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HÖLDER REGULARITY OF HOROCYCLE FOLIATIONS

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1. Introduction

Let M be a C^∞, nonpositively curved manifold. A *horosphere* in M is the projection to M of a limit of metric spheres in the universal cover $\tilde{M}$ (see §2). A *horospherical foliation* $\mathcal{H}$ is a foliation of the unit tangent bundle T^1M whose leaves consist of unit normal vector fields to horospheres.[1]

While regularity of horospherical foliations has been studied extensively for negatively curved manifolds M, considerably less is known in the nonpositively curved case. The most general result is due to P. Eberlein: if M is complete and nonpositively curved, then horospheres are C^2, which implies that the individual leaves of $\mathcal{H}$ are C^1. Further, the tangent distribution $T\mathcal{H}$ is continuous on T^1M (see [9]).

Beyond Eberlein's theorem, smoothness results have consisted mainly of counterexamples ([2], [5]); in particular, the best one could hope for in the case of a general compact, nonpositively curved M is for $T\mathcal{H}$ to be Hölder-continuous. In this paper we prove

Theorem I′. *Let S be a compact, real-analytic, nonpositively curved surface. Then $T\mathcal{H}$ is Hölder.*

Theorem I′ is actually a corollary of a more general result, Theorem I below.

The problem of finding the regularity of horospherical foliations has a long history, which we briefly summarize here.

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[1]As we explain in §2, there are two such foliations, $\mathcal{H}^-$ and $\mathcal{H}^+$, called *stable* and *unstable* horospherical foliations, respectively. In this discussion, we use $\mathcal{H}$ to denote either of these.

E. Hopf showed in [7] that if M is a compact, negatively curved surface, then $T\mathcal{H}$ is C^1. Under the assumption that the sectional curvatures of M are 1/4-pinched, Hopf's result was generalized by M. Hirsch and C. Pugh [10] to any dimension. D.V. Anosov [1] showed that the stable and unstable foliations are always Hölder for what are now called Anosov flows. In particular, this implies that $T\mathcal{H}$ is Hölder, when M is compact and negatively curved.

In Anosov's theorem, the conclusion "Hölder" cannot be improved to "C^1" [1]. In fact, B. Hasselblatt showed that C^1 fails even for geodesic flows. He found open sets of metrics, with negative curvature arbitrarily close to 1/4-pinched, for which the horospherical foliations fail to be C^1 [8]. Related bounds on the smoothness of $T\mathcal{H}$ beyond C^1, in the context of 3-dimensional Anosov flows, were found by S. Hurder and A. Katok [11]. An example of W. Ballmann, M. Brin and K. Burns shows that compactness is necessary in Anosov's result; they construct in [2] a complete, finite volume surface whose curvature is arbitrarily close to -1 but for which $T\mathcal{H}$ is not Hölder.

Returning to the compact, nonpositive curvature case, Gerber and V. Niţică [5] have examples of real-analytic surfaces showing that $T\mathcal{H}$ in Theorem I′ can fail to have a Hölder exponent greater than 1/2. In particular, $T\mathcal{H}$ can be non-Lipschitz. (See also Lemma 3.3 in the present paper.) A related issue is that of the regularity of $T\mathcal{H}$ along the leaves of $\mathcal{H}$; that is, how smooth are the leaves of $\mathcal{H}$? For M compact and negatively curved, Anosov [1] showed that the leaves of $\mathcal{H}$ are C^∞. In the case of nonpositive curvature, Eberlein's "C^1" conclusion cannot be improved to "C^2"; Ballman, Brin and Burns construct in [2] a compact, real-analytic surface of nonpositive curvature for which the leaves of $\mathcal{H}$ fail to be C^2. However, the non-C^2 leaves in their example are $C^{1+\text{Lipschitz}}$, (i.e., $T\mathcal{H}$ is Lipschitz along leaves) and this suggested to us the question of whether this is always the case for compact, real-analytic surfaces of nonpositive curvature. As a corollary of our Theorem II below, we have

Theorem II′. *Let S be a compact, real-analytic, nonpositively curved surface. Then the leaves of $\mathcal{H}$ are uniformly $C^{1+\mathit{Lipschitz}}$.*

Our interest in these questions arose while studying the ergodic properties of the geodesic flow for analytic, nonpositively curved surfaces. We asked whether the time-one map of such a flow remains ergodic under suitable perturbations. Related results for negatively curved manifolds use Hölder continuity of the horospherical foliations in a central way ([6], [13], [12]). We hope that Theorems I and II can be used to establish similar results for certain nonpositively curved surfaces.

1.1. Statement of results

Throughout this paper we always assume that manifolds are boundaryless. We follow the usual convention of referring to horospheres as "horocycles" when M has dimension 2.

Theorem I. *Let S be a compact surface with a C^∞ metric of nonpositive curvature K satisfying the following conditions:*

1) *If γ is a geodesic that is not closed, then there is no infinite time interval I for which $K(\gamma(t)) = 0$, for all $t \in I$.*

2) *If γ is a closed geodesic, then there exists a t such that K does not vanish to infinite order at $\gamma(t)$.*

Then the tangent distributions $T\mathcal{H}^+$ and $T\mathcal{H}^-$ of the horocycle foliations are Hölder-continuous.

Theorem II. *Let S be a compact surface with a C^∞ metric of nonpositive curvature satisfying the conditions of Theorem I. Then the leaves of $\mathcal{H}^+$ and $\mathcal{H}^-$ are uniformly $C^{1+Lipschitz}$.*

Proof of Theorems I′ and II′ from I and II. If S is real-analytic, then the set of points in S where K vanishes is a real-analytic subvariety in S. In particular, K cannot vanish on an infinite time interval on a non-closed geodesic nor can K vanish to all orders at a point, unless it vanishes identically on the surface. In this case, S is a flat torus or Klein bottle and the horocycle foliations are analytic. q.e.d.

Remarks. It is an open question whether Theorems I and II hold without hypotheses 1) and 2). It is also not known whether there exist C^∞ surfaces that fail to satisfy hypothesis 1), except if the curvature vanishes identically. There are Lipschitz metrics with this property [3]. At the end of §3 we give an example to show that the estimates on the curvatures of the horocycles that are used in our proofs do not hold without hypothesis 2). The C^∞ assumption in Theorems I and II can be replaced by C^r, where $r \geq 4$ and K vanishes to order at most $r - 3$ along any closed geodesic.

We also have an easier version of Theorem I, with a weaker conclusion, but which holds without the assumptions 1) and 2).

Proposition III. *Let S be a compact surface with a C^3 metric of nonpositive curvature. Then the leaves of the horocylic foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ are uniformly $C^{1+1/2}$; that is, $T\mathcal{H}^{\pm}$ is uniformly $1/2$-Hölder along leaves.*

As a corollary to Theorem II and Proposition III, we obtain an improvement to previously known regularity results for the Busemann functions (see §2).

Corollary IV. *Under the hypotheses of Proposition III, the Busemann functions are uniformly $C^{2+1/2}$, and under the hypotheses of Theorem II, the Busemann functions are uniformly $C^{2+\mathit{Lipschitz}}$.*

1.2. Outline of the proofs

To prove these results, we study the dependence on $v \in T^1S$ of solutions to the scalar Riccati equation

$$u'(t) + u(t)^2 + K(\sigma_v(t)) = 0,$$

where $\sigma_v(t)$ is the unit-speed geodesic determined by v, and $K : S \to \mathbf{R}$ is the curvature. In §2 we explain how Hölder regularity of $T\mathcal{H}$ amounts to Hölder dependence on v of the "unstable" solutions to the Riccati equations.

In §3 we turn to a study of these Riccati equations. The analysis begins by taking the difference of two Riccati solutions u_0 and u_1 along geodesics determined by v_0 and v_1, to obtain

$$(u_1 - u_0)' = -(u_1 + u_0)(u_1 - u_0) + (K_0 - K_1), \tag{1.1}$$

where K_0 and K_1 are the curvatures of S along these geodesics. To obtain our regularity results, we need $|(u_1 - u_0)(0)|$ to be small relative to the distance between v_0 and v_1. It is apparent from (1.1) that $|u_1 - u_0|$ decreases rapidly if $u_1 + u_0$ is large relative to $|K_0 - K_1|$. The remainder of the proofs is devoted to estimating the sizes of these terms.

The proof of Proposition III depends only on Lemma 3.1. For the proofs of Theorems I and II, we need the additional Lemmas 3.2 - 3.7. The proof of the lower bound in Lemma 3.3 is presented in §4.

1.3. Acknowledgements

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2. Preliminaries

Let M be a complete n-dimensional manifold of nonpositive sectional curvatures and let $\tilde{M}$ be its universal cover. We now define the horospherical foliations discussed in the introduction. For a unit vector v let σ_v denote the geodesic in M (or $\tilde{M}$) with $\sigma_v'(0) = v$. Vectors $v, w \in T^1\tilde{M}$ are *asymptotic* if there exists a constant $C > 0$ such that for all $t > 0$, $\operatorname{dist}(\sigma_v(t), \sigma_w(t)) \leq C$. Nonpositive curvature and simple connectivity imply that for every $v \in T^1\tilde{M}$ and $p \in \tilde{M}$, there is a unique vector $Z_v(p) \in T_p^1\tilde{M}$ such that $Z_v(p)$ is asymptotic to v. Fixing v, this defines a radial vector field Z_v on $\tilde{M}$. The vector v also determines a *Busemann function* $F_v : \tilde{M} \to \mathbf{R}$ by:

$$F_v(p) = \lim_{t\to\infty} (\operatorname{dist}(p, \sigma_v(t)) - t).$$

It is well-known (see, e.g. [9]) that F_v is C^1, Z_v is the gradient of $-F_v$, and each level set $F_v^{-1}(c)$ is the limit of geodesic spheres of radius $t+c$ centered at $\sigma_v(t)$. Moreover, as was shown by Eberlein, Busemann functions are C^2, and consequently their level sets are C^2 [9].

