i \omega_j$$

has a special meaning related to the immersion x. It is easy to see that $II_N = 0$ if and only if $N = 0$. Thus the immersion is minimal if and only if II_N vanishes identically.

If the form II_N is proportional to the Riemannian metric ds^2 on M, that is, if

$$II_N = \rho ds^2 = \rho \sum_i \omega_i \omega_i ,$$

then due to Ōtsuki [4] the immersion is said to be *pseudo-umbilical*. If this is the case we have

$$\rho = \frac{1}{n} \sum_r A_r^2 = \frac{1}{n} \|N\|^2 .$$

Let us consider the quadratic differential form

$$\Psi = \sum_{j,k} K_{jk} \omega_j \omega_k ,$$

called the *Ricci form* of M, where we have put

$$K_{jk} = \sum_i K_{ijki} .$$

The Ricci form is independent of the choice of the frame and therefore is a quadratic differential form on M. We have, from (4),

$$K_{jk} = \varepsilon \frac{n-1}{a^2} \delta_{jk} - \sum_{r,i} A_{rik} A_{rji} + \sum_r A_r A_{rjk} ,$$

and hence

$$\Psi = \varepsilon \frac{n-1}{a^2} \sum_i \omega_i \omega_i - \sum_{i,r} (\omega_{ir})^2 + II_N . \tag{5}$$

We shall next consider the meaning of the term $\sum_{i,r} (\omega_{ir})^2$.

2. The Gauss map

Let Q be the set of all the totally geodesic n-spaces in V. Then the group $G(N)$ acts on Q transitively. Take a point p in Q. Then the isotropy subgroup at p is identified with $G(n) \times O(N-n)$, where $G(n)$ is viewed as acting on the totally geodesic n-space V_0 representing the point p in Q and $O(N-n)$ on the totally geodesic $(N-n)$-space orthogonal to V_0 at the point of intersection which is kept fixed. Therefore Q is identified with a homogeneous space

$$Q = G(N)/G(n) \times O(N-n) .$$

By using the Maurer-Cartan forms θ_{AB} of $G(N)$ we introduce a quadratic differential form $d\Sigma^2$ on Q:

$$d\Sigma^2 = \sum_r (\theta_{0r})^2 + \sum_{i,r} (\theta_{ir})^2 ,$$

which is obviously invariant under the action of $G(N)$.

In the case $G(N) = O(N+1)$, Q is the Grassmann manifold $G_{n+1,N+1}$ of the $(n+1)$-spaces through the origin in the Euclidean $(N+1)$-space and $d\Sigma^2$ is the standard Riemannian metric on it with respect to which Q is a Riemannian symmetric space.

In the case $G(N) = O(1, N)$, $d\Sigma^2$ is the standard pseudo-Riemannian metric with respect to which Q is a pseudo-Riemannian symmetric space.

In the case $G(N) = E(N)$, $d\Sigma^2$ is obviously degenerate. However, if we consider the natural projection of Q onto the Grassman manifold $G_{n,N}$, by identifying the parallel planes, $d\Sigma^2$ coincides with the quadratic differential form induced from the standard Riemannian metric on $G_{n,N}$ by the projection.

With an immersion $x: M \to V$ we associate the (generalized) *Gauss map* $f: M \to Q$, where $f(p)$, $p \varepsilon M$, is totally geodesic n-space tangent to $x(M)$ at $x(p)$, and consider the following diagram:

$$\begin{array}{ccl} B & \xrightarrow{F} & F(V) = G(N) \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & Q = G(N)/G(n) \times O(N-n) , \end{array}$$

where π is the natural projection and F is the natural identification of a frame in B with an element of $G(N)$ mentioned in § 1.

The quadratic differential form III induced from $d\Sigma^2$ on Q by the Gauss map f is written as

$$III = f^*d\Sigma^2 = \sum_{i,r} (\omega_{ir})^2 = \sum_{r,i,j,k} A_{rij}A_{rik}\omega_j\omega_k , \tag{6}$$

since $\omega_{0r} = \omega_r = 0$ by (3). III is called the *third fundamental form* of x.

The Gauss map is a constant map if and only if III vanishes identically, i.e. $\omega_{ir} = 0$ *and therefore if and only if the immersion x is totally geodesic.*

Combining (5) and (6), we obtain

$$\Psi - II_N + III = \varepsilon\frac{n-1}{a^2}ds^2 , \tag{7}$$

and therefore

Theorem 1. *Suppose that a Riemannian n-manifold M is isometrically immersed into a simply-connected complete space of constant curvature* ε/a^2.

Then the relation (7) *holds among the Ricci form* Ψ *on* M, *the second fundamental form* II_N *in the direction of the mean curvature vector, and the third fundamental form* III *of the immersion.*

Suppose that M is Einsteinian. Then from (7) II_N is proportional to ds^2 if and only if III is. Thus we obtain

Theorem 2. *Let* x *be an isometric immersion of an Einstein space into a* V. *Then* x *is pseudo-umbilical if and only if the Gauss map is conformal.*

Suppose that II_N is proportional to ds^2. Then Ψ is proportional to ds^2 if and only if III is. In particular, if II_N vanishes identically, Ψ is proportional to ds^2 if and only if III is. In this case if furthermore $\dim M > 2$, then the proportional factor of Ψ is constant and the same holds for III. Hence

Theorem 3. *Let* x *be a pseudo-umbilical immersion of a Riemannian manifold* M *into a* V. *Then the Gauss map is conformal if and only if* M *is Einsteinian. In the case* $\dim M > 2$, *the Gauss map is homothetic if and only if* M *is Einsteinian.*

In a similar way, we have

Theorem 4. *Let* x *be an isometric immersion of a Riemannian manifold* M *into a* V. *Then* x *is pseudo-umbilical if and only if* M *is Einsteinian.*

By a result of Mostow [3], every compact homogeneous space G/H can be equivariantly imbedded in a certain Euclidean sphere. If the isotropy subgroup H is irreducible on the tangent space, then we may naturally assume that the imbedding is isometric, and Ψ, II_N and III are proportional to the Riemannian metric with constant factors, since they are all invariant by G.

Theorem 5. *An equivariant imbedding of a compact homogeneous space with irreducible isotropy subgroup into a Euclidean sphere is pseudo-umbilical and the Gauss map is homothetic.*

By results of Hsiang [2] and Takahashi [5] such a space can be minimally immersed into a certain Euclidean sphere. In this case the Gauss map is obviously homothetic.

References

[1] S. S. Chern, *Minimal surfaces in a Euclidean space of N dimensions*, Differential and Combinatorial Topology, Morse Jubilee Volume, Princeton University Press, Princeton, 1965, 187–198.

[2] W. Y. Hsiang, *On the compact homogeneous minimal submanifolds*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966) 5–6.

[3] G. D. Mostow, *Equivariant embeddings in Euclidean spaces*, Ann. of Math. **65** (1957) 432–446.

[4] T. Otsuki, *A theory of Riemannian submanifolds*, to appear.

[5] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966) 380–385.

[6] J. A. Wolf, *Spaces of constant curvature*, McGraw-Hill, New York, 1967.

UNIVERSITY OF SOUTHAMPTON
TOKYO METROPOLITAN UNIVERSITY