J. DIFFERENTIAL GEOMETRY
51 (1999) 517-550

THE MODULI SPACE OF HYPERBOLIC CONE STRUCTURES

QING ZHOU

Introduction

A topological cone manifold is a manifold together with a link each of whose component has a cone angle attached. If each cone angle is of the form $2\pi/n$, for some integer n, the cone manifold becomes an orbifold. Unlike an orbifold, the cone manifold is not a natural concept, but it turns out to be very important in the study of geometrization of orbifolds.

In this paper, we will consider 3-dimensional geometric cone structures. The main result in this paper is an existence and uniqueness theorem for 3-dimensional hyperbolic cone structures.

Theorem A. *Let Σ be a hyperbolic link with m components in a 3-dimensional manifold X. Then the moduli space of marked hyperbolic cone structures on the pair (X, Σ) with all cone angles less than $2\pi/3$ is an m-dimensional open cube $(0, 2\pi/3)^m$, parameterized naturally by the m cone angles.*

The uniqueness part of Theorem A is an analogue of Mostow's rigidity theorem. Namely if we have two hyperbolic cone structures C_1 and C_2 with cone angles less than $2\pi/3$, then C_1 and C_2 are isometric if and only if there is a homeomorphism between (X_1, Σ_1) and (X_2, Σ_2) so that the corresponding cone angles are the same. This rigidity theorem has been improved by Kojima to hyperbolic cone manifolds with cone

Received November 11, 1997, and, in revised form, September 16, 1998. The project is partially supported by grants of NNSFC, FEYUT of SEDC and HYTEF.

angles at most π recently [10]. The proof of the theorem goes in a similar way as what Thurston proposed for the proof of his geometrization theorem for orbifolds [16]. As a corollary, we will give a proof of the following special case of Thurston's geometrization theorem.

Corollary B. *Suppose M is an irreducible, closed, atoroidal, non-Seifert fibered 3-manifolds, and G is a group acting effectively on M. If the order of G is odd and the action is not fixed-point-free, then the quotient M/G is a geometric orbifold.*

This paper is organized in seven sections. §1 collects some basic facts for geometric cone manifolds and their limits; we refer the readers to somewhere else for details. Our first goal is using equivariant Ricci flow to study the topology and geometry of compact Euclidean cone manifolds. A briefly review of Hamilton's work on Ricci flow is given in §2, and we also establish a version of Ricci flow on orbifolds. Using Ricci flow on orbifolds we study compact Euclidean cone manifolds in §3. Our next goal is a compactness theorem for hyperbolic cone structures. For the purpose, we show that the compact hyperbolic cone manifolds with cone angles less than $2\pi/3$ cannot become thinner and thinner everywhere in §4, and the compactness theorem will be given in §5. In §6, we review the deformation theory of hyperbolic cone structures, and the proofs of Theorem A and Corollary B will be given in the last section.

Essentially this paper is a rewriting of my Ph.D. thesis [17]. I would like to express my gratitude to my thesis advisor Robert D. Edwards here. Without his encouragement, this work could not have been done.

1. Preliminaries

We collect some basic facts about geometric cone manifolds and their limits in this section; for details we refer the readers to [6] and [14].

Let C be a 3-dimensional geometric cone manifold, X_C the underlying space, and Σ_C the singular locus. The pair (X_C, Σ_C) is called the combinatorial type of the geometric cone manifold C. Two cone manifolds C_1 and C_2 are said to be isomorphic if there is an isometry between them. The isometry of two geometric cone manifolds induces an homeomorphism between their combinatorial types.

A geometric cone manifold C is a geometric orbifold if each of the cone angles is of the form $2\pi/n$ for some integer $n \geqslant 2$. The concept of the cone manifold is a generalization of orbifold.

A geometric cone structure is a geometric structure in the sense of Thurston [15] but in a more general setting. So we can talk about the developing map and the holonomy representation. It is clear that holonomy representations of two isomorphic cone manifolds C_1 and C_2 are conjugate in the sense that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X_1 - \Sigma_1) & \rightarrow & PSL_2(\mathbb{C}) \\ \downarrow & & \downarrow \\ \pi_1(X_2 - \Sigma_2) & \rightarrow & PSL_2(\mathbb{C}) \end{array}$$

An ε-ball $B_\varepsilon(x)$ in a geometric cone manifold C is standard if it is standard in the usual sense or the ball has a sigular diameter and x is on the sigular diameter. For a point $x \in C$, The injectivity radius of x is defined as follows:

If $x \in \Sigma_C$, then

$$\mathrm{inj}(x) = \sup\{\varepsilon | \varepsilon\text{-ball } B_\varepsilon(x) \text{ is standard}\};$$

If $x \notin \Sigma_C$, then

$$\begin{aligned} \mathrm{inj}(x) \quad = \quad & \max(\sup\{\varepsilon | \varepsilon\text{-ball } B_\varepsilon(x) \text{ is standard}\}, \\ & \qquad \sup\{\delta | \text{there is } y \in \Sigma_C \text{ such that } \mathrm{inj}(y) > 2\delta \\ & \qquad \text{and } d(x,y) < \delta\}). \end{aligned}$$

It is easy to see that the injectivity radius is a positive lower semi-continuous function on C.

Using injectitivty radius we can divide a geometric cone manifold into the thin part and the thick part. For any $\varepsilon > 0$, the ε-thin part of a geometric cone manifold C is the set $C_{\mathrm{thin},\varepsilon} = \{x | \mathrm{inj}(x) \leqslant \varepsilon\}$ and the ε-thick part is the complement of the ε-thin part, i.e., $C_{\mathrm{thick},\varepsilon} = C - C_{\mathrm{thin},\varepsilon}$.

Just like the injectivity radius on a Riemannian manifold, the injectivity radius on a cone manifold with nonpositive constant sectional curvature cannot decrease too fast. More precisely, we have

Theorem 1.1 ([14, Proposition 6.1]). *Let C be a 3-dimensional cone manifold with nonpositive constant sectional curvature. Suppose that there is a positive constant ω such that all cone angles are at least ω and at most π. Then, for any $R, \varepsilon > 0$, there is a $\delta > 0$ which depends only on R, ε, ω and not on C, such that, if inj$(x) > \varepsilon$, then $B_R(x, C) \subset C_{thick,\delta}$.*

A useful tool to analyze the geometric cone manifold is the pointed Gromov limit. For the definition of Gromov limit and the proofs of the following theorems we refer readers to [3] and [14].

Theorem 1.2. (See [14, Proposition 3.6 and Theorem 4.2]). *Let (C_n, x_n) be a sequence of cone manifolds with nonpositive constant sectional curvature and*

1) the sectional curvatures K_n has a limit K_∞,

2) all cone angles have a uniform lower bound ω, and

3) there is an $\varepsilon > 0$, such that $x_n \in C_{n,thick,\varepsilon}$ for all n.

Then it has a convergent subsequence (C_{n_k}, x_{n_k}), and the limit space (C, x) is a cone manifold with constant sectional curvature K_∞.

In general, even if all the C_n have the same fixed combinatorial type, the limit space C need not have the same combinatorial type. However C does have the same topology as C_n locally.

Theorem 1.3. (See [14, Theorem 4.2 and Proposition 8.1]). *Let (C_n, x_n) be a convergent sequence of cone manifolds with nonpositive constant sectional curvature such that $\lim(C_n, x_n) = (C, x)$ is still a cone manifold. For any $R > 0$, if $B_R(x, C)$ is a proper set and n large then enough, $(B_R(x_n, C_n), B_R(x_n, C_n) \cap \Sigma_n)$ is homeomorphic to $(B_R(x, C), B_R(x, C) \cap \Sigma)$. Furthermore, this homeomorphism can be chosen to be almost isometry.*

This also shows that the holonomy of $B_R(x_n, C_n)$ can be chosen to converge to a holonomy of $B_R(x, C)$ if all C_n are hyperbolic cone manifolds.

To understand the "local picture" near a point in a hyperbolic cone manifold with a small injectivity radius δ, we can rescale the hyperbolic metric by multiplying δ^{-1}, which is a large number. Then the new structure has constant curvature $-\delta^2$. If δ is small enough, the new structure is very close to a Euclidean structure. Theorem 1.3 says that locally the hyperbolic cone manifold has the topology of a Euclidean cone manifold. More precisely, we have the following analogue of the Kazhdan-Margulis theorem.

Theorem 1.4. ([14, Proposition 8.1]). *For any $R > 0$ and $\omega > 0$, there is a $0 < \delta < 1/R$ so that, for any 3-dimensional hyperbolic cone manifold C with cone angle at least ω and $x \in C_{thin,\,\delta}$, we have that $(B_{Rinj(x)}(x, C), B_{Rinj(x)}(x, C) \cap \Sigma_C)$ is homeomorphic to $(B_R(y, E), B_R(y, E) \cap \Sigma_E)$ for some 3-dimensional Euclidean cone manifold E with $inj(y) = 1$, and after a rescale, the homeomorphism is an almost isometry.*

This theorem tells us that 3-dimensional Euclidean cone manifolds play an important role in analyzing the thin part of 3-dimensional hyperbolic cone manifolds. A complete classification of noncompact Euclidean cone manifolds is known, and we state the conclusion here and refer readers to [14, §5] for the discussion.

Theorem 1.5. *Let E be a noncompact complete 3-dimensional Euclidean cone manifold with cone angles at most π, then E is one of the following:*

1) $\mathbb{E}^3$ or a product of $\mathbb{E}^1$ and an infinite disk with a cone;

2) a product of $\mathbb{E}^1$ with a torus or a compact 2-dimensional Euclidean cone manifold;

3) a twisted $\mathbb{E}^1$ bundle over a Klein bottle or a projective plane with two cones of angle π;

4) a bundle over S^1 whose fiber is $\mathbb{E}^2$ or an infinite disk with a cone; or

5) a quotient of $\mathbb{E}^3$ modulo one of the following groups: Denote by $T_{\mathbf{a}}$ the translation in a vertor $\mathbf{a}$ and by $R_{l,\varphi}$ the rotation through an angle φ about an axis l, i) the group generated by $R_{l_1,\pi}, R_{l_2,\pi}$ for two nonintersect lines l_1, l_2; ii) the group generated by $R_{l_1,\pi}, R_{l_2,\pi}$ and $T_{\mathbf{a}}$, where $l_2 = T_{\mathbf{b}} l_1$, $\mathbf{a}$ is parallel to l_1 and perpendicular to $\mathbf{b}$, iii) the group generated by $R_{l_1,\pi}, R_{l_2,\pi}$ and $T_{\mathbf{a}}$, where l_1, l_2 are nonintersect lines, perpendicular to each other and $\mathbf{a}$ is parallel to l_1.

Remark. All of the compact 2-dimensional Euclidean cone manifolds are easily to be classified. They are a flat torus, a double of an acute angled Euclidean triangle, a double of a rectangle or the boundary of a Euclidean tetrahedron with equal opposite edges. In the last case that the boundary of a Euclidean tetrahedron, since all cone angles are at most π, the opposite edges have the same lengths.