For $v \in T^1\tilde{M}$, define the *stable* and *unstable horospheres* $h^-(v)$ and $h^+(v)$ determined by v to be the level sets $F_v^{-1}(0)$ and $F_{-v}^{-1}(0)$, respectively. The leaves of the stable and unstable horospherical foliations $\mathcal{H}^-$ and $\mathcal{H}^+$ of $T^1\tilde{M}$ are defined by:

$$\mathcal{H}^-(v) = \{Z_v(p) \,:\, p \in h^-(v)\}$$

and

$$\mathcal{H}^+(v) = \{-Z_{-v}(p) \,:\, p \in h^+(v)\}.$$

Since Busemann functions are C^2, the leaves of $\mathcal{H}^\pm$ are C^1, and the tangent distributions $T\mathcal{H}^\pm$ are defined.

We project the horospheres from $\tilde{M}$ into M to obtain horospheres for vectors in T^1M. Similarly, we obtain the horospherical foliation of T^1M.

We are interested in the regularity of $T\mathcal{H}^\pm$, which reduces to the regularity of the sectional curvature of the horospheres. These sectional curvatures are determined by solutions to certain Riccati and Jacobi equations. We now restrict to the case where M is a surface, S, and these equations can be reduced to scalar ones.

Let $v \in T_p^1\tilde{S}$ and $w \in T_p\tilde{S}$, and let J_- [J_+] be the stable [unstable] Jacobi field along σ_v with $J_-(0) = w$ [$J_+(0) = w$]. (The stable Jacobi field is defined by $J_- = \lim_{n\to\infty} J_n$, where J_n is the Jacobi field along σ_v with $J_n(0) = w$ and $J_n(n) = 0$. The unstable Jacobi field J_+ is defined by the same formula, except replacing $\lim_{n\to\infty}$ by $\lim_{n\to-\infty}$.) If Z_v is the radial vector field defined above, then $\nabla_w Z_v = J_-'(0)$, by Proposition 3.1 in [9]. Now assume w is a unit vector perpendicular to v and let E be the continuous, unit-length vector field along σ_v that is perpendicular to σ_v and satisfies $E(0) = w$. Then $J_-(t) = j_-(t)E(t)$, where j_- is a real-valued function that satisfies the scalar Jacobi equation:

$$j_-''(t) = -K(\sigma_v(t))j_-(t).$$

Let $u_- = j_-'/j_-$. Then u_- satisfies the scalar Riccati equation

$$u_-'(t) + u_-(t)^2 + K(\sigma_v(t)) = 0.$$

Since $j_-(0) = 1$,

$$u_-(0) = j_-'(0) = \langle \nabla_w Z_v, w\rangle = -k_-(v), \tag{2.1}$$

where $k_-(v)$ is the geodesic curvature of $h^-(v)$ at v. The function u_- is called the *stable* solution to the Riccati equation along σ_v; since J_- was constructed as $\lim_{n\to\infty} J_n$, it follows that $u_-(t) = \lim_{n\to\infty} u_n(t)$, where, for $n > 0$, u_n is the solution to the Riccati equation along σ_v with $u_n(n) = -\infty$. The unstable Riccati solution u_+ along σ_v is similarly defined in terms of J_+ and satisfies $u_+(t) = \lim_{n\to-\infty} u_n$, where, for $n < 0$, u_n is the solution to the Riccati equation along σ_v with $u_n(n) = +\infty$. A similar argument to the one summarized in equation (2.1) shows that $u_+(0) = k_+(v)$, where $k_+(v)$ is the geodesic curvature of $h^+(v)$ at v. Since $K \le 0$, it follows that $u_-(t) \le 0$ for all t, and $u_+(t) \ge 0$ for all t. Moreover, if $K(\sigma_v(t_0)) < 0$ for some t_0, then $u_-(t) < 0$, for all $t < t_0$, and $u_+(t) > 0$, for all $t > t_0$. (These inequalities are easy consequences of Lemma 3.1 below.)

A function f from a metric space (X_1, d_1) to a metric space (X_2, d_2) is *Hölder-continuous of exponent* $\alpha \in (0, 1]$ if there exists a constant $C > 0$ such that for all $p, q \in X_1$,

$$d_2(f(p), f(q)) \quad \le \quad C(d_1(p, q))^\alpha. \tag{2.2}$$

The function f is *Lipschitz* if it is Hölder with exponent 1. We say that f is *Hölder (or Lipschitz) at a point* $p \in X_1$ if there is a constant $C = C(p) > 0$ such that inequality (2.2) holds for all $q \in X_1$. A family of functions $\mathcal{F}$ from X_1 to X_2 is *uniformly Hölder (or Lipschitz)* if there is a single constant C such that (2.2) holds for all $p, q \in X_1$ and for all $f \in \mathcal{F}$.

Throughout this paper all geodesics have unit speed. We will use Fermi coordinates (s, x) along a geodesic γ in $\tilde{S}$, where s is the time parameter along γ, and x is the signed distance to γ. Then the curves $s =$ constant are unit-speed geodesics perpendicular to γ. We will frequently use ϕ to denote the angle between a vector v and the curve $x =$ constant; unless stated otherwise, such angles will be signed angles in $[-\pi/2, \pi/2]$ chosen so that $\sphericalangle(\partial/\partial x, x = a) = \pi/2$.

3. Proofs of Theorems I and II

This section contains the proofs of Theorems I and II, with the exception of the proof of the lower bound on the curvatures of horocycles in Lemma 3.3. This lower bound is proved in §4.

The following lemma contains facts which are routinely used in the study of Riccati and Jacobi equations. For example, part (iv) is the Comparison Lemma in [2] and it is also a special case of a well–known differential inequality ([7], Chapter III, Corollary 4.2). Part (vi) is a special case of the Sturm Comparison Theorem.

Lemma 3.1. *Let $K, K_0, K_1 : [A, B] \to \mathbf{R}$ be continuous functions, and suppose u, u_0, u_1 are solutions to the Riccati equations $u' = -u^2 - K$, $u_i' = -u_i^2 - K_i$, $i = 0, 1$, respectively, that are finite on the interval $[A, B]$. Let $y = u_1 - u_0$. Let j_0, j_1 satisfy the Jacobi equations $j_i'' = -K_i j_i$, $i = 0, 1$. Then the following hold:*

(i) $y' = -(u_0 + u_1)y + K_0 - K_1$.

(ii) *If* $\hat{\jmath}_i(t) = exp\left[-\int_t^B u_i(\tau)\, d\tau\right]$, *for* $i = 0, 1$, *then*

$$y(B) = \int_A^B [K_0(t)) - K_1(t)]\hat{\jmath}_0(t)\hat{\jmath}_1(t)\, dt + y(A)\hat{\jmath}_0(A)\hat{\jmath}_1(A).$$

(iii) *If $\hat{\jmath}_i(t)$ is as defined in (ii), then $\hat{\jmath}_i(B) = 1$ and $\hat{\jmath}_i$ satisfies the Jacobi equation $\hat{\jmath}_i'' = -K_i\hat{\jmath}_i$ for which $u_i = \hat{\jmath}_i'/\hat{\jmath}_i$. Moreover, if u_i is nonnegative throughout $[A, B]$, then $0 \le \hat{\jmath}_i \le 1$ for $A \le t \le B$ and $\hat{\jmath}_i'(A) \le 1/(B - A)$.*

(iv) *If $u_0(A) \le u_1(A)$ and $K_1(t) \le K_0(t)$ for $A \le t \le B$, then $u_0(B) \le u_1(B)$.*

(v) *If* $K(t) \leq 0$ *for* $A \leq t \leq B$, *and* $u(A) \geq 0$, *then*

$$u(B) \geq \frac{u(A)}{(B-A)u(A)+1}$$

and this inequality can be replaced by equality if $K(t) = 0$ *for* $A \leq t \leq B$.

(vi) *If* $0 \leq j_0(A) \leq j_1(A)$, $0 \leq j_0'(A) \leq j_1'(A)$ *and* $K_1(t) \leq K_0(t) \leq 0$ *for* $A \leq t \leq B$, *then* $j_0(B) \leq j_1(B)$.

Proof. Property (i) is obtained by subtracting the Riccati equation for u_0 from the Riccati equation for u_1. By the formula for the solution of first order linear differential equations, we have

$$y(B) = \textstyle\int_A^B (K_0(t)) - K_1(t)) \exp\left[-\int_t^B (u_0(\tau)+u_1(\tau))\,d\tau\right] dt$$
$$+y(A)\exp\left[-\textstyle\int_A^B (u_0(\tau)+u_1(\tau))\,d\tau\right],$$

and (ii) follows. The first statement in (iii) is an immediate consequence of the definition of $\hat{\jmath}_i$. Now if u_i is nonnegative throughout $[A,B]$, then $\hat{\jmath}_i$ will be convex, nondecreasing and positive on $[A,B]$, and the second statement in (iii) follows. It is clear from (ii) and (iii) that if $y(A) \geq 0$ and $K_0(t) - K_1(t) \geq 0$ for $t \in [A,B]$, then $y(B) \geq 0$. This proves (iv). The inequality in (v) is a special case of (iv), because if $K_0 \equiv 0$ and $u_0(A) = u(A)$, then $u_0(t) = u(A)/((t-A)u(A)+1)$. For the proof of (vi), see Chapter 10 of [4]. q.e.d.

Proof of Proposition III. Let S be a compact surface with a C^3 metric of nonpositive curvature. Then K is C^1 and there exists a constant $L > 0$ such that $|K(p) - K(q)| \leq L\ \mathrm{dist}_S(p,q)$.

Let γ_0 and γ_1 be two geodesics on $\tilde{S}$ such that $\gamma_0'(t)$ and $\gamma_1'(t)$ are on the same unstable horocycle; i.e.,

$$\lim_{t\to-\infty} \mathrm{dist}_{\tilde{S}}(\gamma_0(t),\gamma_1(t)) = 0.$$

To prove that the leaves of the unstable horocycle foliation of $T^1\tilde{S}$ (or T^1S) are uniformly $C^{1+1/2}$, it suffices to show that there exists a constant $C > 0$ (depending only on S) such that

$$|k_+(\gamma_1'(0)) - k_+(\gamma_0'(0))| \leq C\sqrt{\epsilon}, \tag{3.1}$$

where $\epsilon = \mathrm{dist}_{\tilde{S}}(\gamma_0(0),\gamma_1(0))$. Since $K \leq 0$, $\mathrm{dist}_{\tilde{S}}(\gamma_0(t),\gamma_1(t)) \leq \epsilon$, for $t \leq 0$. Let u_i be unstable Riccati solutions along $\gamma_i, i = 0,1$. Let $y = u_1 - u_0$. Then $y(0) = k_+(\gamma_1'(0)) - k_+(\gamma_0'(0))$.

