

## A LOCAL PROOF OF PETRI'S CONJECTURE AT THE GENERAL CURVE

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### Abstract

A proof of Petri's general conjecture on the unobstructedness of linear systems on a general curve is given, using only the local properties of the deformation space of the pair (curve, line bundle).

### 1. Introduction

Let  $L_0$  denote a holomorphic line bundle of degree  $d$  over a compact Riemann surface  $C_0$ . The Petri conjecture stated that, if  $C_0$  is a curve of general moduli, the mapping

$$\mu_0 : H^0(L_0) \otimes H^0(\omega_{C_0} \otimes L_0^\vee) \rightarrow H^0(\omega_{C_0})$$

is injective. Later, this assertion was given a more modern interpretation making it a central question in the study of curves and their linear series—what is now called Brill-Noether theory.

To recap the modern formulation we proceed as in [1]. Let  $C_0^{(d)}$  denote the  $d$ -th symmetric product of  $C_0$  and let  $\Delta \subseteq C_0^{(d)} \times C_0$  denote the tautological divisor. Let

$$\mathbb{P}^r = \mathbb{P}(H^0(L_0)).$$

For the projection

$$p_* : C_0^{(d)} \times C_0 \rightarrow C_0^{(d)}$$

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and exact sequence

$$0 \rightarrow O_{C_0^{(d)} \times C_0} \rightarrow O_{C_0^{(d)} \times C_0}(\Delta) \rightarrow O_{C_0^{(d)} \times C_0}(\Delta) \Big|_{\Delta} \rightarrow 0,$$

one has that

$$T_{C_0^{(d)}} = p_* \left( O_{C_0^{(d)} \times C_0}(\Delta) \Big|_{\Delta} \right).$$

Applying the derived functor  $Rp_* \circ \mathcal{O}_{\mathbb{P}^r \times C_0}$  to the above exact sequence as in (2.6) of [1], one obtains an exact sequence

$$0 \rightarrow N_{\mathbb{P}^r \setminus C_0^{(d)}} \rightarrow O_{\mathbb{P}^r} \otimes H^1(O_{C_0}) \rightarrow O_{\mathbb{P}^r}(1) \otimes H^1(L_0) \rightarrow 0,$$

where  $N_{A \setminus B}$  denotes the normal bundle of  $A$  in  $B$ . So the dual of the kernel of  $\mu_0$  above is exactly

$$H^1 \left( N_{\mathbb{P}^r \setminus C_0^{(d)}} \Big|_{\mathbb{P}^r} \right).$$

Via the standard short exact sequence of normal bundles, Petri's conjecture becomes the assertion

$$H^1 \left( N_{\mathbb{P}^r \setminus C_0^{(d)}} \right) = 0,$$

that is, the deformation theory of linear series is unobstructed at a curve of general moduli.

There are several proofs of Petri's conjecture, proofs via degeneration by Gieseker [7] and Eisenbud-Harris [6] and a proof via specialization to the locus of curves on a general  $K3$ -surface due to Lazarsfeld [11] (see also [12]). However the only proof based on properties of the infinitesimal deformation of the general curve, as opposed to some specialization of it, is a proof for  $r \leq 2$  by Arbarello and Cornalba in [1]. In conversations concerning his joint work with Cornalba, Arbarello explained to the author the viewpoint of [2] that there should exist a generalization to higher order of the following result (which appears both in [2] and [1]):

Let

$$\mathfrak{D}_n(L_0)$$

denote the sheaf of holomorphic differential operators of order  $\leq n$  on sections of the line bundle  $L_0$ . (If

$$L_0 = \mathcal{O}_{C_0}$$

we denote this sheaf simply as  $\mathcal{D}_n$ .) The first-order deformations the pair  $(L_0, C_0)$  are in natural one-to-one correspondence with the elements

$$\psi \in H^1(\mathcal{D}_1(L_0))$$

in such a way that a section  $s_0$  of  $L_0$  deforms to first order with the deformation  $\psi$  if and only if the element

$$\psi(s_0) \in H^1(L_0)$$

is zero.

Furthermore he pointed out that an appropriate higher-order generalization of this fact and a simple Wronskian argument would immediately yield a "local" proof of Petri's general conjecture at the general curve (see §4 below). The purpose of this paper is to carry out that generalization.

The general idea of the proof is to use the Kuranishi theory of (curvilinear)  $C^\infty$ -trivializations of deformations of complex manifolds as it applies to the total space the dual line bundle  $L_0^\vee$ . Roughly speaking, if we denote the  $t$ -disk as  $\Delta$  and are given a  $C^\infty$ -trivialization

$$F_\sigma = (\sigma, \pi) : M \rightarrow M_0 \times \Delta$$

of a deformation  $M/\Delta$  of a complex manifold  $M_0$ , Kuranishi associated to this situation a power series

$$\xi = \xi_1 t + \xi_2 t^2 + \dots,$$

where each  $\xi_j$  is a  $(0, 1)$ -form with coefficients in (a subsheaf of) the tangent bundle of  $M_0$ .  $F_\sigma$  is not allowed to be an arbitrary  $C^\infty$ -isomorphism over  $\Delta$ . The relevant restriction is that trajectory of each point on  $M_0$  must be holomorphic, that is,