2. The Ricci flow on orbifolds

In this section, we give a brief review of Hamilton's work ([4] and [5]) on Ricci flow of 3-dimensional nonnegatively curved Riemannian manifolds. As Hamilton pointed out that the flow is invariant under an action by isometries, we can establish an orbifold version of Ricci flow. These results will be used to study the topology and the geometry of compact 3-dimensional Euclidean cone manifolds in the next section.

Let M be a closed 3-dimensional manifold and g_0 a Riemannian metric with positive Ricci curvature. Hamilton considered a partial differential equation

$$\begin{cases} \partial_t g = \frac{2}{3} r g - 2\mathrm{Ric}, \\ g(\cdot, 0) = g_0 \end{cases}$$

on M, where $r = \int R / \int 1$ is the average of the scalar curvature R. Hamilton, using the Nash-Morse inverse function theorem, showed that the equation has a short time solution for any initial metric g_0 and then estimated the curvatures by the maximum principle for parabolic equations under the assumption that the initial metric g_0 has positive Ricci curvature. The main result in [4] is the following theorem.

Theorem 2.1. *For a compact 3-dimensional manifold M with positive Ricci curvature, the metric evolution*

$$\partial_t g = \frac{2}{3} r g - 2Ric$$

has a solution for all the time and converges to a metric of positive constant Riemannian curvature as $t \to \infty$.

The equation is invariant under the full diffeomorphism group of M, so any isometry for the initial metric are preserved as the metric evolves. This fact allows us to establish an orbifold version of Hamilton's theorem.

Definition 2.2. Let O be an orbifold and g a Riemannian metric on the complement of the singular locus; we say that it is a Riemannian metric on the orbifold O if, passing to a local manifold cover, the lift of g can be extended to a smooth Riemannian metric.

For a Riemannian orbifold O, we can talk about the connection and the curvature. On the singular locus, the curvature is defined by the curvature of the local smooth lift.

An orbifold is said to be very good, if it is the quotient of a manifold modulo a finite group.

Theorem 2.3. *Suppose that a compact very good orbifold O admits a Riemannian metric with positive Ricci curvature, then it is a spherical orbifold.*

Proof. Since O is very good, O is a quotient of a manifold M modulo a finite group G. We lift the Riemannian metric on O to get a metric g_0 with positive Ricci curvature on M. Obviously, G consists of isometries of g_0. Applying Hamilton's theorem to (M, g_0), we have a Ricci flow g_t, which converges to a metric g_∞ of a positive constant Riemannian curvature as $t \to \infty$, and G is a group of isometries on g_t for all t and g_∞. So $(M, g_\infty)/G$ is a spherical structure on the orbifold O. q.e.d.

Hamilton also developed a method to deal with the case of nonnegatively curved manifold in [5]. He considered the unnormalized equation

$$\partial_t g = -2\mathrm{Ric}.$$

The equation has a short time solution for any initial metric g_0. This equation for the metric implies a heat equation for the Riemannian curvature tensor R_{ijkl}:

$$\begin{aligned}\partial_t R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &\quad - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}),\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{ipjq}R_{krls}$.

Fixing a Riemannian bundle (V, h) over M, and choosing a family of bundle isometries $u_t : (V, h) \to (TM, g_t)$, Hamilton pullbacked the Levi-Civita connections and the curvatures on (TM, g_t) and defined the covariant derivatives and the Laplacian on (V, h). Then the pullback of the heat equation for Riemannian curvature tensor can be written simply as

$$\partial_t R_{abcd} = \Delta R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}).$$

The Riemannian curvature tensor can be regarded as a symmetric bilinear form K, which is called the curvature operator, on the 2-forms $\Lambda^2(V)$. For a 2-form $\alpha \wedge \beta$, $K(\alpha \wedge \beta)$ is the product of $|\alpha \wedge \beta|$ and the sectional curvature of the plane with the Plücker coordinates $\alpha \wedge \beta$. The curvature operator contains the information as much as the Riemannian curvature tensor, and the positive curvature operator implies positive sectional curvature. However in general, the positive curature operator is not equivalent to the positive sectional curvature. In dimension 3, any 2-form is pure, which means that any 2-form can be written as a wedge product of two 1-forms. So the positive curvature operator and the positive sectional curvature are the same thing in this dimension. Now the heat equation for the Riemannian curvature tensor can be rewritten as an equation for the curvature operator

$$\partial_t K = \Delta K + K^2 + 2\mathrm{Adj}(K).$$

The crucial lemma in [5] is the following.

Lemma 2.4 ([5, Lemma 8.2.]). *Let Q be a symmetric bilinear form on V. Suppose Q satisfies a heat equation $\partial_t Q = \Delta Q + \phi(Q)$, where the matrix $\phi(Q) \geqslant 0$ for all $Q \geqslant 0$. If $Q \geqslant 0$ at time $t = 0$, then it remains so for $t \geqslant 0$. Moreover there exists an interval $0 < t < \delta$ on which the rank of Q is a constant, the null space of Q is invariant under parallel translation and invariant in the time and also lies in the null space of $\phi(Q)$.*

Suppose that M is a 3-dimensional Riemannian manifold with nonnegative curvature operator, we solved the unnormalized Ricci flow

$$\partial_t g = -2\mathrm{Ric}.$$

Applying Lemma 2.4 to the evolving equation for the curvature operator, we can see that, if we start with a nonnegative curvature operator, after the flow proceeds for a while we will get $K = 0$, $K > 0$ or that the rank of K is constant one or two. If $K = 0$, then we have a Euclidean structure which is the same as the metric we started with. If $K > 0$, we can use Theorem 2.1 to deform the metric to a spherical structure further. If the rank of K is one, then the image of K, which is the orthogonal complement of the null space, has dimension 1 and also is invariant under the parallel translation. This one dimensional space is generated by a 2-form φ. We know that φ is a pure element, i.e., φ is the Plücker coordinates of a plane field in TM, which is invariant under parallel translation, and the sectional curvature of each plane is the positive eigenvalue of the curvature operator. This gives an orthogonal decomposition $TM = V_1 \oplus V_2$, with V_i being invariant under parallel translation. The universal covering space $\tilde{M}$ also has such a decomposition. Applying the de Rham decomposition theorem to the universal cover $\tilde{M}$, $\tilde{M}$ splits isometrically into a product $N \times \mathbb{E}^1$, where N is a positively curved surface and its curvature has a positive lower bound since M is compact. This implies that N is a sphere. The metric on N may not be standard, but we will show that we can replace this metric by a standard one such that any given group of isometries of the original metric on $N \times \mathbb{E}^1$ remains a group of isometries of $S^2 \times \mathbb{E}^1$.

It is clear that isometry group acting on $N \times \mathbb{E}^1$ splits into a product of two isometric actions of N and $\mathbb{E}^1$. The action on N induces a map into the conformal group $\mathrm{Conf}(S^2)$ of S^2. If this action can be conjugated to a subgroup of $O(3) \subset \mathrm{Conf}(S^2)$, then we can replace the metric on N by a standard one such that the given group of isometries of the original metric on $N \times \mathbb{E}^1$ remains a group of isometries of $S^2 \times \mathbb{E}^1$. To do this, we need a lemma to characterize those subgroup of $\mathrm{Conf}(S^2)$ which can be conjugated into $O(3)$. Note that the $\mathrm{Conf}(S^2)$ action on S^2 has a natural extension to an action on $\mathbb{H}^3$, and a subgroup of $\mathrm{Conf}(S^2)$ can be conjugated into $O(3)$ if and only if the action has a common fixed point in $\mathbb{H}^3$.

Lemma 2.5. *A subgroup $\Gamma \subset Conf(S^2)$ fixes a point in $\mathbb{H}^3$ if and only if $\Gamma \cap PSL_2(\mathbb{C})$ consists of only elliptic elements (and the identity element).*

Proof. It is clear that if Γ fixes a point in $\mathbb{H}^3$, then $\Gamma \cap PSL_2(\mathbb{C})$ consists of only elliptic elements. On the other hand, it is known that if $\Gamma \cap PSL_2(\mathbb{C})$ consists of only elliptic elements, $\Gamma \cap PSL_2(\mathbb{C})$ fixes a point in $\mathbb{H}^3$ (see [1, Theorem 4.3.7]). So if $\Gamma \subset PSL_2(\mathbb{C})$, then we are done. Otherwise, $\Gamma_1 = \Gamma \cap PSL_2(\mathbb{C})$ is a normal subgroup in Γ with index 2. Fix a $g \in \Gamma - \Gamma_1$, we have $g(\mathrm{Fix}(\Gamma_1)) = \mathrm{Fix}(\Gamma_1)$ where $\mathrm{Fix}(\Gamma_1)$ is the fixed point set of Γ_1 in $\mathbb{H}^3$, which is $\mathbb{H}^3$, a geodesic or a point. If $\mathrm{Fix}(\Gamma_1) = \mathbb{H}^3$, then Γ is a group of order 2, and consists of I and a reflection g. Of course Γ has a fixed point in $\mathbb{H}^3$. If $\mathrm{Fix}(\Gamma_1)$ is a geodesic and g havs no fixed point on it, then $g^2 \in \Gamma_1$ also has no fixed point on it. This is impossible, so g must fix a point on $\mathrm{Fix}(\Gamma_1)$. If $\mathrm{Fix}(\Gamma_1)$ is only a point, then g fixes this point. This shows that $\Gamma = \Gamma_1 \cup g\Gamma_1$ fixes this point in $\mathbb{H}^3$. q.e.d.

Now we only need to show that any orientation preserving element in a group of isometries of N must be elliptic. If g is parabolic, then g has a unique fixed point $p \in N$. For any point $q \in N - \{p\}$, $g^n(q) \to p$, this contradicts the hypothesis that g is an isometry on N. A similar argument shows that there is no hyperbolic element in a group of orientation preserving isometries of N. Now Lemma 2.5 says that we can replace the original metric on N by a standard one such that any given group of isometries of $N \times \mathbb{E}^1$ remains a group of isometries of $S^2 \times \mathbb{E}^1$.

Hamilton showed that the case that the rank of K is two cannot really happen. Diagonalize K, then K^2 and $\mathrm{Adj}(K)$ are also diagonalized, say

$$K = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}, \text{ then } K^2 = \begin{pmatrix} \lambda^2 & & \\ & \mu^2 & \\ & & \nu^2 \end{pmatrix},$$

$$\text{and } \mathrm{Adj}(K) = \begin{pmatrix} \mu\nu & & \\ & \nu\lambda & \\ & & \lambda\mu \end{pmatrix}.$$

If the rank of K is two, then we can assume that $\lambda = 0$ but $\mu\nu \neq 0$, Lemma 2.4 implies that the null space of K lies in the null space of $K^2 + 2\mathrm{Adj}(K)$; in other words, $\mu\nu = 0$. We get a contradiction.