Now apply Lemma 3.1(ii) with $A = -1/\sqrt{\epsilon}$ and $B = 0$. We obtain

$$(3.2) \qquad |y(0)| \leq \int_A^0 |K_0(t) - K_1(t)|\hat{\jmath}_0(t)\hat{\jmath}_1(t)\,dt + |y(A)|\hat{\jmath}_0(A)\hat{\jmath}_1(A),$$

where $\hat{\jmath}_0$ and $\hat{\jmath}_1$ are as in Lemma 3.1(ii). Since u_0 and u_1 are both nonnegative throughout $[A, B]$, we have $0 \leq \hat{\jmath}_i(t) \leq 1$ for $A \leq t \leq B$, and $\hat{\jmath}_i'(A) \leq 1/(B - A)$, for $i = 1, 2$, by Lemma 3.1(iii). The first term on the right-hand side of (3.2) is bounded from above by $(1/\sqrt{\epsilon})(\epsilon L) = \sqrt{\epsilon}L$. The estimate on the second term of the right-hand side of (3.2) is given by

$$(3.3) \qquad |y(A)|\hat{\jmath}_0(A)\hat{\jmath}_1(A) \leq u_i(A)\hat{\jmath}_i(A) = \hat{\jmath}_i'(A) \leq \frac{1}{(B-A)} = \sqrt{\epsilon},$$

where i is chosen so that $u_i(A)$ is the larger of $u_0(A)$ and $u_1(A)$. This proves that the leaves of $\mathcal{H}^+$ are uniformly $C^{1+1/2}$. q.e.d.

The following lemma will be applied to the curvature function $f = K$. In this lemma the complete surface S could easily be replaced by a complete Riemannian manifold.

Lemma 3.2. *If f is a nonpositive function on a complete surface S such that $|(d^2/dt^2)(f(\sigma(t)))|$ exists and is uniformly bounded from above along all geodesics σ, then there exist constants $L_1, L_2 > 0$ such that for all $p, q \in S$,*

$$(3.4) \qquad |f(p) - f(q)| \leq L_1\epsilon\sqrt{-f(p)} + L_2\epsilon^2,$$

where $\epsilon = dist(p, q)$.

Proof. Let $L = \sup\{|(d^2/dt^2)|_{t=0}(f(\sigma(t)))| : \sigma \text{ is a geodesic on } S\}$. We only need to consider the case $L > 0$. Let $p \in S$, and let σ be a geodesic on S such that $\sigma(0) = p$ and $\sigma'(0)$ is in a direction of the greatest increase of f at p; i.e., $(d/dt)|_{t=0}(f(\sigma(t))) = \|Df_p\|$. Let $g : \mathbf{R} \to \mathbf{R}$ satisfy $g(0) = f(\sigma(0)), g'(0) = (d/dt)|_{t=0}(f(\sigma(t)))$ and $g''(t) = -L$ for all t.

Then for $t \geq 0$,

$$0 \geq f(t) \geq g(t) = f(p) + \|Df_p\|t - \frac{1}{2}Lt^2.$$

Setting $t = \|Df_p\|/L$, we obtain $\|Df_p\| \leq \sqrt{-2Lf(p)}$. Let $L_1 = \sqrt{2L}, L_2 = L/2$. Then the lemma follows from Taylor's Theorem.

q.e.d.

In the following lemmas, we begin invoking our hypotheses 1) and 2) on the surface S.

Lemma 3.3. *Suppose S is a complete surface of nonpositive curvature K and $\gamma(s)$ is a closed geodesic on S such that K vanishes to order $m-1$ on γ, where $m \in \{2,4,6,\dots\}$; i.e., if (s,x) are the Fermi coordinates along the lift $\tilde{\gamma}$ of γ to $\tilde{S}$, then*

$$\frac{\partial^k}{\partial x^k}\bigg|_{x=0} K(s,x) = 0$$

for $k = 0, 1, \dots, m-1$ and all s. Also assume that there is at least one point, say $\gamma(0)$, such that K does not vanish to order m at $\gamma(0)$; i.e., $(\partial^m/\partial x^m)K(0,x) \neq 0$. Then there exist a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ in $T^1\tilde{S}$ and a positive constant C such that for any $v \in \mathcal{U}$ with footpoint having second Fermi coordinate $x = a$ the curvatures $k_-(v)$ and $k_+(v)$ of the stable and unstable horocycles satisfy

$$C \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right) \leq k_-(v) \leq C^{-1} \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right)$$
$$C \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right) \leq k_+(v) \leq C^{-1} \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right),$$

where $\phi = \sphericalangle(v, x = a)$.

The upper bounds on $k_-(v)$ and $k_+(v)$ are proved in [5], Theorem 3.1. The assumption that there is a point on γ where the curvature does not vanish to order m is not needed to obtain these upper bounds. The lower bounds are proved in §4.

Note that the hypothesis of Lemma 3.3 could not hold for odd m, because K does not change sign.

Lemma 3.4. *If S is a surface satisfying the hypotheses of Theorems I and II, then there is a constant $C > 0$ such that for all $v \in T^1S$,*

$$k_-(v) \geq C\sqrt{-K(p)} \tag{3.5}$$

and

$$k_+(v) \geq C\sqrt{-K(p)}, \tag{3.6}$$

where p is the footpoint of v.

Proof. If γ is a closed geodesic on S along which K vanishes to order $m-1$, for $m \in \{2,4,6,\dots\}$, then there is a constant $C_1 > 0$ such that $-K(p) \leq C_1|a|^m$ for p in a neighborhood of γ with Fermi coordinates (s,a). If, in addition, there is a point on γ at which K does not vanish to order m, then the lower bounds on $k_-(v)$ and $k_+(v)$ in Lemma 3.3 imply that (3.5) and (3.6) are satisfied for v in some neighborhood of $T^1\gamma$, for some constant $C > 0$.

By hypotheses 1) and 2) of Theorems I and II, there are at most finitely many closed geodesics along which K vanishes. Therefore (3.5) and (3.6) hold for v in a neighborhood $\mathcal{U}$ of the union of the unit tangent bundles of such geodesics.

Now for $v \in T^1S$, $k_-(v)$ vanishes only if $K(\sigma_v(t)) = 0$ for all $t \geq 0$, and $k_+(v)$ vanishes only if $K(\sigma_v(t)) = 0$ for all $t \leq 0$. Thus by hypothesis 1), $k_-(v)$ and $k_+(v)$ each vanish only for v in the unit tangent bundle of a closed geodesic along which K vanishes. Then (3.5) and (3.6) extend to the complement of $\mathcal{U}$ in T^1S for some $C > 0$ by the continuity of k_-, k_+, and K. q.e.d.

Lemma 3.5. *Let S be a surface satisfying the hypotheses of Theorems I and II. Let γ_0 and γ_1 be geodesics on S or $\tilde{S}$, let $K_i(t) = K(\gamma_i(t))$, for $i = 0, 1$ and $A \leq t \leq B$. Let u_i be a solution to the Riccati equation $u_i' = -u_i^2 - K_i, i = 0, 1$, where u_0 is greater than or equal to the unstable solution, u_+, and $u_1(A) \geq 0$. Let $y = u_1 - u_0$. Then there exist positive constants C_1 and C_2, which depend only on S, such that*

$$|y(B)| \leq C_1\epsilon + C_2(B-A)\epsilon^2 + |y(A)|\hat{\jmath}_0(A)\hat{\jmath}_1(A),$$

where $\epsilon = \max\{dist(\gamma_0(t), \gamma_1(t)) : A \leq t \leq B\}$ and $\hat{\jmath}_0$ and $\hat{\jmath}_1$ are defined as in Lemma 3.1(ii).

Proof. By parts (ii) and (iii) of Lemma 3.1, we have

$$|y(B)| \leq \int_A^B |K_0(t) - K_1(t)|\hat{\jmath}_0(t)\hat{\jmath}_1(t)dt + |y(A)|\hat{\jmath}_0(A)\hat{\jmath}_1(A), \tag{3.7}$$

where $0 \leq \hat{\jmath}_i(t) \leq 1$ for $i = 0, 1$. Moreover, if we apply Lemma 3.2 to $f = K$, we obtain constants $L_1, L_2 > 0$ such that

$$|K_0(t) - K_1(t)| \leq L_1\epsilon\sqrt{-K_0(t)} + L_2\epsilon^2,$$

where ϵ is as in the statement of the present lemma. Thus

$$\begin{aligned} \int_A^B &|K_0(t) - K_1(t)|\hat{\jmath}_0(t)\hat{\jmath}_1(t)dt \\ &\leq \int_A^B (L_1\epsilon\sqrt{-K_0(t)} + L_2\epsilon^2)\hat{\jmath}_0(t)dt \\ &\leq L_2(B-A)\epsilon^2 + L_1\int_A^B \epsilon\sqrt{-K_0(t)}\hat{\jmath}_0(t)dt. \end{aligned} \tag{3.8}$$

Also, by Lemma 3.4, there is a constant $C_3 > 0$ such that for $A \leq t \leq B$,

$$\sqrt{-K_0(t)} \leq C_3 k_+(\gamma_0'(t)) = C_3 u_+(t) \leq C_3 u_0(t).$$

Then

$$\begin{aligned} L_1 \int_A^B \epsilon \sqrt{-K_0(t)} \hat{\jmath}_0(t) dt &\leq C_3 L_1 \epsilon \int_A^B u_0(t) \hat{\jmath}_0(t) dt \\ &= C_3 L_1 \epsilon \int_A^B \hat{\jmath}_0'(t) dt = C_3 L_1 \epsilon [1 - \hat{\jmath}(A)] \leq C_3 L_1 \epsilon \ . \end{aligned} \tag{3.9}$$

The lemma follows by combining (3.7),(3.8) and (3.9). q.e.d.