$$\sigma^{-1}(x_0) \subseteq M$$

must be a holomorphic disk for each  $x_0 \in M_0$ . This is of course just a restriction on the choice of trivialization; it implies no restriction on the deformation  $M/\Delta$ . For such a trivialization, the holomorphic functions  $f$  on  $M$  have a very nice form; namely we can write power-series expansions

$$f \circ F_\sigma^{-1} = f_0 + f_1 t + f_2 t^2 + \dots$$

such that the holomorphicity condition

$$\bar{\partial}_M f = 0$$

becomes just

$$(\bar{\partial}_{M_0} - \xi) (f_0 + f_1 t + f_2 t^2 + \dots) = 0.$$

Although later on we will actually need to consider a slightly more general case in the body of this paper, it is perhaps helpful as an introduction to give the line of reasoning of the paper in the case in which  $M_0$  happens to be the total space of a holomorphic line bundle

$$q_0 : L_0^\vee \rightarrow C_0$$

over a compact Riemann surface  $C_0$ . One easily sees that the deformation is a deformation of holomorphic line bundles if and only if the Kuranishi data  $\xi^L$  are invariant under the action of the  $\mathbb{C}^*$ -action on  $L_0^\vee$ . In fact, if  $\chi$  denotes the  $(1,0)$  Euler vector field on  $L_0^\vee$  associated with the natural  $\mathbb{C}^*$ -action on the line bundle, this is just the condition

$$[\chi, \xi_j^L] = 0$$

for all  $j$ , that is, that the  $\xi_j^L$  can be written everywhere locally in the form

$$(1) \quad q_0^*(\alpha) \cdot \chi + q_0^*(\beta) \cdot \tau_L$$

where  $\alpha$  and  $\beta$  are  $(0,1)$ -forms on  $C_0$  and  $\tau_L$  is a lifting of a  $(1,0)$ -vector-field  $\tau_C$  on  $C_0$  such that

$$[\chi, \tau_L] = 0.$$

(The “associated” or “compatible” Kuranishi data for the deformation of  $C_0$  is just given by  $\xi_j^C = \beta \cdot \tau_C$ .) Sections  $s$  of  $L$  are just functions  $f$  on  $L_0^\vee$  for which

$$L_\chi(f) = f,$$

where  $L_\chi$  denotes Lie differentiation with respect to the vector field  $\chi$ .

Suppose now we have a line-bundle deformation  $(L/\Delta, C/\Delta)$  of  $(L_0, C_0)$  with compatible trivializations

$$\begin{aligned} \sigma & : C \rightarrow C_0, \\ \lambda & : L^\vee \rightarrow L_0^\vee, \end{aligned}$$

and a section  $s$  of  $L$  whose zeros are given by

$$\sigma^{-1}(\text{zeros}(s_0)).$$

Rescaling  $\lambda$  in the fiber direction we arrive at a trivialization of the deformation  $L^\vee$  of  $L_0^\vee$  for which  $s$  is constant, that is,

$$s = s_0 \circ \lambda.$$

We call such compatible trivializations of  $C_0$  and  $L_0^\vee$  “adapted” to the section  $s$ .

Of course we have twisted the almost complex structure on  $C_0$  and  $L_0^\vee$  to achieve this trivialization. To keep track of this twisting, we consider only “Schiffer-type” deformations  $C$  of  $C_0$ , for which the twist in almost complex structure is given almost everywhere by a gauge transformation, that is, by a power series

$$\beta = \beta_1 t + \beta_2 t^2 + \dots,$$

where the  $\beta_j$  are  $C^\infty$ -vector-fields of type  $(1, 0)$  on  $C_0 - \{p\}$  and meromorphic in a small analytic neighborhood of  $p$ . Then we take

$$\xi^C = \frac{e^{[\beta, \cdot]} - 1}{[\beta, \cdot]} (\bar{\partial}_{C_0} \beta)$$

(see [5]) and get a compatible trivialization of  $L^\vee/\Delta$  by lifting the  $\beta_j$  to vector fields  $\tilde{\beta}_j$  on  $L_0^\vee$  with

$$[\tilde{\beta}_j, \chi] = 0$$

with the same meromorphic property near  $q_0^{-1}(p)$ . Holomorphicity of a section  $s$  becomes the condition

$$\left( \bar{\partial}_{L_0^\vee} (e^{L-\beta}(f)) \right) = 0$$

on the power series

$$f = f_0 + f_1 t + f_2 t^2 + \dots$$

representing  $s$  as a function on  $L_0^\vee \times \Delta$ . That is, the condition is simply that the pull-back of  $f$  via the gauge transformation is a power series whose coefficients are meromorphic sections of  $L_0$ .