The equation

$$\partial_t g = -2\mathrm{Ric}$$

is also invariant under the full diffeomorphism group of M, so we can also establish an orbifold version. To summarize, we have

Theorem 2.6. *Let O be a very good orbifold which admits a Riemannian metric with nonnegative curvature operator. Then it is a geometric orbifold locally modelled in S^3, $S^2 \times \mathbb{E}^1$ or $\mathbb{E}^3$. Furthermore, if O has an Euclidean structure, then the original Riemannian metric must be Euclidean.*

Remark. Hamilton actually proved a stronger result that, if M has nonnegative Ricci curvature, then M admits a Euclidean geometry, a spherical geometry or a geometry locally modelled in $S^2 \times \mathbb{E}^1$. We can get an orbifold version for this stronger result in the same manner, but Theorem 2.6 is enough for our purpose.

3. Compact Euclidean cone manifolds

Using the Ricci flow on orbifolds, we will discuss the topological type of compact 3-dimensional Euclidean cone manifolds with cone angles at most π in this section, and show that at least one cone angle in a compact Euclidean cone manifolds is not less than $2\pi/3$.

Suppose E is a compact 3-dimensional Euclidean cone manifold with cone angle at most π, and let X_E be the underlying space, and Σ_E be the singular locus. If Σ_E is empty or all cone angles are exactly π, then E is a Euclidean manifold or orbifold, and they are a quotient of $\mathbb{E}^3$ by a 3-dimensional crystallographic group. All 3-dimensional crystallographic groups have been classified (see for example [7]). So, without loss of generality, we can assume that $\Sigma_E \neq \varnothing$ and that at least one cone angle is strictly less than π.

Near the singular locus, we use Fermi coordinates

$$ds^2 = dt^2 + dr^2 + r^2\alpha^2 d\theta^2/4\pi^2,$$

where (t, r, θ) are the cylindrical coordinates with t-axis to be the singular locus, and α to be the cone angle. Geometrically, it is a product of $\mathbb{E}^1$ and a disk with a cone. For a metric in the form

$$ds^2 = dt^2 + dr_1^2 + f^2(r_1)d\theta^2,$$

the curvature operator is

$$\begin{pmatrix} 0 & & \\ & 0 & \\ & & -f''(r_1)/f(r_1) \end{pmatrix}.$$

If near 0, $f(r_1) = \beta \sin r_1$ and $f(r_1) > 0$ for all $r_1 > 0$, this is a singular Riemannian metric with cone angle $2\pi\beta$ at $r_1 = 0$. The metric is smooth at $r_1 = 0$ if $\beta = 1$. Pick a function $f(r_1)$ such that $f(r_1) = \sin r_1$ for r_1 near 0, $f''(r_1) \leqslant 0$ and

$$f(r_1) = (r_1 + (2\pi - \alpha)\varepsilon/2\alpha)\alpha/2\pi \text{ for } r_1 > \varepsilon > 0.$$

This can be done because $\alpha < 2\pi$ and in this case $f''(r_1)$ cannot be identically zero. This is a metric with nonnegative curvature operator and, in the region $r_1 > \varepsilon$, the metric is globally isometric to $ds^2 = dt^2 + dr^2 + r^2\alpha^2 d\theta^2/4\pi^2$ under the coordinates change $r = r_1 + (2\pi - \alpha)\varepsilon/2\alpha$. So we get a metric with nonnegative curvature operator on the underlying space X_E. Note that this metric is not flat. Now we can apply Theorem 2.6 to conclude that X_E must be a spherical manifold or covered by $S^2 \times \mathbb{E}^1$.

Remark. Acturally we can increase the cone angle to any angle at most 2π by deforming the Euclidean cone structure to a metric with nonnegative curature operator in the same fashion. If all cone angles are in the form of $2\pi/n$, we get a Riemannian orbifold with nonnegative curvature operator.

If X_E is covered by $S^2 \times \mathbb{E}^1$, then we can pullback the singular Euclidean structure on X_E to the universal cover $S^2 \times \mathbb{E}^1$. Denote this Euclidean cone structure by $\tilde{E}$. Since $\tilde{E}$ has two ends and at least one cone angle is less than π, by the classification, $\tilde{E}$ is a product of $\mathbb{E}^1$ and a double of an acute angled Euclidean triangle. There are two possible deck transformation groups $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$. Suppose that the deck group is $\mathbb{Z}$, the $\mathbb{Z}$ acts on $\mathbb{E}^1$ as a translation group. This shows that E can be regarded as a quotient of a product of an interval I and a double of a Euclidean triangle with an isometric mondromy identifying the top and the bottom. If the deck group is $\mathbb{Z}_2 * \mathbb{Z}_2$, then it must act on $\mathbb{E}^1$ as an isometry group generated by two reflections. Since some elements have a fixed point on E^1, these elements will keep a cross intersection invariant. On the other hand, a double of a Euclidean triangle does not allow an order-2 orientation reversing isometry which permutes three cone points, so the cross intersection cannot be invariant, i.e., this is impossible. In this case, X_E must be $S^2 \times S^1$, Σ_E can be any 3-string braid in $S^2 \times S^1$ and at least one cone angle is not less than $2\pi/3$. Note that, in this case, $X_E - \Sigma_E$ is Seifert fibered, and the following theorem shows that the converse is also true.

Theorem 3.1. *Suppose (X, Σ) is a combinatorial type of a compact Euclidean cone manifold E with nonempty singular locus and cone angle at most π. If there is at least one cone angle less than π, then $N = X - \mathcal{N}(\Sigma)$ is an irreducible, atoroidal 3-manifold with an incompressible boundary, where $\mathcal{N}(\Sigma)$ is a regular neighborhood of Σ.*

Furthermore, N is Seifert fibered if and only if X is $S^1 \times S^2$.

Proof. It is not difficult to see that any essential simple closed loop in ∂N will be sent to a nontrivial element under the holonomy, so ∂N is incompressible.

To show that N is irreducible and atoroidal, we will use minimal surface techniques. We deform the singular Euclidean structure near the singular locus Σ as before. First we can find a number $\varepsilon > 0$ such that $B_\varepsilon(\Sigma)$ is a disjoint union of $B_\varepsilon(C_i)$ for all component C_i of Σ, and each $B_\varepsilon(C_i)$ is a twisted product of S^1 and a disk with a cone. Without loss of generality we can assume that $N = X - B_{\varepsilon/10}(\Sigma)$.

Replace the metric on $B_\varepsilon(\Sigma) - B_{\varepsilon/10}(\Sigma)$ by

$$ds^2 = dt^2 + dr^2 + f^2(r)d\theta^2,$$

where $f''(r) \geqslant 0$, $f(r) = \alpha r/2\pi$ when $r > \varepsilon$, and $f(r) = \text{constant}$ when r near $\varepsilon/10$. Thus we have a Riemannian metric with nonpositive sectional curvature on N such that the boundary is totally geodesic.

If N is reducible, we can find a least area immersed sphere S. The sectional curvature is nonpositive and the determinant of the second fundamental form on S is also nonpositive since S is minimal. So the Gaussian curvature of this immersed sphere is nonpositive, because it is the sum of the sectional curvature of the tangent plane and the determinant of the second fundamental form. On the other hand, $\int_S K d\sigma = 4\pi > 0$ by Gauss-Bonnet theorem. This is a contradiction and it shows that N is irreducible.

If we start with an incompressible torus in N, we can also find an immersed least area torus T in the same homotopy class. In fact this T

is either embedded or a 2-fold cover onto an embedded Klein bottle K. By the same argument as above, we know that the Gaussian curvature of this torus is nonpositive, and the Gauss-Bonnet theorem implies that the Gaussian curvature of this torus will be identically equal to 0 and, therefore this T is totally geodesic, and all sectional curvatures on the tangent planes of T is zero. Now it is easy to see that T is either ∂N or contained in $X - B_\varepsilon(\Sigma)$. We will argue that T is ∂N. If T is contained in $X - B_\varepsilon(\Sigma)$, we get a totally geodesic embedded torus T or a Klein bottle K in E. Cutting E along T or K, we can get an E' which is a compact Euclidean cone manifold with one or two totally geodesic flat tori as boundary. Then we can extend E' to a noncompact Euclidean cone manifold with one or two ends look like a product of $\mathbb{E}^1_+$ and a torus. On the other hand, we assume that there is at least one cone angle less than π. This is a contradiction, since there is no noncompact Euclidean cone manifolds with ends look like a product of $\mathbb{E}^1_+$ and a torus, and a cone angle less than π.

To prove the second statement, we look at the holonomy

$$\rho : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{E}^3)$$

of E. There are three possibilities for a nontrivial element α in $\mathrm{Isom}^+(\mathbb{E}^3)$:

a) α has a unique invariant line;

b) α is a π rotation along an axis, and this axis is the unique line which is fixed by α; or

c) α is a pure translation in a direction n; in this case, α has infinitely many invariant lines, and they are all parallel to the direction n.

Suppose that N is Seifert fibered. Then $\pi_1(N)$ has a central infinite cyclic subgroup $\mathbb{Z}$ which is generated by an element a represented by a regular fiber. We can choose this regular fiber on ∂N, so $\rho(a)$ is nontrivial in $\mathrm{Isom}^+(\mathbb{E}^3)$. If $\rho(a)$ is type a) or b), there is a unique line l_a which is invariant or fixed under $\rho(a)$. a is a central element, so for any element b, l_a is also invariant or fixed under $\rho(b)$. This is impossible, since we cannot get a compact Euclidean cone manifold in such way.

Now we know that $\rho(a)$ must be a pure translation along the direction n. For any other element b, $\rho(b)$ commutes with $\rho(a)$. If $\rho(b)$ is type a) or b), the unique invariant line or fixed line will be parallel to the direction n. This shows that the foliation which arises by all planes perpendicular to n is invariant under $\rho(\pi_1(N))$, so it gives us a foliation on E. Each leaf of this foliation must be a compact 2-dimensional Euclidean cone manifold, otherwise we will have a noncompact 2-dimensional cone manifold with infinitely many cone points.

This is impossible, since the only noncompact 2-dimensional cone manifold with cone angles less than π is an infinite disk with a cone. This foliation gives us an S^2 bundle structure on X, so X must be $S^1 \times S^2$. q.e.d.

Finally, we show that there is no compact Euclidean cone manifold with all cone angles less than $2\pi/3$.

Theorem 3.2. *For any compact Euclidean cone manifold with cone angle at most π, there is at least one cone angle not less than $2\pi/3$.*

Proof. We have already known that at least one cone angle must be not less than $2\pi/3$, if the underlying space is $S^1 \times S^2$. If the theroem does not hold, then we can assume that there is a Euclidean cone manifold E with all cone angles less than $2\pi/3$ and $X_E = S^3$. This is because any Euclidean cone structure on a spherical manifold can be lifted to a structure on S^3.