Proof of Theorem II. Let γ_0, γ_1, u_0, u_1,$\hat{\jmath}_0$,$\hat{\jmath}_1$, y and ϵ be as in the proof of Proposition III. The beginning of the proof of the present theorem is the same as the second paragraph of the proof of Proposition III, except that in order to obtain uniform $C^{1+\text{Lipschitz}}$ leaves, $\sqrt{\epsilon}$ must be replaced by ϵ in the desired inequality (3.1). Now apply Lemma 3.5 with $A = -1/\epsilon$ and $B = 0$. We obtain

$$|y(0)| \leq C_1 \epsilon + C_2 \epsilon + |y(A)| \hat{\jmath}_0(A) \hat{\jmath}_1(A).$$

By the same estimate as in (3.3), we have

$$|y(A)| \hat{\jmath}_0(A) \hat{\jmath}_1(A) \leq \frac{1}{B - A} = \epsilon.$$

This completes the proof. q.e.d.

Remark. In the proofs of Proposition III and Theorem II, it is not necessary to assume that $\gamma_0'(0)$ and $\gamma_1'(0)$ are on the same unstable horocycle. It suffices to assume that $\gamma_0'(0)$ and $\gamma_1'(0)$ are negatively asymptotic; i.e, they belong to the gradient field of the same Busemann function. This is enough to give us the property that $\mathrm{dist}_{\tilde{S}}(\gamma_0(t), \gamma_1(t))$ is nondecreasing, which is all that our proofs use.

Proof of Corollary IV. Assume that the hypotheses of Theorem II hold. Let $v \in T^1\tilde{S}$ and let $F = F_v$ be the Busemann function corresponding to v, as defined in §2. Then the level sets of F are stable horocycles and the integral curves to $-\nabla F$ are geodesics asymptotic to σ_v. The derivative of ∇F in the direction of these geodesics is identically 0. Let $(\nabla F)_{\mathcal{H}}$ denote the derivative of ∇F in the direction of the stable horocycles. Then $(\nabla F)_{\mathcal{H}}$ consists of vectors tangent to these horocycles whose lengths are equal to the curvatures of the horocycles. Thus, by

Theorem II, $\nabla F_{\mathcal{H}}$ is Lipschitz in the direction of the horocycles. Along each asymptotic geodesic, $(\nabla F)_{\mathcal{H}}$ consists of vectors perpendicular to that geodesic whose lengths are equal to the absolute value of the stable solution to the Riccati equation. Thus $(\nabla F)_{\mathcal{H}}$ is smooth in the direction of the asymptotic geodesics. Therefore $(\nabla F)_{\mathcal{H}}$ is uniformly Lipschitz on $\tilde{S}$, and F is uniformly $C^{2+\text{Lipschitz}}$. (The Lipschitz constant does not depend on v.) A similar argument shows that the Busemann functions are uniformly $C^{2+1/2}$ under the assumptions of Proposition III.

We now turn to the proof of Theorem I. We need two additional lemmas.

Lemma 3.6. *If S is a surface satisfying the hypotheses of Theorems I and II, then there is a constant $C > 0$ such that for all $v \in T^1S$,*

$$Ck_+(v) \leq k_-(v) \leq C^{-1}k_+(v).$$

Proof. This lemma follows from Lemma 3.3 in the same way that Lemma 3.4 follows from Lemma 3.3. q.e.d.

Lemma 3.7. *Suppose $K(t) \leq 0$ for $A \leq t \leq B$. Let u_0 and u_1 be solutions of the Riccati equation $u' = -u^2 - K$ that satisfy $u_1(t) \geq u_0(t) > 0$ for $A \leq t \leq B$. Then*

$$\frac{exp\Big[\int_A^B u_1(t)\,dt\Big]}{exp\Big[\int_A^B u_0(t)\,dt\Big]} \leq \frac{u_1(A)}{u_0(A)}.$$

Proof. Let $j_i(t) = \exp[\int_A^t u_i(\tau)\,d\tau]$, for $i = 1, 2$. Then $j_i(A) = 1$ and $j_i'(A) = u_i(A)$. Let $\bar{\jmath} = (u_1(A)/u_0(A))j_0$. Then $\bar{\jmath}(A) \geq j_1(A)$ and $\bar{\jmath}'(A) = j_1'(A)$. It now follows from Lemma 3.1(vi) that $\bar{\jmath}(B) \geq j_1(B)$. This inequality can be rewritten as $(u_1(A)/u_0(A)) \geq j_1(B)/j_0(B)$, which proves the lemma. q.e.d.

Proof of Theorem I. We must show that there are constants α, C with $C > 0$ and $0 < \alpha < 1$ such that

$$|k_+(v_1) - k_+(v_0)| \leq C(\text{dist}(v_0, v_1))^\alpha, \text{ for all } v_0, v_1 \in T^1S. \tag{3.10}$$

Step 1. We first show that it suffices to prove (3.10) in the case v_0 and v_1 have the same footpoint; i.e., we will show that (3.10) will follow for some $C > 0$ if there is a constant $\tilde{C} > 0$ such that for all $p \in S$ and v_0, $v_1 \in T^1_p S$,

$$|k_+(v_1) - k_+(v_0)| \le \tilde{C}\theta^\alpha, \text{ where } \theta = \sphericalangle(v_0, v_1). \tag{3.11}$$

Suppose $v_0, v_1 \in T^1 S$ with $\operatorname{dist}(v_0, v_1) \le 1$, and let p_0, p_1 be the footpoints of v_0, v_1. Let $W = W_{v_0} = -Z_{-v_0}$, where Z_{-v_0} is the radial vector field consisting of vectors asymptotic to $-v_0$ (see §2). Then W is Lipschitz, with Lipschitz constant independent of v_0. (This follows from Busemann functions being uniformly C^2, and does not use Corollary IV.) Let $v_1' = W(p_1)$. By the remark following the proof of Theorem II, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} |k_+(v_1') - k_+(v_0)| &\le C_1 \operatorname{dist}(p_0, p_1) \\ &\le C_1 \operatorname{dist}(v_0, v_1) \le C_1(\operatorname{dist}(v_0, v_1))^\alpha. \end{aligned} \tag{3.12}$$

Moreover, since W is Lipschitz, there is a constant $C_2 > 0$ such that

$$\sphericalangle(v_1, v_1') \le C_2 \operatorname{dist}(v_0, v_1). \tag{3.13}$$

The constants C_1, C_2 depend only on S. Now suppose (3.11) holds (with v_0 replaced by v_1'). Then by (3.11),(3.12) and (3.13), we have

$$\begin{aligned} |k_+(v_0) - k_+(v_1)| &\le |k_+(v_1') - k_+(v_0)| + |k_+(v_1) - k_+(v_1')| \\ &\le C_1(\operatorname{dist}(v_0, v_1))^\alpha + \tilde{C}(\sphericalangle(v_1, v_1'))^\alpha \\ &\le (C_1 + \tilde{C}C_2^\alpha)(\operatorname{dist}(v_0, v_1))^\alpha. \end{aligned}$$

This completes the reduction of (3.10) to (3.11), and we proceed with the proof of (3.11).

Step 2. Application of Lemmas 3.5 and 3.6.

Let $p \in S$, let $v_0, v_1 \in T^1_p S$ and let $\theta = \sphericalangle(v_0, v_1)$. For $0 \le r \le 1$, let v_r be a continuous curve in $T^1_p S$ such that $\sphericalangle(v_0, v_r) = r\theta$, and let γ_r be the smooth variation of geodesics with $\gamma_r(0) = \gamma_0(0) = \gamma_1(0) = p$ and $\gamma_r'(0) = -v_r$ Let

$$\begin{aligned} T = \max\{T_0 : &\text{ length of curve } r \to \gamma_r(t),\ 0 \le r \le 1, \\ &\text{is less than or equal to } \sqrt{\theta}\ ,\ \text{for } 0 \le t \le T_0\}. \end{aligned} \tag{3.14}$$

Let $K_r(t) = K(\gamma_r(t))$ and let J_r be the perpendicular Jacobi field along γ_r defined by $J_r(t) = (d/dr)(\gamma_r(t))$. Let $J_r(t) = j_r(t)E_r(t)$, where E_r's are unit normal fields along γ_r's oriented so that $j_r(t) > 0$ for $t > 0$.

Then $j_r(0) = 0$ and $j_r'(0) = \theta$. By comparing with the $K \equiv 0$ case and applying Lemma 3.1 (vi), we have $j_r(T) \geq \theta T$. From the definition of T it follows that

$$\sqrt{\theta} = \int_0^1 j_r(T)dr \geq \theta T. \tag{3.15}$$

Therefore $T \leq 1/\sqrt{\theta}$. (See Figure 3.1.) Similarly, by comparing with the $K \equiv K_{\min}$ case, where $K_{\min} < 0$ is the minimum value of the curvature function on S, we obtain $j_r(T) \leq (\theta/\sqrt{|K_{\min}|})\sinh(\sqrt{|K_{\min}|}T)$. Thus there exists $\theta_0 > 0$ such that if $\theta < \theta_0$, then $T > 1$. Since it is clear that there is a $\tilde{C}$ such that (3.11) holds for $\theta \geq \theta_0$, we will henceforth assume that $T > 1$. (This will be used in (3.17).) We will also assume that $\theta < 1$.