Now we increase all cone angles to $2\pi/3$ to get a nonnegatively curved Riemannian orbifold O as we did at the beginning of the section. O can be 3-fold covered by a manifold N, since such an N can be constructed by using a Seifert surface of Σ_E. By Theorem 1.6, now we can conclude that O is a geometric orbifold with spherical geometry or geometry of $S^2 \times \mathbb{E}^1$. In fact, O cannot admit $S^2 \times \mathbb{E}^1$ geometry, otherwise, N will be $RP^3 \sharp RP^3$ or $S^2 \times S^1$. By the equivariant minimal sphere theorem, the quotient of $RP^3 \sharp RP^3$ or $S^2 \times S^1$ modulo a $\mathbb{Z}_3$ action will never yield an S^3. So, in this case, O is a spherical orbifold. The Riemannian universal cover of N is S^3 and, if we lift the identity on O, we get the fundamental group $\pi^{\mathrm{orb}}(O) = G$ of O which is a subgroup of $SO(4)$ and $O = S^3/G$.

To analyze this group G, we regard $SO(4)$ as $S^3 \times S^3/\mathbb{Z}_2$. There is a natural map $p : SO(4) \to SO(3) \times SO(3)$. Let $H = p(G)$ and let H_1, H_2 be the projections of H into the two factors of $SO(3) \times SO(3)$. Note that all nontrivial elements in G which have a fixed point in S^3 have order three, this allows us to establish the following lemma.

Lemma 3.3. *H_1 or H_2 is cyclic.*

Proof of the lemma. If the order of G is odd, both H_1 and H_2 have odd order. All finite subgroups of $SO(3)$ with odd order are cyclic, so the lemma follows.

We suppose that G has even order. Then G must have an element of order two. By our assumption that all cone angles in O is $2\pi/3$, any elements in G of order two are fixed point free and the only fixed point free involution in $SO(4)$ is $-I$ which lies in the kernel of p, so we have $G = p^{-1}(H)$. Therefore, any nontrivial elements which act on $SO(3)$ with a fixed point must have order three as well, because we can lift such an element to one in G which has a fixed point in S^3. In fact, we only need that any element of order two acts on $SO(3)$ freely. Another fact we need is that $(u_1, u_2) \in SO(3) \times SO(3)$ acts on $SO(3)$ freely if and only if u_1 is not conjugate to u_2 in $SO(3)$. In particular, H cannot contain an element (u_1, u_2) where both u_1 and u_2 have order two, since all elements of order two are fixed point free and therefore they are conjugate in $SO(3)$.

We can assume that H has even order, otherwise, both H_1 and H_2 have odd order and, therefore, they are cyclic. Let $H_i' = H \cap H_i$. Then H_i' is normal in H_i and $H_1/H_1' = H/(H_1 \times H_2) = H_2/H_2'$.

Since $H_1' \times H_2'$ does not contain an element (u_1, u_2) with both u_1 and u_2 having order two, one of H_1' or H_2' must have odd order. Say H_1' has odd order, so it must be a cyclic group. We also want to show that H_2' contains all elements of order two in H_2. Suppose that $u_2 \in H_2$ has order two. We can find an element $u_1 \in H_1$ such that $(u_1, u_2) \in H$. $(u_1, u_2)^2 = (u_1^2, I)$ implies that $u_1^2 \in H_1'$, so u_1^2 has odd order n. Hence either u_1^n is trivial or has order two. On the other hand, since $(u_1, u_2)^n = (u_1^n, u_2)$, u_1^n cannot have order two. So $u_1^n = I$ and $u_2 \in H_2'$.

Suppose that H_2 is not cyclic. Then H_2 must be a dihedral group, A_4, S_4 or A_5. If H_2 is not an A_4, then $H_2 = H_2'$, since those groups are generated by elements of order two. Thus $H_1 = H_1'$ is also cyclic. If H_2 is an A_4, either $H_2 = H_2'$ which implies $H_1 = H_1'$, or $H_2' = \mathbb{Z}_2 \times \mathbb{Z}_2$ and, then H_2/H_2' has order three and H_1 also has odd order. So we finish the proof. q.e.d.

Now we return to the proof of Theorem 3.2. We can suppose that H_1 is cyclic and then G can be conjugated in $SO(4)$ to a subgroup of $S^1 \times S^3/\mathbb{Z}_2$. The group $S^1 \times S^3/\mathbb{Z}_2$ preserves the Hopf fiberation on S^3 and the orientation on fibers, so does G. This shows that S^3/G is Seifert fibered and the singular locus is a union of fibers. It is in contradiction with Theorem 3.1, which says that $X_E - \Sigma_E$ is not Seifert fibered. So we cannot have a Euclidean cone manifold with all cone angles less than $2\pi/3$. q.e.d.

Finally we want to point out that Theorem 3.2 is sharp; $(S^3, \text{figure eight})$ does have a Euclidean cone structure with cone angles equal to $2\pi/3$.

4. The foliation on thin parts

In this section, we will show that the injectivity radius cannot be very small everywhere on a compact 3-dimensional hyperbolic cone manifold.

Theorem 4.1. *Suppose that Σ is a link in X and all ends of X are cusps. If (X, Σ) is a combinatorial type of a 3-dimensional hyperbolic cone manifold with finite volume, then $X - \Sigma$ supports a complete hyperbolic structure with finite volume.*

Remark. In fact, the condition that $X - \Sigma$ is hyperbolic is also sufficient for the pair (X, Σ) to be a combinatorial type of a 3-dimensional hyperbolic cone manifold. We will see that in §6.

Proof. To prove the theorem, it is enough to show that the compact core N of $X - \Sigma$ is an irreducible, atoroidal 3-manifold with incompressible boundary and it is not Seifert fibered.

Since any simple essential closed loop on the ∂N will be sent to a nontrivial element in $PSL_2(\mathbb{C})$ under holonomy, N must have an incompressible boundary. Now suppose that N is Seifert fibered. Then $\pi_1(N)$ has a central infinite cyclic group $\mathbb{Z}$ and the holonomy image of the generator of this central cyclic group is nontrivial, since the generator is represented by an essential loop on the ∂N. The fact that any element in $\pi_1(N)$ commutes with this generator implies that the holonomy image of $\pi_1(N)$ has a common fixed point in S^∞. This is impossible, since we cannot get a cone manifold C with finite volume in such a way.

To prove that N is irreducible and atoroidal, we use an argument similar to the one used in the proof of Theorem 3.1. First we deform the hyperbolic structure on C to a Riemannian metric with nonpositive sectional curvature on N, such that, away from a small regular neighborhood of the boundary, we have constant sectional curvature -1; and near the boundary, the metric is flat and the boundary is totally geodesic.

Near a singular component, we use Fermi coordinates

$$ds^2 = \cosh^2 r dt^2 + dr^2 + \sinh^2 r \alpha^2 d\theta^2 / 4\pi^2.$$

In general, if a metric in a cylindrical coordinates system is given by

$$ds^2 = g^2(r) dt^2 + dr^2 + f^2(r) d\theta^2,$$

then the curvature operator will be

$$\begin{pmatrix} -g''(r)/g(r) & & \\ & -g'(r)f'(r)/f(r)g(r) & \\ & & -f''(r)/f(r) \end{pmatrix}.$$

Now we can choose positive functions $f(r)$ and $g(r)$ nondecreasing and concave upward, such that, near $r = 0$, both of them are constant and, away from 0, $f(r)$ coincides with $\sinh^2 r\alpha^2/4\pi^2$, and $g(r)$ coincides with $\cosh^2 r$.

For a cusp end, we have the metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}, \quad z > z_0.$$

We will deform this metric to a new metric which has the following form:

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{f^2(z)}.$$

The curvature operator of this metric is given by

$$\begin{pmatrix} -f'^2(z) & & \\ & f''(z)f(z) - f'^2(z) & \\ & & f''(z)f(z) - f'^2(z) \end{pmatrix}.$$

Now the question of getting the desired metric becomes finding a positive function $f(z)$ such that $f(z) = z$ near $z = z_0$, $f(z)$ is constant for $z \gg z_0$, and $f''(z)f(z) - f'^2(z) \leqslant 0$. Note that $f''(z)f(z) - f'^2(z)$ is the numerator in $(f'(z)/f(z))' = (f''(z)f(z) - f'^2(z))/f^2(z)$. Choose a function $g(z)$ so that $g(z) = 1/z$ near $z = z_0$, $g(z) = 0$ for $z \gg z_0$ and $g(z)$ is nonincreasing. Solve $(f'(z)/f(z))' = g(z)$ we can get the desired $f(z)$.

Thus we have a metric on N as expected. The irreducibility of N is followed from the fact that there is no minimal immersed sphere in N. If we start with an incompressible torus in N, we will end up which an immersed least area torus T in the same homotopy class. This T is either embedded or a 2-fold cover to an embedded Klein bottle K. By the same argument as before, this can happen only if the torus lies in a regular neighborhood of ∂N, and then we can homotopy it into ∂N. We complete the proof of the theorem. q.e.d.

Theorem 4.2. *For any $\omega < 2\pi/3$, there is a universal constant $\delta > 0$, so that, for any 3-dimensional hyperbolic cone manifold C with all cone angles less than ω, we have $C_{thick,\,\delta} \neq \varnothing$.*

Proof. Suppose the contrary that we have a compact hyperbolic cone manifold C with all cone angles between α and ω, so that $C_{\text{thick},\delta} = \varnothing$, where the δ is the constant in Theorem 1.4 for $R = 120$ (later we will see why pick $R = 120$ here). By Theorem 1.4, for any point $x \in C$, $(B_{120\text{inj}(x)}(x, C), B_{120\text{inj}(x)}(x, C) \cap \Sigma_C)$ is homeomorphic to $(B_{120}(y, E), B_{120}(y, E) \cap \Sigma_E)$ for a Euclidean cone manifold E, $y \in E$ and $\text{inj}(y) = 1$. Recall the proof of Theorem 4.1, these Euclidean cone manifolds are limits of the rescaled hyperbolic cone manifolds. The hyperbolic cone manifolds considered here all have cone angles less than $\omega < 2\pi/3$, and Theorem 3.2 implies that there is no compact Euclidean cone manifolds with all cone angles less than $2\pi/3$, so we can assume further that these Euclidean cone manifolds which can occur in our case are all noncompact. Now, by the classification of noncompact Euclidean cone manifolds, we can conclude that they are a product of $\mathbb{E}^1$ with a torus, a twisted $\mathbb{E}^1$ bundle over a Klein bottle or a bundle over S^1 whose fiber is $\mathbb{E}^2$ or an infinite disk with a cone. The case that E is a twisted $\mathbb{E}^1$ bundle over a Klein bottle will not happen, otherwise, we could embed this Klein bottle in $X - \Sigma$. $X - \Sigma$ is atoroidal and the boundary of the regular neighborhood of this Klein bottle is incompressible inward, so this torus must be parallel to a singular circle in Σ. This contradicts the fact that $X - \Sigma$ is not Seifert fibered.