Let $u_i, i = 0, 1$, be the unstable solution of the Riccati equation along σ_{v_i}. Let $y = u_1 - u_0$ and apply Lemma 3.5 with $A = -T, B = 0$ and $\epsilon = \sqrt{\theta}$. Then $|k_+(v_1) - k_+(v_0)| = |y(0)|$ and we obtain constants $C_3, C_4, C_5 > 0$ such that

$$\begin{aligned}
&|k_+(v_1) - k_+(v_0)| \\
&\quad = \leq C_3\sqrt{\theta} + C_4 T\theta \\
&\qquad + |y(-T)|\exp\left[-\int_{-T}^0 (u_0(t) + u_1(t))\,dt\right] \\
&\quad \leq (C_3 + C_4)\sqrt{\theta} \\
&\qquad + C_5\Big(\exp\left[-\int_{-T}^0 u_0(t)\,dt\right]\Big)\Big(\exp\left[-\int_{-T}^0 u_1(t)\,dt\right]\Big).
\end{aligned} \tag{3.16}$$

Let $u_i^-, i = 0, 1$ be the stable solution of the Riccati equation along $\gamma_i = \sigma_{-v_i}$. Then $u_i(-t) = -u_i^-(t)$. Moreover, by Lemma 3.6, there is a positive constant β (which is C in Lemma 3.6) such that

$$-u_i^-(t) \geq \beta u_i(t).$$

Thus we can rewrite (3.16) as

$$
\begin{aligned}
&|k_+(v_1) - k_+(v_0)| \\
&\quad \le (C_3 + C_4)\sqrt{\theta} \\
&\qquad + C_5\Big(\exp\Big[-\int_0^T u_0(t)\,dt\Big]\Big)^{\beta}\Big(\exp\Big[-\int_0^T u_1(t)\,dt\Big]\Big)^{\beta} \\
&\quad \le (C_3 + C_4)\sqrt{\theta} \\
&\qquad + C_5\Big(\exp\Big[-\int_1^T u_0(t)\,dt\Big]\Big)^{\beta}\Big(\exp\Big[-\int_1^T u_1(t)\,dt\Big]\Big)^{\beta}.
\end{aligned}
\tag{3.17}
$$

Therefore we must estimate $\exp\Big[-\int_1^T u_0(t)\,dt\Big]$ and $\exp\Big[-\int_1^T u_1(t)\,dt\Big]$ from above. We first estimate a related integral.

Step 3. Let $w_r = j_r'/j_r$. Then $w_r' = -w_r^2 - K_r$, $w_r(0) = \infty$, and $w_r(t) > 0$ for $t > 0$. In this step, we will show that there is a constant $C_6 > 0$ such that

$$
\exp\Big[-\int_1^T w_0(t)\,dt\Big] \le C_6\sqrt{\theta}. \tag{3.18}
$$

We have

$$
\frac{j_r(T)}{j_r(1)} = \exp\Big[\int_1^T \frac{j_r'}{j_r}\,dt\Big] = \exp\Big[\int_1^T w_r\,dt\Big].
$$

Thus

$$
j_r(T) = j_r(1)\exp\Big[\int_1^T w_r\,dt\Big]. \tag{3.19}
$$

Since $j_r(0) = 0$ and $j_r'(0) = \theta$, we see that $j_r(1) \le C_7\theta$, for some $C_7 > 0$ (by comparing with the case $K \equiv K_{\min}$ and applying Lemma 3.1(vi)). Combining this fact with (3.15) and (3.19), we obtain

$$
\theta^{-1/2} \le C_7 \int_0^1 \exp\Big[\int_1^T w_r\,dt\Big]\,dr. \tag{3.20}
$$

If the average of the quantities $\exp\Big[\int_1^T w_r\,dt\Big]$ for $0 \le r \le 1$ in (3.20) could be replaced by $\exp\Big[\int_1^T w_0\,dt\Big]$, the desired inequality (3.18) would follow. To make this type of replacement we now show that there is a constant $C_8 > 0$ such that

$$\int_1^T |w_r - w_0|\,dt \le C_8, \quad \text{for all } r,\ 0 \le r \le 1. \tag{3.21}$$

Fix r, $0 \le r \le 1$. Let $\tilde{y} = w_r - w_0$, and let t satisfy $1 \le t \le T$. Since w_r and w_0 are greater than the unstable Riccati solutions, it follows from Lemma 3.5 that there are positive constants C_9 and C_{10} such that

$$|\tilde{y}(t)| \le C_9\sqrt{\theta} + C_{10}t\theta + |\tilde{y}(1)| \le (C_9 + C_{10})\sqrt{\theta} + |\tilde{y}(1)|. \tag{3.22}$$

Now consider the function from vectors v in $T^1\tilde{S}$ to the values at $t = 1$ of the Riccati solutions along σ_v which have value ∞ at $t = 0$. This function is smooth, and $w_r(1)$ and $w_0(1)$ are the values of this function for $v = \gamma_r'(0) = -v_r$ and $v = \gamma_0'(0) = -v_0$, respectively. Thus there is a constant $C_{11} > 0$ (depending only on S) such that $|\tilde{y}(1)| \le C_{11}\theta$. By combining this inequality with (3.22), we obtain $|w_r(t) - w_0(t)| \le (C_9 + C_{10})\sqrt{\theta} + C_{11}\theta$. Since $T \le 1/\sqrt{\theta}$, (3.21) follows. Rewriting (3.20) and applying (3.21) yields

$$\begin{aligned}\theta^{-1/2} &\le C_7 \int_0^1 \exp\left[\int_1^T w_0\,dt\right] \exp\left[\int_1^T w_r - w_0\,dt\right] dr \\ &\le C_7 \exp(C_8) \exp\left[\int_1^T w_0\,dt\right].\end{aligned}$$

This proves (3.18).

Step 4. Comparison of u_0 and w_0. Let $\Gamma \subset T^1S$ be the set of unit vectors which are tangent to closed geodesics along which K vanishes. Suppose that $k_+(v_0) \ne 0$ (i.e., $v_0 \notin \Gamma$). Since $w_0(1) > u_0(1)$, Lemma 3.7 applies, and we have

$$\frac{\exp\left[\int_1^T w_0(t)\,dt\right]}{\exp\left[\int_1^T u_0(t)\,dt\right]} \le \frac{w_0(1)}{u_0(1)}. \tag{3.23}$$

By Lemma 3.1(v),

$$u_0(1) \ge \frac{u_0(0)}{u_0(0) + 1}. \tag{3.24}$$

Also, since K is bounded from below, both $u_0(0)$ and $w_0(1)$ are bounded from above by a constant. Therefore, from (3.23) and (3.24) it follows that

$$\exp\left[-\int_1^T u_0(t)\,dt\right] \le \frac{C_{12}}{k_+(v_0)}\exp\left[-\int_1^T w_0(t)\,dt\right], \tag{3.25}$$

for some $C_{12} > 0$.

Step 5. Completion of the proof. Combining the results of steps 3 and 4, we obtain

$$\left(\exp\left[-\int_1^T u_0(t)\,dt\right]\right)^\beta \le \left(\frac{C_6 C_{10}}{k_+(v_0)}\right)^\beta \theta^{\beta/2}.$$

The same argument shows that this inequality also holds with u_0 and v_0 replaced by u_1 and v_1, respectively, if $k_+(v_1) \neq 0$. These inequalities, together with (3.17), imply that

$$\begin{aligned}|k_+(v_1) - k_+(v_0)| \le &(C_3 + C_4)\sqrt{\theta} \\ &+ C_5\Big(C_6 C_{12}\Big)^\beta \min((k_+(v_0))^{-\beta}, \quad (k_+(v_1))^{-\beta})\theta^{\beta/2}.\end{aligned}$$

If $k_+(v_0)$ and $k_+(v_1)$ are both less than or equal to $\theta^{1/4}$, then

$$|k_+(v_1) - k_+(v_0)| \le 2\theta^{1/4}.$$

If at least one of $k_+(v_0)$ and $k_+(v_1)$ is greater than $\theta^{1/4}$, then

$$|k_+(v_1) - k_+(v_0)| \le (C_3 + C_4)\sqrt{\theta} + C_5\Big(C_6 C_{12}\Big)^\beta \theta^{\beta/4}.$$

In both cases, (3.11) holds for $\alpha = \min(1/4, \beta/4)$ and some positive constant $\tilde{C}$. q.e.d.

Although we do not have counterexamples to Theorems I and II if hypothesis 2) is omitted, we now give an example to show that the crucial Lemmas 3.4 and 3.6 fail to hold without hypothesis 2). This example satisfies hypothesis 1).

Example. Let S be a compact surface containing a closed right circular cylinder $\mathcal{C}$ with negative curvature on $S \setminus \mathcal{C}$. Let γ be a closed geodesic along the boundary of $\mathcal{C}$ and let S be constructed so that for some $\epsilon > 0$, the ϵ neighborhood of γ in S is a surface of revolution, and in Fermi coordinates (s, x) along γ, we have

$$K(s,x) = \begin{cases} -e^{-1/x}, & \text{for } 0 < x < \epsilon, -\infty < s < \infty, \\ 0, & \text{for } -\epsilon < x \leq 0, -\infty < s < \infty. \end{cases}$$

It follows from a minor modification of Theorem 2.3 in [5] that there is a constant $C_1 > 0$ such that if v_ϕ is a vector with footpoint on γ, which makes an angle ϕ with γ and has a positive component in the $\partial/\partial x$ direction, then

$$k_-(v_\phi) > C_1 \phi |\ln \phi|. \tag{3.26}$$

Let $\sigma = \sigma_{v_\phi}$ and let $T_1 = T_1(\phi)$ be chosen so that $\{\sigma(t) : -T_1 \leq t \leq 0\}$ is a component of the intersection of σ and $\mathcal{C}$. Then there is a constant $C_2 > 0$ such that $T_1 > C_2/\phi$. Let u_+ be the unstable Riccati solution along σ. Since $K(\sigma(t)) = 0$ for $-T_1 \leq t \leq 0$,

$$k_+(v_\phi) = u_+(0) = \frac{u_+(-T_1)}{T_1 u_+(-T_1) + 1} \leq \frac{1}{T_1} < C_3 \phi, \tag{3.27}$$

where $C_3 = 1/C_2$. By (3.26) and (3.27) we see that the second inequality in the conclusion of Lemma 3.6 does not hold for any constant C.