Now we take a closer look at Euclidean case first; the picture in this case is the same when we look at a point in C which has a small injectivity radius.

Case 1). E is a product of $\mathbb{E}^1$ with a torus. The isometry group acts on E transitively, so we only need to work at one point $y \in E$. Injectivity radius of y in E is the same as the injectivity radius of y in the cross section, $B_{\text{inj}(y)}(y, E)$ is ball with one, two or three pairs of points identified on the boundary. The image of $\pi_1(B_{\text{inj}(y)}(y, E))$ in $\pi_1(B_{3\text{inj}(y)}(y, E))$ is a free abelian group of rank one or two. This group is called the local fundamental group of y and denoted by $\pi_{1,y}$. If the rank of $\pi_{1,y}$ is one, $B_{\text{inj}(y)}(y, E)$ is a ball with only one pair of points identitied on the boundary, i.e., there is a unique shortest closed geodesic goes through y representing the generator of the local fundamental group. If the rank of $\pi_{1,y}$ is two, then $B_{3\text{inj}(y)}(y, E)$ contains the whole cross section which is an essential embedded torus in E.

Case 2). E is a boundle over S^1 whose fiber is $\mathbb{E}^2$. If the boundle is a product bundle, then the injectivity radius is a constant function on E, the local fundamental group $\pi_{1,y}$ has rank one and there is a unique shortest closed geodesic goes through y representing the generator of the local fundamental group. Otherwise E is a quotient of $\mathbb{E}^3 = \mathbb{E}^1 \times \mathbb{E}^2$ by an infinite cyclic group generated by g which is equal to a translation by d in $\mathbb{E}^1$ times a rotation with angle $\theta \neq 0$ on $\mathbb{E}^2$. In this case, the isometry group of E acts transitively on the equi-distant tori from the unique closed geodesic γ whose length is d. Take a point y on an equi-distant torus with distance $r \geqslant 60d$. We can find an element g^i $(1 \leqslant i \leqslant 20)$ such that the absolute value of its rotation angle is less than $\pi/10$; then we have that

$$\mathrm{inj}(y) \leqslant \frac{1}{2}\left(\frac{\pi r}{10} + 20d\right) \leqslant \frac{r}{3}.$$

This means that if $y \notin \cup_{y' \in \gamma} B_{60d}(y', E)$, then

$$B_{3\mathrm{inj}(y)}(y, E) \subset (\mathbb{E}^1 \times (\mathbb{E}^2 - \{0\}))/\langle g \rangle.$$

As same as in Case i), whether the local fundamental group $\pi_{1,y}$ of y has rank one or two depends on we have only one or more pairs of points identitied on the boundary of $B_{\mathrm{inj}(y)}(y, E)$. If the rank is one, there is a unique shortest broken closed geodesic going through y and representing the generator. If the rank is two, then $B_{3\mathrm{inj}(y)}(y, E)$ contains the whole equi-distant torus.

Case 3). E is a boundle over S^1 whose fiber is an infinite disk with a cone point.Take a point y on an equi-distant torus with distance r at least 60 times of the length of the singular locus. The same argument as in the Case 2) yields the same conculsion.

For hyperbolic cone manifold C, we can argue similarly. Let $N = \cup_{x \in \Sigma} B_{60\mathrm{inj}(x)}(x, C)$. This is a disjoint union of solid torus with a singular core, because that $B_{120\mathrm{inj}(x)}(x, C) \cap \Sigma$ has only one sigular core for any $x \in \Sigma$. Now let $D = C - N$. For any point $x \in D$, $B_{3\mathrm{inj}(x)}(x, C) \cap \Sigma = \varnothing$. Define the image of $\pi_1(B_{\mathrm{inj}(x)}(x, C)) \subset \pi_1(B_{3\mathrm{inj}(x)}(x, C))$ the local fundamental group $\pi_{1,x}$ of x, which is a free abelian group of rank one or two depends on there is one or more pairs of points identified on the boundary of $B_{\mathrm{inj}(x)}(x, C)$. If the rank is one, there is a unique shortest broken closed geodesic going through x and representing the generator of $\pi_{1,x}$. If the rank is two, the holonomy image of $\pi_{1,x}$ is abelian, implying that the holonomy image of $\pi_{1,x}$ is generated by one hyperbolic element or two loxodromic elements with a common axis because we have no parabolic elements if C is compact. In either case, $B_{3\mathrm{inj}(x)}(x, C)$ contains the whole equi-distant torus from the singular circle or the closed geodesic with length $\leqslant \delta$, and once a point on such an equi-distant torus has rank-two local fundament group, all local fundamental groups of other points on the same equi-distant torus have rank two as well.

Consider all of the tubes such of which is in form of

$$T_{\gamma,r} = \cup_{y\in\gamma} B_r(y, C)$$

and γ is a component in Σ or a closed geodesic with length at most δ with the property that any point on the equi-distant torus has rank two local fundamental group. All these γ have only finitely many choices. For a fixed γ_1, if there exists such a $T_{\gamma_1,r}$, we can define a maximal one. Let $\{T_i\}$ be all of these maximal tubes. Then we can see that they are pairs-wisely disjoint, otherwise $T_i \cap T_j \neq \varnothing$, and ∂T_i and ∂T_j are parallel. This shows that $X - \Sigma$ is either a solid torus or a product of $\mathbb{E}^1$ and a torus, contradict to Theorem 4.1.

Denote $D' = D - \cup_i T_i$ which is homeomorphic to C minus those tubes whose core are not a singular componenmt. Now for any point $x \in D'$, the local fundamental group $\pi_{1,x}$ is isomorphic to $\mathbb{Z}$. The generator α_x of $\pi_{1,x}$ is realized by the unique shortest broken geodesic through x. If we fix a point x_0, take any closed curve c with base point x_0 and move the unique shortest broken geodesic through x along c, then we will end up with the same broken geodesic. This shows that π_{1,x_0} is a normal subgroup of $\pi_1(D')$. We know that D' is irreducible and also that $\partial D' \neq \varnothing$, so D' is Haken. These two facts together imply that D' is Seifert fibered, and π_{1,x_0} is carried by the fiber.

If T_i is a maximal tube with core being a closed geodesic, then the 1-dimensional fiber on the boundary of T_i is not a meridian of T_i, since the fiber represents a nontrivial element in the local fundamental group and has nontrivial holonomy image. So we can extend the foliation on D' to a Seifert fiberation on $X - \Sigma$. Again this is in contradiction with Theorem 4.1, so we complete the proof of the theorem. q.e.d.

Remark. In the proof of the theorem, we first rule out the case that sequences of rescaled hyperbolic cone manifolds converge to a compact Euclidean cone manifold, and then argue that only the following three possible noncompact Euclidean manifolds can occur in the limit: a product of $\mathbb{E}^1$ with a torus, a bundle over S^1 whose fiber is $\mathbb{E}^2$, and an infinite disk with a cone, all by the condition $\omega < 2\pi/3$. If these things can be done, then the rest of the proof goes through. For example, we can also prove the following theorem.

Theorem 4.3. *Let $\omega < \pi$, (X, Σ) a combinatorial type of a hyperbolic cone structure, X is not a spherical manifold and there is no* *embedded sphere* $S^2 \subset X$ *which intersects* Σ *transversely, so that* $S^2 \cap \Sigma$ *is the set of three points. Then there is a constant* $\delta > 0$*, so that for any 3-dimensional hyperbolic cone structure* C *on* (X, Σ) *with all cone angles less than* ω*, we have* $C_{thick,\ \delta} \neq \varnothing$.

Proof. If one sequence of the rescaled hyperbolic cone structures on (X, Σ) converges to a compact Euclidean cone structure, then (X, Σ) is a combinatorial type of a compact Euclidean cone manifold. We know that X is either a spherial manifold or $X - \Sigma$ is Seifert fibered. Both are impossible. For the noncompact Euclidean cone manifold, which may occur in the limit, we have one more case to consider, that is a product of $\mathbb{E}^1$ with a double of an acute angled Euclidean triangle. If it happens, by Theorem 1.3, we can find an embedded sphere $S^2 \subset X$ which intersects Σ transversely, so that $S^2 \cap \Sigma$ is the set of three points. That is also ruled out by our hypothesis. The rest of the proof is the same as the proof of Theorem 4.2. q.e.d.

5. Compactness of hyperbolic cone structure

In this section, we will prove a compactness theorem for hyperbolic cone structures. It is the key step toward Theorem A.

For a fixed combinatorial type (X, Σ), let K be a triangulation of (X, Σ) such that the simplicial neighborhood of Σ in K is a regular neighborhood of Σ. Note that this is always true, if we replace K by its second barycentric derived subdivision. Then the volume of hyperbolic cone manifolds with the same combinatorial type (X, Σ) has a uniform bound.

Theorem 5.1 ([14, Proposition 4.1]). *For all 3-dimensional hyperbolic cone manifolds with the same combinatorial type* (X, Σ)*, the volumes have an upper bound* νn_3*, where* $\nu = -3\int_0^{\pi/3} \log|\sin 2u|du$ *is the maximal volums of a hyperbolic simplex (see* [11]*) and* n_3 *is the number of 3-simplexes in* K.

Corollary 5.2. *Let* C *be a compact 3-dimensional hyperbolic cone manifold, and* $\delta > 0$ *be any positive number. Then the diameter of any component of* $C_{thick,\delta}$ *has a uniform upper bound which only depends on the combinatorial type of* C*,* δ *and the lower bound of cone angles* ω.

Proof. Let V be the upper bound of the volume, ω the lower bound of the cone angles and C_1 a component of $C_{\text{thick},\ \delta}$. For each point $x \in C_1 \cap \Sigma$, $B_{\delta/4}(x)$ is standard. We put disjoint standard $\delta/4$ balls centered in $C_1 \cap \Sigma$ as many as we can and denote the centers of the balls by $\{x_i\}$. Note that $N_{\delta/2}(\Sigma) = \{x \in C_1 | d(x, C_1 \cap \Sigma) \leqslant \delta/2\} \subset \cup_i B_\delta(x_i)$. So $\{x_i\}$ is a δ-net of $N_{\delta/2}(\Sigma)$, and for those points $x \in C_1 - N_{\delta/2}$, balls centered at these points with radius $\frac{\delta}{2} \sin \frac{\omega}{2}$ are standard. Use these balls to pack $C_1 - N_{\delta/2}(\Sigma)$, and the centers of these balls together with x_i's form a δ-net of C_1. The total number of points in this δ-net is bounded, so the diameter of C_1 is bounded. q.e.d.

Another theorem associated with the volume is the following:

Theorem 5.3. *The volume of hyperbolic cone manifolds is lower semicontinuous, i.e., if (C_n, x_n) is a convergent sequence and the limit (C, x) is still a cone manifold, then*

$$\mathrm{vol}(C, x) \leqslant \liminf\ \mathrm{vol}(C_n, x_n).$$

Proof. We can suppose that $\lim \mathrm{vol}(C_n, x_n)$ exists, say it is equal to v. We only need to show that $\mathrm{vol}(B_R(x, C)) \leqslant v$ for all R. $B_R(x, C) = \lim B_R(x_n, C_n)$, and the $B_R(x, C)$ and $B_R(x_n, C_n)$ are compact. So $\mathrm{vol}(B_R(x)) = \lim\ \mathrm{vol}(B_R(x_n))$, which is less than v. q.e.d.

Corollary 5.4. *Let (C_n, x_n) be a convergent sequence of compact hyperbolic cone manifolds with fixed combinatorial type. If the limit is still a hyperbolic cone manifold, then $\lim(C_n, x_n)$ has finite volume.*

Proof. Applying Theorem 5.1, we get that $\liminf \mathrm{vol}(C_n, x_n) < \infty$. Then the conclusion follows from Theorem 5.3. q.e.d.

The main theorem in this section is the following:

Theorem 5.5. *For any $0 < \varepsilon < \omega < 2\pi/3$, the space of all 3-dimensional hyperbolic cone structures with the same combinatorial type (X, Σ) and the set of all cone angles between ε and ω is compact under Hausdorff distance.*

Proof. Let C_n be a sequence of compact 3-dimensional hyperbolic cone manifolds with the same combinatorial type (X, Σ) and all cone angles between ε and ω. By Theorem 4.2, there is a $\delta > 0$, so that $C_{n,\mathrm{thick},\delta} \neq \varnothing$. Pick $x_n \in C_{n,\mathrm{thick},\delta}$ and consider the sequence (C_n, x_n), by Theorem 1.2, it has a convergent subsequence and the limit (C, x) of this subsequence is still a hyperbolic cone manifold. If C is compact, it will have the same combinatorial type as C_n by Theorem 1.3. This is the conclusion of our theorem.

Suppose that C is noncompact, and we will show that it leads to a contradiction. Corollary 5.4 says that C has finite volume. Without loss of generality, we also can assume that (C_n, x_n) is convergent itself. The number of circle components in Σ_C is less than the number of components in Σ. We can find a number $M_0 > 0$, so that all the circle components in Σ_C are contained in $B_{M_0}(x, C)$. The rest of the proof will be divided into two cases.

Case 1). Suppose that $[C - B_m(x, C)] \cap \Sigma_C \neq \varnothing$ for all $m > M_0$. Let $y_m \in [C - B_m(x, C)] \cap \Sigma_C$. Then $\lim_{m\to\infty}\mathrm{inj}(y_m) = 0$, as C has finite volume. For m large enough and applying Theorem 1.4, we know that C looks like a noncompact Euclidean cone manifold E near y_m. Since all cone angles are less than ω, E must be a bundle over S^1 whose fiber is an infinite disk with a cone. In the Euclidean case, the singular component is a circle, so the singular component which contains y_m is a circle too and it contradicts the hypothesis that all circle components of Σ_C are contained in $B_{M_0}(x, C)$.

Case 2). $[C - B_m(x, C)] \cap \Sigma_C = \varnothing$ for some $m > M_0$. This really means that all singular component are circles. Then, far away from x, C looks like a noncompact Euclidean manifolds. By the same reason as in the proof of Theorem 4.2, E cannot be a twisted line bundle over a Klein bottle. So we can do thin-thick decomposition for C, and all ends of C are cusps.

Let (N, Σ_C) be a compact core of (X_C, Σ_C). By Theorem 1.3, we can embed N back into C_n for n large enough. Let g_n be the embeddings and N_n the image $g_n(N)$. ∂N_n is a bunch of tori, and each torus either bounds a solid torus in $X - \Sigma$ or is parallel to a singular circle. For each boundary torus T of N, we have a sequence $\alpha_n^T \in \mathbf{H}_1(T)$, where α_n^T is represented by the preimage of the essential circle β_n in $g_n(T)$ which is null homotopic or a meridian. α_n^T must tend to infinity in $\mathbf{H}_1(T)$, otherwise, α_n^T has a subsequence; we still call it α_n^T, which converges to α. The holonomy image of β_n will converge to the holonomy image of α, which is a parabolic element. The holonomy of β_n is either a trivial element or an elliptic element with rotation angle between ε and ω. Such a sequence cannot converge to a parabolic element.

Now we can apply Thurston's hyperbolic Dehn surgery theorem to this situation. For n large enough, the Dehn filling along all these α_n^T's will give us a hyperbolic manifold $\hat{N}_n$. Note that all these results of Dehn filling are the same, they are all homeomorphic to X. On the other hand, the volume of a hyperbolic manifold is an invariant and $\mathrm{vol}(N) > \mathrm{vol}(X) = \mathrm{vol}(\hat{N}_n) \to \mathrm{vol}(N)$ (see Theorem 6.5.6, [15] and

Theorem 1, [12]). This is impossible.

So C must be compact, and we finish the proof of the theorem.

q.e.d.

The same as the Remark after Theroem 4.2 and the proof of Theorem 4.3, we can prove the following theorem.

Theorem 5.6. *Let $0 < \varepsilon < \omega < \pi$, (X, Σ) a combinatorial type of a hyperbolic cone structure, X is not a spherical manifold and there is no embedded sphere $S^2 \subset X$ which intersects Σ transversely, so that $S^2 \cap \Sigma$ is the set of three points. Then the space of all 3-dimensional hyperbolic cone structures on (X, Σ) with all cone angles between ε and ω is compact under Hausdorff distance.*

6. Deformation theory of hyperbolic cone structures

We will discuss the deformation of hyperbolic cone structures in this section. Most of the theorems in this section are known to Thurston.

Theorem 6.1. *Let C be a hyperbolic cone manifold. The holonomy represetation $\rho : \pi_1(X_C - \Sigma_C) \to PSL_2(\mathbb{C})$ can be lifted to a representation in $SL_2(\mathbb{C})$.*

Proof. Our proof goes the same fashion as the proof of [2, Proposition 3.1.1]. Let $\rho : \pi_1(X_C - \Sigma_C) \to PSL_2(\mathbb{C})$ be the representation. Then $\pi_1(X_C - \Sigma_C)$ is isomorphic to the subgroup $\Gamma = \{(g, \rho(g))\}$ in $\pi_1(X_C - \Sigma_C) \times PSL_2(\mathbb{C})$. Γ acts on $(\widetilde{X_C - \Sigma_C}) \times PSL_2(\mathbb{C})$ freely and proper discontinuousely, and the quotient Q is a principle $PSL_2(\mathbb{C})$ boundle over $X_C - \Sigma_C$. $PSL_2(\mathbb{C})$ can be identified with the principal bundle of orthonormal tangent frames to $\mathbb{H}^3$. Use the fact that $X_C - \Sigma_C$ has trivial tangent bundle, it is not difficult to see that Q is homeomorphic to $(X_C - \Sigma_C) \times \mathbb{H}^3 \times SO(3)$.

On the other hand, let $\tilde{\Gamma}$ be the preimage of Γ under the homomorphism $\pi_1(X_C - \Sigma_C) \times SL_2(\mathbb{C}) \to \pi_1(X_C - \Sigma_C) \times PSL_2(\mathbb{C})$. Then $Q = (\widetilde{X_C - \Sigma_C}) \times SL_2(\mathbb{C})/\tilde{\Gamma}$. Since $SL_2(\mathbb{C})$ is simply connected, $\tilde{\Gamma} = \Gamma \times \mathbb{Z}_2$. This shows that ρ can be lifted to a representation in $SL_2(\mathbb{C})$. q.e.d.

Later on, the lift $\rho : \pi_1(X_C - \Sigma_C) \to SL_2(\mathbb{C})$ in Theorem 6.1 will be also called holonomy representation.

Let (X, Σ) be the combinatorial type of a compact hyperbolic cone structure, let Σ have m components $S_1, \dots, S_m$, and let μ_i be the meridian of S_i.

Theorem 6.2. *Let C_0 be a hyperbolic cone manifold structure on (X, Σ) or a complete hyperbolic structure on $X - \Sigma$ and ρ_0 its holonomy representation. If ρ is sufficiently close to ρ_0 such that all $\rho(\mu_i)$ are elliptic, then there is a hyperbolic cone manifold structure C_1 on (X, Σ) whose holonomy representation is ρ. Furthermore, if C_0 is a hyperbolic cone structure, then C_1 can be chosen close to C_0 and such nearby cone structure is determined uniquely by the conjugacy class of ρ.*

Proof. Let $\mathcal{N}(\Sigma)$ be a regular neighborhood of Σ. Then $U = X - \mathcal{N}(\Sigma)$ is compact. By [15, Proposition 5.1], there is a hyperbolic structure C_1' on U which is close to $C_0|_U$ with holonomy ρ. Since each $\rho(\mu_i)$ are elliptic, we have a unique way to extend C_1' to a hyperbolic cone structure C_1 on (X, Σ). This C_1 is determined uniquely by the conjugacy class of ρ. q.e.d.

The representation space plays an essential role in the deformation theory for hyperbolic cone structures. We denote by $R(X, \Sigma)$ the representation space $R(X, \Sigma) = \{\rho : \pi_1(X - \Sigma) \to SL_2(\mathbb{C})\}$, which is a complex affine algebraic set. When we consider different hyperbolic cone structures on a fixed combinatorial type (X, Σ), it is convenient to define marked hyperbolic cone structure. A marked hyperbolic cone structure on (X, Σ) is an equivalent class of hyperbolic cone structures on (X, Σ) with equivalent relation that there is an isometry which is homotopic relΣ to the identity map. It is similar to the marked Riemann surface in Teichmüller theory. Theorem 6.2 says that near a marked hyperbolic cone structure, the space of marked hyperbolic cone structures is parametrized by conjugacy classes of their holonomy representations. A deformation of conjugacy class of the holonomy representation of a cone structure in $R(X, \Sigma)/SL_2(\mathbb{C})$ is called an algebraic deformation of the cone structure. Locally a geometric deformation is equivalent to an algebraic deformation, and increasing cone angles is the same as increasing the angle of elliptic elements $\rho_0(\mu_i)$.

The basic fact about the algebraic deformation is the following theorem.

Theorem 6.3. *Let R_0 be an irreducible component of $R(X, \Sigma)$ which contains the holonomy representation ρ_0 of a compact hyperbolic cone structure with combinatrial type (X, Σ) or the complete hyperbolic structure on $X - \Sigma$. Then*

$$\dim_{\mathbb{C}} R_0 \geqslant m + 3,$$

where m is the number of circle components of Σ.

Proof. Let N be a compact core of $X - C$ since ρ_0 is irreducible, for each torus component T of ∂N, $\rho_0(\pi_1(T)) \not\subset \{\pm 1\}$. [2, Proposition 3.2.1] says that $\dim_{\mathbb{C}} R_0 \geqslant m + 3$. q.e.d.