We now show that the same example also fails to satisfy the conclusion of Lemma 3.4. Let $0 < x_0 < \epsilon$, let $0 < \phi < \pi/2$, and let v_ϕ and $\sigma = \sigma_{v_\phi}$ be as above. Let $T_2 = \sup\{T > 0 : \operatorname{dist}(\sigma(t), \gamma)) \leq x_0 \text{ for } 0 \leq t \leq T\}$. By comparison with the case of curvature 0 (see Lemma 2.1 in [5]), we have $T_2 \leq x_0/\sin\phi \leq 2x_0/\phi$. Since $u_+' = -u_+^2 - K$, it follows that

$$\begin{aligned} u_+(T_2) &\leq u_+(0) + \int_0^{T_2} -K(\sigma(t))\, dt \\ &\leq u_+(0) + T_2 e^{-1/x_0} \\ &\leq C_3\phi + \frac{2x_0 e^{-1/x_0}}{\phi}. \end{aligned}$$

Let $\phi = \sqrt{x_0} e^{-1/(2x_0)}$, let $w = \sigma'(T_2)$, and let p be the footpoint of w. Then

$$k_+(w) = u_+(T_2) \leq \sqrt{x_0}(C_3 + 2) e^{-\frac{1}{2x_0}}, \tag{3.28}$$

while $\sqrt{K(p)} = e^{-1/(2x_0)}$. Since the coefficient $\sqrt{x_0}(C_3 + 2)$ in (3.28) can be made arbitrarily small, the second inequality in the conclusion of Lemma 3.4 does not hold for any constant $C > 0$.

4. Lower bounds on curvatures of horocycles

In this section we establish the lower bound given in Lemma 3.3 on the curvatures, $k_+(v)$, of the unstable horocycles. We consider the curvatures of these horocycles at vectors v that are close to $T^1\gamma$, where γ is a closed geodesic along which the curvature K of S vanishes identically. As in hypothesis 2) of Theorems I and II, we will assume that there is a point q on γ such that K does not vanish to infinite order at q. Assume that q is chosen so that the order to which K vanishes is minimized (over points of γ) at q. The geodesic σ_v determined by v wraps around S many times very close to γ. (See Figure 4.1.) In those time intervals when σ_v passes close to q, K is bounded from above by a negative function of the distance from σ_v to γ. On the complements of these intervals we only assume that K is nonpositive. The following lemma gives an upper bound for solutions to the Riccati equation which will apply in this situation.

Lemma 4.1. *(Estimate for intervals of alternating curvatures.) Let A, B be positive constants and let K_1 be a negative constant. Let n be a positive integer and let $I_0, I_0', I_1, I_1', \dots, I_n, I_n'$ be closed intervals (arranged in the natural order from left to right) that partition $[-T, 0]$, where $T = \sum_{i=0}^{n}(|I_i| + |I_i'|)$. Assume the following:*

1) *All intervals I_i are of positive length, except possibly I_0, which may be empty.*
2) *If $n > 1$, then $|I_i| \geq A$ for $i = 1, \dots, n-1$.*
3) *If $|I_n'| > 0$, then $|I_n| \geq A$.*
4) *In the case $n = 1$, at least one of the inequalities $|I_0| \geq A$, $|I_1| \geq A$ holds.*
5) *$|I_i'| \leq B$ for $i = 0, \dots, n$.*

Let K_0 be a constant such that $K_1 < K_0 < 0$ and let u be a solution to the Riccati equation $u' = -u^2 - K$, where $K \leq K_0$ on I_i, and $K \leq 0$ on I_i', for $i = 0, \dots, n$.

Then for every $\eta > 0$, there exists a positive constant C which depends only on η, A, B and K_1 such that:

i) *If $T \geq \eta(-K_0)^{-1/2}$ and $u(-T) \geq 0$, then $u(0) \geq C\sqrt{-K_0}$.*

ii) If $u(-T) \geq \sqrt{-K_0}$, *then* $u(0) \geq C\sqrt{-K_0}$.

(The assertion ii) is still true if we delete the assumption 4).)

Proof. Let $a, b > 0$ and consider a piecewise smooth solution $U(t)$ to the Riccati equation $U' = -U^2 - \bar{K}$, where $\bar{K}(t) = K_0$ for $-a < t < 0$ and $\bar{K}(t) = 0$ for $0 \leq t < b$. Suppose $0 < U(0) \leq \sqrt{-cK_0}$, where $0 < c \leq 1$, and $U(-a) \geq 0$. Since $U(0) \leq \sqrt{-K_0}$, $U(t)$ is nondecreasing for $-a \leq t \leq 0$. Therefore $0 < U(t) \leq \sqrt{-cK_0}$, for $-a \leq t \leq 0$. Also, $U(t) \leq \sqrt{-cK_0}$ for $0 \leq t \leq b$, because U is decreasing on $[0, b]$. We then see from the Riccati equation that $U'(t) \geq cK_0 - K_0 = (1-c)(-K_0)$ for $-a < t < 0$ and $U'(t) \geq cK_0$ for $0 < t < b$. Hence $U(t) \geq U(-a)$ for $-a \leq t \leq b$ and

$$
\begin{aligned}
U(b) - U(-a) \geq& a(1-c)(-K_0) + bcK_0 \\
=& -K_0[a - c(a+b)] \\
\geq& D(-K_0)(a+b),
\end{aligned}
\tag{4.1}
$$

provided $D + c \leq a/(a+b)$. Note that $A/(A+B) \leq a/(a+b)$ whenever $a \geq A$ and $b \leq B$. Now fix $c > 0$ and $D > 0$ with $c+D \leq A/(A+B)$. The inequality (4.1) remains true if we assume $\bar{K}(t) \leq K_0$ for $-a < t < 0$ and $\bar{K}(t) \leq 0$ for $0 \leq t < b$.

Proof of i). Assume $T \geq \eta(-K_0)^{-1/2}$ and $u(-T) \geq 0$.

Case 1. Suppose that $u \leq \sqrt{-cK_0}$ at all the right endpoints of the I_i intervals. Then we repeatedly compare u with U and apply (4.1) on pairs of intervals I_i, I_i', starting with the first i ($i = 0$ or 1) such that $|I_i| \geq A$. Thus we obtain $u(0) \geq D(-K_0)TA/(2A+B)$, which gives the desired estimate, since $T \geq \eta(-K_0)^{-1/2}$. The reason for replacing T by $TA/(2A+B)$ is that this is a lower bound for the length of the intervals that come after I_0 and I_0', in case $|I_0| < A$.

Case 2. Suppose $u \geq \sqrt{-cK_0}$ at the right endpoint of some I_i interval. Let j be the largest index i for which this happens. By Lemma 3.1(iv), at the right endpoint of I_j' we have

$$
u \geq \frac{\sqrt{-cK_0}}{B\sqrt{-cK_0}+1} \geq \frac{\sqrt{-cK_0}}{B\sqrt{-cK_1}+1} \geq E\sqrt{-K_0},
$$

for some positive constant E. By applying inequality (4.1) on any remaining pairs of intervals I_i, I_i', where $i > j$, we see that $u(0)$ is greater than or equal to the value of u at the right endpoint of I_j'. (If $i = n$ and

$|I_n'| = 0$, then we just use the fact that u is increasing on I_n.) Therefore $u(0) \geq E\sqrt{-K_0}$.

Proof of ii). Assume $u(-T) \geq \sqrt{-K_0}$. Then $u \geq \sqrt{-K_0} \geq \sqrt{-cK_0}$ at the right endpoint of I_0, and by the same argument as in Case 2 of the proof of i), $u(0) \geq E\sqrt{-K_0}$. q.e.d.

The next lemma is due to Keith Burns. This lemma will enable us to estimate the lengths of time intervals that geodesics σ_v in S (or in $\tilde{S}$) spend in certain regions near geodesics along which the curvature vanishes.

Lemma 4.2. *Let S be a complete surface with curvature K. Let $\gamma(s)$ be a unit-speed geodesic in S, and let (s,x) be Fermi coordinates along γ. Let I and J be open intervals in $\mathbf{R}$, with $0 \in J$, such that the map*

$$p \mapsto (s(p), x(p))$$

is a diffeomorphism from a neighborhood $\mathcal{N}$ of $\gamma(I)$ onto $I \times J$.

Assume that for some constant $C > 0$ and some positive integer m, the following condition holds:

$$\textit{For all } p \in \mathcal{N},\ 0 \leq -K(p) \leq C|x(p)|^m.$$

Let $\sigma : [0,T] \to \mathcal{N}$ be a unit-speed geodesic segment. For $t \in [0,T]$, let $\phi(t)$ be the signed angle between $\sigma(t)$ and the curve $x = x(\sigma(t))$, chosen to lie in the interval $[-\pi/2, \pi/2]$, and consistent with $\sphericalangle(\partial/\partial x, \partial/\partial s) = \pi/2$. Let $d(t) = x(\sigma(t))$ be the signed distance from $\sigma(t)$ to γ. Then

i) $d'(t) = \sin\phi(t)$ and

ii) $|\phi'(t)| \leq C|x(\sigma(t))|^{m+1}$,

for all $t \in [0,T]$.

Proof. Conclusion (i) follows immediately from the definitions of $\phi(t)$ and $d(t)$, and we proceed with the proof of (ii). We may assume that $d(t) \geq 0$ for all $t \in [0,T]$.

Let $\mathcal{S}$ and $\mathcal{X}$ be the unit-speed vector fields in the directions of $\partial/\partial s$ and $\partial/\partial x$, respectively. For $p \in \mathcal{N}$ with $x(p) \geq 0$, let $A(p)$ be the geodesic curvature at the point p of the curve $x = x(p)$ in $\mathcal{N}$, so that

$$\nabla_{\mathcal{S}}\mathcal{X} = A\mathcal{S}.$$

Observe that if $\beta : J \to \mathcal{N}$ is a geodesic segment tangent to $\partial/\partial x$ with $x(\beta(0)) = 0$, then for $x \geq 0$, the function $w(x) = A(\beta(x))$ is the solution to the Riccati equation

$$w'(x) = -w(x)^2 - K(\beta(x)),$$

with initial condition $w(0) = 0$. Then, for $x \geq 0$, we obtain

$$\begin{aligned} w(x) &= \int_0^x w'(\tau)\, d\tau \\ &= \int_0^x -(w(\tau))^2 - K(\beta(\tau))\, d\tau \\ &\leq \int_0^x -K(\beta(\tau)), d\tau \\ &\leq Cx^{m+1}, \end{aligned}$$

for all nonnegative $x \in J$. Therefore

$$A(p) \leq C(x(p))^{m+1}. \tag{4.2}$$

If $\phi(t_0) = \pm\pi/2$ for some $t_0 \in [0, T]$, then σ is contained in a geodesic segment perpendicular to γ, and $\phi(t)$ is constant. In this case (ii) is clearly satisfied. Thus we may assume that $\phi(t) \in (-\pi/2, \pi/2)$ for $t \in [0, T]$. Then $\langle \sigma'(t), \mathcal{S}\rangle$ is never zero for $t \in [0, T]$, and we may assume that $\langle \sigma'(t), \mathcal{S}\rangle > 0$, for $t \in [0, T]$.

We now calculate $\phi'(t)$, for $t \in [0, T]$. Notice that

$$\sigma'(t) = \cos\phi(t)\mathcal{S}(\sigma(t)) + \sin\phi(t)\mathcal{X}(\sigma(t)),$$

and so

$$\begin{aligned} \phi' \cos\phi &= \frac{d}{dt}\sin\phi \\ &= \frac{d}{dt}\langle \mathcal{X}(\sigma), \sigma'\rangle \\ &= \langle \nabla_{\sigma'}\mathcal{X}(\sigma), \sigma'\rangle \\ &= \langle \nabla_{\cos\phi\mathcal{S}(\sigma)+\sin\phi\mathcal{X}(\sigma)}\mathcal{X}(\sigma), \sigma'\rangle \\ &= \cos\phi\langle \nabla_{\mathcal{S}(\sigma)}\mathcal{X}(\sigma), \sigma'\rangle \ + \ \sin\phi\langle \nabla_{\mathcal{X}(\sigma)}\mathcal{X}(\sigma), \sigma'\rangle \\ &= \cos\phi\langle \nabla_{\mathcal{S}(\sigma)}\mathcal{X}(\sigma), \sigma'\rangle \\ &= \cos\phi\langle A(\sigma)\mathcal{S}(\sigma), \sigma'\rangle \\ &= A(\sigma)\cos^2\phi \end{aligned}$$

on $[0,T]$. Since $\cos\phi(t) \neq 0$ for $t \in [0,T]$, we obtain

$$\phi'(t) \quad = \quad A(\sigma(t))\cos\phi(t). \tag{4.3}$$

Conclusion (ii) then follows from (4.2) and (4.3). q.e.d.

The following lemma, together with its analog for stable horocycles, provides the last step in the proof of Lemma 3.3, thereby completing the proofs of Theorems I and II.

Lemma 4.3. *Under the hypothesis of Lemma 3.3, there exists a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ in $T^1\tilde{S}$ such that for any $v \in \mathcal{U}$ with footpoint having second Fermi coordinate $x = a$, the curvature $k_+(v)$ of the unstable horocycle satisfies*

$$k_+(v) \geq C\max(|a|^{m/2}, |\phi_0|^{m/(m+2)}),$$

where $\phi_0 = \sphericalangle(v, x = a)$.

Proof. Let s_0 be the length of γ, and let $P : \mathbf{R} \to [0, s_0]$ be the covering map with $P(0) = 0$. For a set $A \subseteq [0, s_0]$, let $\tilde{A}$ denote $P^{-1}(A)$. Since K vanishes to order $m-1$ on $\tilde{\gamma}$, but does not vanish to order m at $\tilde{\gamma}(0)$, there exist positive constants C_1, C_2 and ϵ and an interval $L = [0, s_1]$ for some $s_1 \in (0, s_0)$ such that

$$-C_1 x^m \leq K(s,x), \text{ for } |x| < \epsilon, \text{ for all } s,$$

and

$$-C_1 x^m \leq K(s,x) \leq -C_2 x^m, \text{ for } |x| < \epsilon, \text{ for } s \in \tilde{L}.$$

Let L' be the closure of $[0, s_0] \setminus L$.

In this proof we modify our previous convention and let σ_v denote the maximal geodesic segment (possibly of infinite length) in $\tilde{S}$ with initial tangent vector v (with footpoint in the region where $|x| < \epsilon$) which remains in the region where $|x| < \epsilon$.

The first part of our proof is concerned with the choice of a neighborhood $\mathcal{U}$ such that for $v \in \mathcal{U}$, σ_v will assume all s values in one component of $\tilde{L}$, while taking x values in the interval $[\delta, 2\delta]$ (or $[-2\delta, -\delta]$), for some suitably chosen $\delta > 0$.

Let δ and a be such that $0 < \delta < \epsilon/2$ and $0 \leq a \leq \delta/2$, and suppose σ is a geodesic segment in the region of $\tilde{S}$ where $a \leq x \leq 2\delta$ such that $\sigma(0)$ lies on $x = a$ and $\sigma(-T)$ lies on $x = 2\delta$. Let $\phi(t) = \sphericalangle(\sigma'(t), x = \text{const})$, for $t \in [-T, 0]$, and let $\phi_1 = \phi(-T)$ and $\phi_0 = \phi(0)$. If we let

$x = x(\sigma(t))$ and we consider ϕ to be a function of x, as well as a function of t, then $(dx/dt)(d\phi/dx) = d\phi/dt$. By Lemma 4.2, $dx/dt = \sin\phi(t)$ and

$$|d\phi/dt| \leq C_1(2\delta)^{m+1},$$

whence

$$|\phi/2||d\phi/dx| \leq |\sin\phi(t)||d\phi/dx| = |d\phi/dt| \leq C_1(2\delta)^{m+1}.$$

Thus $d(\phi)^2/dx \leq 4C_1(2\delta)^{m+1}$ and

$$\phi_1^2 \leq \phi_0^2 + 4C_1(2\delta)^{m+2}.$$

Hence there are positive constants β and C_3, where β depends on δ, but C_3 can be chosen independently of δ, such that if $-\beta < \phi_0 \leq 0$, then $|\phi_1| \leq C_3\delta^{(m+2)/2}$. Let

$$\ell = \max\{||(\partial/\partial s)_p|| : |x(p)| \leq \epsilon\},$$

and choose $\delta > 0$ such that

$$C_3\delta^{(m+2)/2} \leq \min(\delta/(4\ell s_0), \pi/4).$$

Then

$$|\phi(t)| \leq \min(\delta/(4\ell s_0), \pi/4) \tag{4.4}$$

for $-T \leq t \leq 0$. Thus σ assumes all s values in some interval of length at least $2s_0$ during the time when it is in the region $\delta \leq x \leq 2\delta$. Reason: If we let

$$\nu(c) = |\sphericalangle(\sigma', x = c)|,$$

then $0 < \nu(c) < \pi/4$ and

$$\begin{aligned} |(ds/dx)_{x=c}| &= (\cot\ \nu(c))/||\partial/\partial s|| \geq (2||\partial/\partial s||\sin\ \nu(c))^{-1} \\ &\geq (2\ell\nu(c))^{-1} \geq 2s_0/\delta. \end{aligned}$$

In particular, the interval of s values so obtained would include at least one component of $\tilde{L}$.

It follows from the argument in the preceding paragraph that there exist a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ and a $\delta \in (0, \epsilon/2)$ such that if $v \in \mathcal{U}$ then any geodesic segment contained in σ_v, which goes from $x = 2\delta$ to $x = \delta$ (or from $x = -2\delta$ to $x = -\delta$), satisfies (4.4).

For the rest of the proof fix a choice of $v \in \mathcal{U}$, let $x = a$ be the second Fermi coordinate of the footpoint of v, and let $\phi_0 = \sphericalangle(v, x = a)$. We may assume that $a \geq 0$.

Case 1. Suppose $|\phi_0| \leq C_0 a^{(m+2)/2}$ for some $C_0 > 0$. (In Case 1, C_0 may be *any* positive constant.) If $a = 0$, then $\phi_0 = 0$ and $k_+(v) = 0$; so assume $a > 0$. Let $T > 0$ be such that $\sigma_v(t)$ is in the region $(a/2) \leq x \leq a$ for $t \in [-T, 0]$ and $\sigma_v(-T)$ is on $x = a/2$ or $x = 2a$. (Three possible ways this can happen are indicated in Figures 4.2 - 4.4.) We will show that $T \geq C_4 a^{-m/2}$ for some positive constant C_4 (to be specified below). We argue by contradiction and assume $T < C_4 a^{-m/2}$. For $t \in [-T, 0]$, let $\phi(t) = \sphericalangle(\sigma'(t), x = \text{const})$. By Lemma 4.2, we have

$$\begin{aligned} |\phi(t)| \leq & C_0 a^{(m+2)/2} + C_1 (2a)^{m+1} T \\ \leq & (C_0 + 2^{m+1} C_1 C_4) a^{(m+2)/2}. \end{aligned} \tag{4.5}$$

Thus

$$T \geq \frac{a/2}{\sup\{\sin(|\phi(t)|) : t \in [-T, 0]\}} \geq \frac{a^{-m/2}}{2(C_0 + 2^{m+1} C_1 C_4)}.$$

If

$$2^{-1}(C_0 + 2^{m+1} C_1 C_4)^{-1} \geq C_4, \tag{4.6}$$

then we have a contradiction to the assumption that $T < C_4 a^{-m/2}$. Choose $C_4 > 0$ such that (4.6) holds. Then $T \geq C_4 a^{-m/2}$.