For a compact marked hyperbolic cone structure, the holonomy representation is always irreducible and unique up to conjugacy. The space of characters will be more interesting, because the conjugacy classes of irreducible representations are in one-to-one corresponding to their characters. Let R_0 be an irreducible component of $R(X, \Sigma)$ which contains the holonomy representation ρ_0 of a compact hyperbolic cone structure with combinatrial type (X, Σ) or the complete hyperbolic structure on $X - \Sigma$. Let $\mathcal{C}_0$ be the space of all characters of representations in R_0.

Proposition 6.4. *$\mathcal{C}_0$ is an affine variety and*

$$\dim_{\mathbb{C}} \mathcal{C}_0 = \dim_{\mathbb{C}} R_0 - 3 \geqslant m.$$

See [2, Section 1, Proposition 3.2.1] for a detailed proof.

Let $f : \mathcal{C}_0 \to \mathbb{C}^m$ be the trace map defined by

$$f([\rho]) = (\operatorname{tr}(\rho(\mu_1)), \dots , \operatorname{tr}(\rho(\mu_m))).$$

f is a regular map and $f([\rho_0]) = (\epsilon_1 2 \cos \theta_1, \dots , \epsilon_m 2 \cos \theta_m)$, where $\epsilon_i = \pm 1$, since each $\rho_0(\mu_i)$ is elliptic with a rotation angle θ_i.

Suppose that ρ_0 is a holonomy representation of the complete hyperbolic structure on $X - \Sigma$, Mostow's rigidity theorem says that $f^{-1}(f([\rho_0]))$ is a single point in $\mathcal{C}_0$, which implies that $\dim_{\mathbb{C}} \mathcal{C}_0 = m$ and the trace map f is onto near $[\rho_0]$. For the case that ρ_0 is a holonomy representation of a compact hyperbolic cone structure with all cone angles less than $2\pi/3$, the same conclusion holds.

Proposition 6.5. *Suppose $\theta_i < 2\pi/3$ for all $1 \leqslant i \leqslant m$. Then the irreducible component of $f^{-1}(f([\rho_0]))$ in $\mathcal{C}_0$ which contains $[\rho_0]$ is the single point $[\rho_0]$.*

Proof. Let L be an irreducible component of $f^{-1}(f([\rho_0]))$ which contains $[\rho_0]$ and S be the subset of L which consists of all characters which can be realized by a holonomy representation of hyperbolic cone structures. Theorem 6.2 says that S is open in L. On the other hand, the convergence under the Hausdorff distance implies the convergence of the holonomy representations, so Theorem 5.5 says that S is also compact in L. In particular, $S = L$ is a compact irreducible affine variety, and thus it must be the single point $[\rho_0]$. q.e.d.

The proposition implies that $\dim_{\mathbb{C}} \mathcal{C}_0 = m$ and f is onto near $[\rho_0]$. Now we restrict our attention to the case that all $\theta_i < 2\pi/3$. For any given direction $\vec{a} = (a_1, \dots, a_m)$, we define a ray $l(t) = f([\rho_0]) + t\vec{a}$ in $\mathbb{C}^m$.

Theorem 6.6. *There is a continuous lift of $l(t)$*

$$\tau : [0, T) \to \mathcal{C}_0, 0 < T < \infty,$$

such that $f\tau = l$ and $\tau(0) = [\rho_0]$.

Proof. Let $L \subset \mathbb{C}^m$ be the complex line which contains $l(t)$. Propositions 6.4 and 6.5 then imply that the irreducible component E of $f^{-1}(L)$ which contains $[\rho_0]$ must be an affine curve in $\mathcal{C}_0$ and $f|_E$ is a nonconstant map. Any such $f|_E : E \to L$ is open, so we can lift $l(t)$ to $\tau : [0, T) \to \mathcal{C}_0$. q.e.d.

Remark. Proposition 6.5 and Theorem 6.6 also hold, if we release the restriction $\theta_i < 2\pi/3$ to $\theta_i < \pi$ and put additional condition on (X, Σ) as Theorems 4.3 and 5.6.

Theorem 6.6 tell us that local algebraic deformations of cone structure always exist, so do local geometric deformations by Theorem 6.2. The complete hyperbolic structure can be regarded as a cone structure on (X, Σ) with "zero cone angle". This theorem is also true when we replace the ρ_0 by the representation of the complete hyperbolic structure on $X - \Sigma$. Instead of using Proposition 6.5, we can use Mostow's rigidity theorem in the proof. Everything works fine. However, if we still want to get a geometric deformation of hyperbolic cone structure, we need to assume that $a_i\epsilon_i \leqslant 0$ for all i. This will guarantee that all holonomy images of μ_i are elliptic, therefore we can get the inverse of Theorem 4.1. We leave the details of the proof to the reader.

Theorem 6.7. *Suppose that Σ is a hyperbolic link in a closed mainfold X. There is an $\varepsilon > 0$, so that for any $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$ and $0 < \theta_i < \varepsilon$, we have a hyperbolic cone structure C_θ on (X, Σ) such that the cone angle of S_i is θ_i.*

To finish the section, we prove the following lemma, which says that, just like the complete hyperbolic structures, the orbifold structures are also rigid.

Lemma 6.8. *Let O_1 and O_2 be two hyperbolic orbifold structures on an orbifold O. Then they are isomorphic. In fact the isometry can be chosen to be homotopic to the identity map rel Σ.*

Proof. Since any hyperbolic orbifold is very good, so there is a finite regular branch cover $\tilde{O}$ of O which is a manifold. Denote by G the covering transformation group. Pulling back the structures O_1 and O_2 to $\tilde{O}$, we get two complete hyperbolic structures $\tilde{O}_1$ and $\tilde{O}_2$ on $\tilde{O}$ such that elements of G are isometries of both structures $\tilde{O}_i$. Mostow's rigidity theorem says that we can homotopy the identity on $\tilde{O}$ to an isometry $\tilde{h} : \tilde{O}_1 \to \tilde{O}_2$. For any element $g \in G$, g and $\tilde{h}^{-1} g \tilde{h}$ are two homotopic isometries on the structure $\tilde{O}_1$. This implies that $g = \tilde{h}^{-1} g \tilde{h}$, i.e., $\tilde{h}$ is G-invariant. So $\tilde{h}$ induces an isometry $h : O_1 \to O_2$. q.e.d.

7. Moduli space of hyperbolic cone structures

In this section, we will prove Theorem A and Corollary B mentioned in the begining of the paper.

Denote by ρ_* the representation of the complete hyperbolic structure on $X - \Sigma$, and by R_* the irreducible component of $R(X, \Sigma)$ which contains ρ_* and $\mathcal{C}_*$, the space of characters of representations in R_*. Let C be another hyperbolic cone structure on (X, Σ) with cone angle less than $2\pi/3$, it has a holonomy $\rho \in R(X, \Sigma)$. Let R_ρ be the irreducible component of ρ in $R(X, \Sigma)$, and $\mathcal{C}_\rho$ the corresponding character space. Then we have

Lemma 7.1. *$R_\rho = R_*$ and $\mathcal{C}_\rho = \mathcal{C}_*$.*

Proof. It suffices to prove that $R_\rho = R_*$. By Theorem 6.7, we can choose a hyperbolic orbifold structure O on (X, Σ) with all cone angles equal to $2\pi/n$ and its holonomy representation lying in R_*. Let

$$\begin{aligned} l(t) =& (1-t)(\epsilon_1 2\cos\theta_1, \dots, \epsilon_m 2\cos\theta_m) \\ &+ t(\epsilon_1 2\cos 2\pi/n, \dots, \epsilon_m 2\cos 2\pi/n), \end{aligned}$$

be a line in $\mathbb{C}^m$ and $I \subset [0,1]$ be the largest interval on which there is a lift $\tau : I \to \mathcal{C}_\rho$ of $l(t)$ such that $\tau(0) = [\rho]$ and each $\tau(t)$ is the charactor of a holonomy representation of a hyperbolic cone structure on (X, Σ). Theorems 6.2 and 6.6 imply that I is open in $[0,1]$, and Theorem 5.5 says that I is also compact. Therefore, it is whole interval $[0,1]$.

$\tau(1)$ is the charactor of a holonomy representation ρ' of a hyperbolic cone structure on (X, Σ). $l(1) = (\epsilon_1 2\cos 2\pi/n, \dots, \epsilon_m 2\cos 2\pi/n)$ implies that all cone angles of this structure are $2\pi/n$, so it is a hyperbolic orbifold. Denote this structure by O' and note that the holonomy representation $\rho' \in R_\rho$. Applying Lemma 6.8, we know that O and O' are isometric by an isometry homotopic to the identity map relΣ; this means that they have the same holonomy representation, i.e., $\rho' \in R_\rho \cap R_*$. The irreducible component R_0 which contains ρ' is contained in $R_\rho \cap R_*$, and all these three affine vatieties R_0, R_ρ and R_* have dimension m, so we have $R_0 = R_\rho = R_*$. q.e.d.

Now we are in a position to prove that the trace map $f : \mathcal{C}_* \to \mathbb{C}^m$ is a local homeomorphism near any character of a holonomy representation of a hyperbolic cone structure.

Theorem 7.2. *Let $\mathcal{U}$ be the subset of $\mathcal{C}_*$ consisting of all characters of holonomy representations of hyperbolic cone structures on (X, Σ) with all cone angles less than $2\pi/3$. Then the trace map $f : \mathcal{U} \to \mathbb{C}^m$ is a homeomorphism onto its image.*

Proof. Let $\mathcal{W} = \{(\epsilon_1 2\cos\theta_1, \dots, \epsilon_m 2\cos\theta_m) | 0 < \theta_i < 2\pi/3\} \subset \mathbb{C}^m$. By the proof of Lemma 7.1, we see that the trace map $f(\mathcal{U}) = \mathcal{W}$ is onto.

To show that $f : \mathcal{U} \to \mathcal{W}$ is a homeomorphism. Let $B_0 \subset \mathcal{W}$ be the set

$$\{p | f^{-1}(p) \cap \mathcal{U} \text{ contains at least two different characters}\}$$

and $B = \overline{B}_0$, the closure of B_0 in $\mathcal{W}$. We want to show that B is open in $\mathcal{W}$. If it is true, B is either empty or the whole $\mathcal{W}$. If $B = \mathcal{W}$, all points of the form $\{(\epsilon_1 2\cos 2\pi/n_1, \dots, \epsilon_m 2\cos 2\pi/n_m)\} \subset B - B_0$, since they are corresponding to orbifold structures. So any character of holonomy representation of a orbifold structure is either a singular point of $\mathcal{C}_0$ or a singular point of $f : \mathcal{C}_0 \to \mathbb{C}^m$. This contradicts to the facts that the image of the singular set of $\mathcal{C}_0$ and the singular set of f lies in a lower dimensional algebraic set, and $\{(\epsilon_1 2\cos 2\pi/n_1, \dots, \epsilon_m 2\cos 2\pi/n_m)\}$ is a Zariski dense set in $\mathbb{C}^m$. So B is empty, i.e., f is injective. For any compact set $K \subset \mathcal{W}$, the space of hyperbolic cone structures whose character of the holonomy representation is contained in $f^{-1}(K)$ is compact by Theorem 5.5, so $f^{-1}(K)$ is compact and this shows that $f^{-1} : \mathcal{W} \to \mathcal{U}$, the inverse of f, is also continuous.

Now we prove that B is open. For any $p \in B_0$, $f^{-1}(p) \cap \mathcal{U}$ contains two distinct characters χ^1 and χ^2 both corresponding to hyperbolic cone structures. Theorem 6.2 says that there are two small neighborhood U^i of χ^i such that all points in $U^i \cap f^{-1}(\mathcal{W}) \subset \mathcal{U}$. Without loss of generality, we can assume that $U^1 \cap U^2 = \varnothing$. Also we know that f is onto near a cone structure by Theorem 6.6. Thus $f(U^1) \cap f(U^2) \cap \mathbb{R}^m \subset B_0$ is a neighborhood of p. If $p \in B - B_0$, there is a sequence $p_n \in B_0$ such that $p_n \to p$. For each $p_n \in B_0$, we have two distinct characters χ_n^1 and χ_n^2 corresponding to hyperbolic cone structures C_n^1 and C_n^2, and $f(\chi_n^i) = p_n$. Without loss of generality, we can assume that both C_n^1 and C_n^2 converge. So the two sequences χ_n^1 and χ_n^2 converge. Let $\chi_n^i \to \chi^i$, $f(\chi^i) = p$ and $p \notin B_0$ imply that $\chi^1 = \chi^2 = \chi$ showing that χ is a singular point of $\mathcal{C}_0$ or a singular point of f and the local degree of $f > 1$. All points of an open neighborhood of p except those points in a lower dimensional algebraic set are contained in B_0. So this open set is a neighborhood of p in B, showing that B is open. q.e.d.

The moduli space of marked hyperbolic cone structures with combinatorial type (X, Σ) and cone angles less than $2\pi/3$ will be denoted by $\mathcal{HC}(X, \Sigma)$.

Let $h : \mathcal{HC}(X, \Sigma) \to \mathcal{U}$ be defined by sending each marked hyperbolic cone structure to the character of its holonomy representation. By the definition, h is onto, Theorem 6.2 tell us that h is a local homeomorphism and Theorem 5.5 implies that h has the path lifting property. So h is a covering map and then we get the following corollary, as $\mathcal{U}$ is simply connected.

Corollary 7.3. *$fh : \mathcal{HC}(X, \Sigma) \to \mathcal{W}$ is a homeomorphism.*

Remark. The map

$$\begin{array}{rcl} \sigma : (0, 2\pi/3)^m & \to & \mathcal{W}, \\ (\theta_1, \dots, \theta_m) & \longmapsto & (\epsilon_1 2\cos\theta_1, \dots, \epsilon_m 2\cos\theta_m) \end{array}$$

is a homeomorphism, so $\iota = h^{-1}f^{-1}\sigma : (0, 2\pi/3)^m \to \mathcal{HC}(X, \Sigma)$ is a natural parametrization of $\mathcal{HC}(X, \Sigma)$ by m cone angles. In other words, Corollary 7.3 is another version of Theorem A.

As a simple application of Theorem A, we have the following proposition.

Proposition 7.4. *Let Σ be a hyperbolic link in a homological sphere M^3. Then k-fold cyclic cover of M^3 branched over Σ is a hyperbolic manifold provided $k \geqslant 4$.*

Proof. Let N be the k-fold cyclic cover of M branched over Σ. Then M can be viewed as an orbifold with the singular locus Σ and all cone angles equal to $2\pi/k$. Thus N is a manifold cover of M.

By Theorem 10.4, the orbifold M is hyperbolic, so N is a hyperbolic manifold. q.e.d.

As we said before, if we release the restriction $\theta_i < 2\pi/3$ to $\theta_i < \pi$ and put additional condition on (X, Σ) as Theorem 4.3 or 5.6, the arguements in the proofs of Theorem 7.2 and Corollary 7.3 work fine. So we have

Theorem 7.5. *Let Σ be a hyperbolic link with m components in a 3-dimensional manifold X. Suppose X is not a spherical manifold and there is no embedded sphere $S^2 \subset X$ which intersects Σ transversely, so that $S^2 \cap \Sigma$ is the set of three points. Then the moduli space of marked hyperbolic cone structures on the pair (X, Σ) with all cone angles less than π is an m-dimensional open cube, parameterized naturally by the m cone angles.*

In the rest of section, we will prove Corollary B which is a more general than Proposition 7.4.

Corollary B. *Suppose M is an irreducible, closed, atoroidal 3-manifolds, and is not a Seifert manifold, and G is a group acting effectively on M. If the order of G is odd and the action is not fixed-point-free, then the quotient M/G is a geometric orbifold.*

Remark. Pulling back the singular geometric structure on M/G, we will get a geometric structure on M. The theorem implies that not only M has a geometric structure, but also G consists of isometries.

This is a special case of Thurston's geometrization theroem; we refer [9, Problem 3.46] for the historical remark on the theorem.

A lot of concepts in 3-orbifold theory, such as, irreducibility and incompressibility, are very similar to the corresponding concepts for manifolds. Following Hodgson in [6], we use *sphere* or *torus* to denote a spherical or a Euclidean 2-orbifold, and *disk* or *ball* for a quotient of a disk or a ball under a finite, orientation preserving linear action. A 3-orbifold O is irreducible if every *sphere* bounds a *ball.* A 2-suborbifold F is incompressible if any 1-suborbifold of F which bounds a *disk* in $O - F$ also bounds a *disk* in F. An orbifold is atoroidal if each incompressible *torus* is ∂-parallel. For our convenience, we always assume that all 3-orbifolds contain no bad 2-orbifolds.

M/G is an orbifold and we denote the orbifold structure by $O_{M/G}$. Let X_O be the underlying space of $O_{M/G}$ and Σ_O the singular locus. It is easy to see that Σ_O is a link in X_O since any finite group of $SO(3)$ with odd order is a finite cyclic group. A simple geometric argument shows that $O_{M/G}$ is closed, irreducible and atoroidal under our assumption that M is closed, irreducible and atoroidal. Denote $X_O - \Sigma_O$ by N_O. Thurston's program starts with the following proposition.

Proposition 7.6. *Either $O_{M/G}$ is Seifert fibered or N_O admits a complete hyperbolic structure with finite volume.*

For a proof of the proposition, we refer readers to [14, §2].

It is easy to handle the case that $O_{M/G}$ is Seifert fibered. Applying the method in [13] (also see [8]), we will see that all of these orbifolds have Seifert geometries. So we assume that $O_{M/G}$ is not a Seifert orbifold and this implies that $\Sigma_O \subset X_O$ is a hyperbolic link.

If all cone angles in $O_{M/G}$ are less than $2\pi/3$, Theorem A shows that we can find a required hyperbolic orbifold structure on $O_{M/G}$. So we now can assume that there exists at least one cone angle which is $2\pi/3$. In this case, we can find a sequence of hyperbolic cone structures C_n on (X_O, Σ_O) such that all cone angles tend to desired cone angles. Now we will investigate the limit of C_n. For this purpose, we need following refinements of Theorems 4.2 and 5.5.

Proposition 7.7. *For the sequence of hyperbolic cone structures C_n, we have the following alternative:*

i) there is a sequence $x_i \in C_n$ so that the rescaled sequence $((inj(x_n))^{-1}C_n, x_n)$ has a subsequence which converges to a compact Euclidean cone manifold, or

ii) there is a constant $\delta > 0$, so that $C_{n,thick,\delta} \neq \varnothing$.

Proposition 7.8. *If there is a $\delta > 0$, such that $C_{n,thick,\delta} \neq \varnothing$, then C_n has a subsequence C_{n_k} which converges to a hyperbolic cone structure on (X_O, Σ_O).*

Proof of Propositions 7.7 and 7.8. By Proposition 7.7 i), we can assume that there is no rescaled sequence which has a subsequence converging to a compact Euclidean cone manifold.

For the noncompact Euclidean cone manifold, which may occur in the limit, just as in the proof of Theorem 4.3, we have one more case to consider, that is a product of $\mathbb{E}^1$ with a double of an equilateral triangle. If it is the case, we can embed this *torus* back into $O_{M/G}$. It is easy to see that this *torus* is incompressible and nonseparating in $O_{M/G}$, contradict to the fact that $O_{M/G}$ is atoroidal. The rest of the proofs are the same as the proofs of Theorems 4.2 and 5.5. q.e.d.

Corollary B is followed immediately from these two propositions. If the alternative i) of Proposition 7.7 holds, we get a Euclidean orbifold structure on $O_{M/G}$, otherwise, we use Proposition 7.8 to produce a hyperbolic orbifold structure on $O_{M/G}$. In either way, M/G is a geometric orbifold. Hence we complete the proof of Corollary B.

References

[1] A. Beardon, *The geometry of discrete groups*, Springer, Berlin, 1983.

[2] M. Culler & P. Shalen, *Varieties of group representations and splitting of 3-manifolds*, Ann. of Math. **117** (1983) 109–146.

[3] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981) 53–78.

[4] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982) 255–306.

[5] ———, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986) 153–179.

[6] C. Hodgson, *Geometric structures on 3-dimensional orbifolds: Notes on Thurs-ton's proof*, Preprint.

[7] T. Janssen, *Crystallographic groups*, North-Holland Elsevier, Amsterdam, 1973.

[8] M. Jankins & W. Neumann, *Lectures on Seifert manifolds*, Brandeis lecture notes 2, March 1983.

[9] R. Kirby, *Problems in low-dimensional topology*, 35–474, Geometric Topology, Vol. II, (ed. W. Kazez), AMS/IP, 1997.

[10] S. Kojima, *Deformations of hyperbolic 3-cone-manifolds*, Preprint, 1997.

[11] J. Milnor, *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc. **6** (1982) 9–24.

[12] W. D. Neumann & D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology **24** (1985) 307–332.

[13] P. Scott, *The geometry of 3-manifolds*, Bull. London Math. Soc. **15** (1983) 401–487.

[14] T. Soma, K. Ohshika & S. Kojima, *Towards a proof of Thurston's geometrization theorem for orbifolds*, Res. Inst. Math. Sci., Kokyuroku, **568** (1985) 1–72.

[15] W. Thurston, *The geometry and topology of three-manifolds*, Princeton Univ., 1979.

[16] ———, *Three-manifolds with symmetry*, Preprint, 1982.

[17] Q. Zhou, *Three-dimensional geometric cone structures*, Ph.D. Thesis, UCLA, 1990.

East China Normal University, Shanghai