Let $I'_0, I_0, I'_1, I_1, \ldots, I'_n, I_n$ be a partition of $[-T, 0]$ such that for $t \in I_i$ $[t \in I'_i]$ the s coordinate of σ_v is in $\tilde{L}$ $[\tilde{L}']$. Then for $t \in I_i$, $K(\sigma_v(t)) \leq -C_2 (a/2)^m$. Since $|\sphericalangle(\sigma_v, x = \text{constant})| \leq \pi/4$ for $v \in \mathcal{U}$, there exist positive constants A, B such that the hypothesis of Lemma 4.1 holds with $K_0 = -C_2(a/2)^m$ and $\eta = C_2^{1/2} C_4 2^{-m/2}$. From part i) of this lemma we conclude that there is a constant $C_5 > 0$ such that $k_+(v) \geq C_5 a^{m/2}$. Since $\phi_0 \leq C_0 |a|^{(m+2)/2}$, it follows that there exists $C > 0$ such that

$$k_+(v) \geq C \max(|a|^{m/2}, |\phi_0|^{m/(m+2)}).$$

Case 2. $|\phi_0| \geq C_0 a^{(m+2)/2}$. (Here C_0 is a sufficiently large positive constant, as described in Case 2b. This constant depends only on C_1.) For starters require that $C_0 > 1$. Since ϕ_0 cannot be 0 except in the trivial case, when a is also 0, we will assume $\phi_0 \neq 0$.

Case 2a. Suppose $\phi_0 < 0$. Let $T_0 > 0$ be such that σ_v crosses $x = \delta$ at time $-T_0$. By the choice of $\mathcal{U}$, there exist T_1, $T_2 > 0$ such that $T_0 \le T_1 < T_2$ and during the time interval $[-T_2, -T_1]$ the s values taken by σ_v lie in a component of $\tilde{L}$ and cover this component. (See Figure 4.5.) Assume that T_1 and T_2 are chosen as small as possible, while satisfying these requirements. The interval of s values assumed by σ_v in the time interval $[-T_1, -T_0]$ has length less than s_0. Since $|\sphericalangle(\sigma_v, x = c)| \le \pi/4$ for $\delta \le c \le 2\delta$, this implies that $|T_1 - T_0| \le \ell s_0\sqrt{2}$. Also, we have $|T_2 - T_1| \ge |L|$. (This inequality depends on the fact that $||\partial/\partial s|| \ge 1$, which follows from the nonpositive curvature assumption.) If $t \in [-T_2, -T_1]$, then $K(\sigma_v(t)) \le -C_2\delta^m$. It follows that there is a positive constant C_6 (depending on δ, but not on ϕ_0) such that the unstable Riccati solution along σ_v is at least C_6 at $t = -T_0$. By reducing the size of the neighborhood $\mathcal{U}$, if necessary, we may assume that ϕ_0 satisfies $\sqrt{C_2}|\phi_0|^{m/(m+2)} < C_6$. Then the hypothesis of part (ii) of Lemma 4.1 holds for the time that σ_v is in the region $|\phi_0|^{2/(m+2)} \le x \le \delta$ with $K_0 = -C_2|\phi_0|^{2m/(m+2)}$. From this lemma we conclude that the value of the unstable Riccati solution along σ_v is at least $C_7|\phi_0|^{m/(m+2)}$, for some $C_7 > 0$, at the time σ_v crosses $x = |\phi_0|^{2/(m+2)}$. Since σ_v makes angle of absolute value at least $|\phi_0|$ with $x = c$ for $a \le c \le |\phi_0|^{2/(m+2)}$, the length of time σ_v is in the region where $a \le x \le |\phi_0|^{2/(m+2)}$ is less than or equal to

$$\frac{|\phi_0|^{2/(m+2)}}{\sin|\phi_0|} \le \frac{|\phi_0|^{2/(m+2)}}{|\phi_0|/2} = 2|\phi_0|^{-m/(m+2)}.$$

Then, by Lemma 3.1(v), $k_+(v)$, the value of the unstable Riccati solution along σ_v at time 0, satisfies

$$k_+(v) \ge \frac{C_7|\phi_0|^{m/(m+2)}}{2|\phi_0|^{-m/(m+2)}C_7|\phi_0|^{m/(m+2)} + 1} \ge C_8|\phi_0|^{m/(m+2)}$$

for some $C_8 > 0$.

Case 2b. Suppose $\phi_0 > 0$. Let $C_9 = (2C_1)^{-1}$. We will show that $\sigma_v(t)$ cannot be in the region $0 \le x \le a$ for all $t \in [-T, 0]$, where $T = C_9\phi_0^{-m/(m+2)}$. Suppose this were the case. By Lemma 4.2, we have

$$\phi'(t) \le C_1 a^{m+1} \le C_1\phi_0^{2(m+1)/(m+2)}$$

for $t \in [-T, 0]$, and consequently,

$$\phi(t) \ge \phi_0 - TC_1\phi_0^{2(m+1)/(m+2)} = (1 - C_1C_9)\phi_0 = \phi_0/2 > 0$$

for $t \in [-T, 0]$. But then the x coordinate of $\sigma(-T)$ is less than or equal to

$$\begin{aligned} a - T\sin(\phi_0/2) &\leq C_0^{-2/(m+2)}\phi_0^{2/(m+2)} - (8C_1)^{-1}\phi_0^{2/(m+2)} \\ &= [C_0^{-2/(m+2)} - (8C_1)^{-1}]\phi_0^{2/(m+2)}. \end{aligned}$$

This contradicts the assumptions on σ_v and T if $C_0^{-2/(m+2)} - (8C_1)^{-1} < 0$. Thus we choose C_0 such that $C_0 > 1$ and $C_0^{-2/(m+2)} < (8C_1)^{-1}$. Then $\sigma_v(-\tilde{T})$ lies on $\tilde{\gamma}$ for some $0 \leq \tilde{T} < C_9\phi_0^{-m/(m+2)}$. (By the above argument $\phi(t) > 0$ as long as $\sigma_v(t)$ remains in the region $0 \leq x \leq a$ in negative time. Thus σ_v leaves this region in negative time at $x = 0$.) Moreover, $\sphericalangle(\sigma_v'(-\tilde{T}), \tilde{\gamma}) \geq \phi_0/2$. Then the hypothesis of Case 2a holds for $\sigma_v'(-\tilde{T})$, and by the conclusion of Case 2a, the value of the unstable Riccati solution along σ_v at time $-\tilde{T}$ is at least $C_8(\phi_0/2)^{-m/(m+2)}$. Hence by the same calculation as in the last step of Case 2a, $k_+(v) \geq C_{10}\phi_0^{m/(m+2)}$ for some $C_{10} > 0$. q.e.d.

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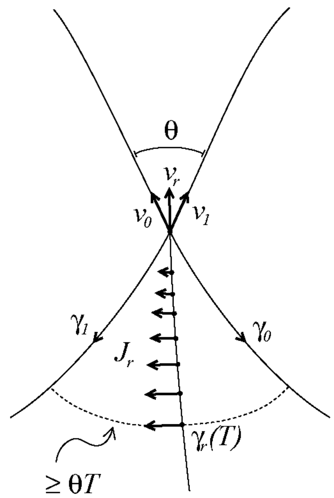

FIGURE 3.1

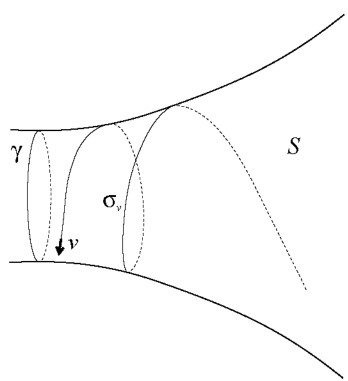

FIGURE 4.1

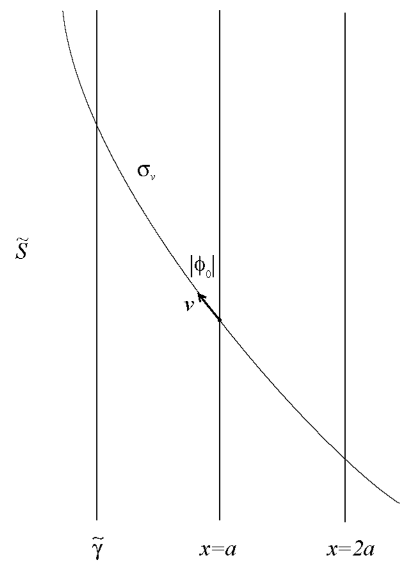

FIGURE 4.2. Case 1a.

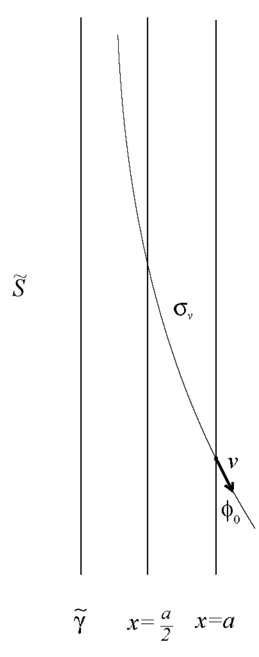

FIGURE 4.3. Case 1b.

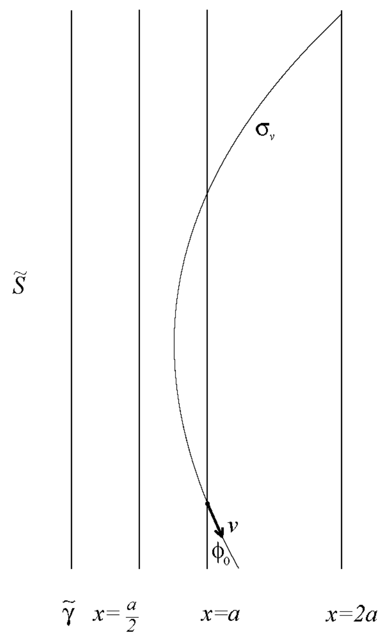

FIGURE 4.4. Case 1c.

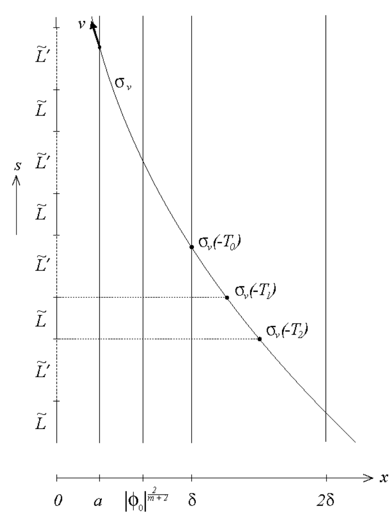

FIGURE 4.5. Case 2a.