If we have a holomorphic section  $s$  of  $L$  whose restriction to  $s_0$  has simple zeros  $D_0$ , and if  $\tilde{\beta}$  is zero in a small analytic neighborhood of

$$D = \text{zero}(s) \subset C$$

then there is a  $C^\infty$ -automorphism

$$\Phi : C_0 \times \Delta \rightarrow C_0 \times \Delta$$

defined over  $\Delta$  such that:

1.  $\Phi$  is holomorphic in a small analytic neighborhood of  $D \cup \{p\}$ .

2.

$$\Phi(\{x_0\} \times \Delta)$$

is a holomorphic disk for each  $x_0 \in C_0$ .

3.

$$\Phi \circ F_\sigma(D) = D_0 \times \Delta.$$

The rough (imprecise) idea is that trivialization  $\Phi \circ F_\sigma$  can also be considered to be of Schiffer type for some vector field

$$\gamma = \gamma_1 t + \gamma_2 t^2 + \dots$$

$\gamma$  lifts to a vector field  $\tilde{\gamma}$  associated to a Schiffer-type trivialization of the deformation  $L^\vee/\Delta$  of  $L_0^\vee$  which is adapted to the section  $s$ . Since by construction  $s$  corresponds to the “constant” power series

$$f_0 + 0 \cdot t + 0 \cdot t^2 + \dots,$$

we have the equation

$$\left( \bar{\partial}_{L_0^\vee} (e^{L-\tilde{\gamma}}(f_0)) \right) = 0,$$

that is,

$$(2) \quad \left[ \bar{\partial}_{L_0^\vee}, e^{L-\tilde{\gamma}} \right] (f_0) = 0.$$

It is in this way that we produce elements of  $H^1(\mathcal{D}_{n+1}(L_0))$  for all  $n \geq 0$  which must annihilate sections  $s_0$  of  $L_0$  which extend to sections of  $L$ . (The difficulty is of course that the elements of  $H^1(\mathcal{D}_{n+1}(L_0))$

depend on the choice of  $s_0$ . To remedy this we will eventually have to replace the deformation  $C/\Delta$  of  $C_0$  with the deformation

$$\mathbb{P}/\Delta = \mathbb{P}(H^0(L/\Delta))$$

of  $\mathbb{P}(H^0(L_0))$  and replace  $L$  with  $\mathcal{O}(1)$ .

As one of the simplest concrete examples, let

$$C_0 = \frac{\mathbb{C}}{\mathbb{Z} + \mathbb{Z}\sqrt{-1}}$$

with linear holomorphic coordinate  $z$  on  $\mathbb{C}$ . For a  $C^\infty$ -function  $\rho$  supported on  $\{z : |z| \leq 1/8\}$  and identically 1 on  $\{z : |z| \leq 1/16\}$ , let

$$\begin{aligned} \beta_1 &= \frac{\rho}{z} \cdot \frac{\partial}{\partial z}, \\ \beta_j &= 0, \quad j > 1. \end{aligned}$$

This is a non-trivial deformation since, to first order it is given by the generator

$$\bar{\partial}_{C_0} \left( \frac{\rho}{z} \cdot \frac{\partial}{\partial z} \right) \in H^1(T_{C_0}).$$

For  $L_0$  we can take the line bundle of degree 2 given by the divisor

$$D_0 = \left\{ \frac{1 + \sqrt{-1}}{4} \right\} + \left\{ \frac{3 + 3\sqrt{-1}}{4} \right\}$$

with corresponding section  $s_0$ . Let  $s$  be some extension of the section  $s_0$ . For a trivialization

$$F_\sigma : C \rightarrow C_0 \times \Delta$$

associated to the above Kuranishi data, the zero set  $D = D' + D''$  of the section  $s$  is given by two power series

$$\begin{aligned} z &= a(t) = \frac{1 + \sqrt{-1}}{4} + a_1 t + \dots, \\ z &= b(t) = \frac{3 + 3\sqrt{-1}}{4} + b_1 t + \dots, \end{aligned}$$

since the deformation of (almost) complex structure is zero near  $D_0$ . So near  $D'$  we recursively solve for

$$\Phi(z, t) = (a'(z, t), t)$$

such that

$$a'(a(t), t) \equiv \frac{1 + \sqrt{-1}}{4},$$

and similarly near  $D''$  for

$$\Phi(z, t) = (b'(z, t), t)$$

such that

$$b'(b(t), t) \equiv \frac{3 + 3\sqrt{-1}}{4}.$$

Near  $\{0\} \times \Delta$  we take

$$\Phi(z, t) = (z, t)$$

and then extend  $\Phi$  to a family of diffeomorphism on all of  $C_0$  by a  $C^\infty$  patching argument. For the new trivialization

$$\Phi \circ F_\sigma : C \rightarrow C_0 \times \Delta$$

the divisor  $D$  giving the line bundle  $L$  is “constant” so that the pull-back of  $s_0$  via the product structure gives rise to a compatible trivialization of  $L$ .

The Petri proof will follow from doing this process (for a line-bundle deformation of  $L_0$  for which all sections extend) for every Schiffer-type variation of a generic curve  $C_0$ . We show that the set of equations (2) we obtain implies that the higher  $\mu$ -maps

$$\mu_{n+1} : \ker(\mu_n) \rightarrow H^0(\omega_{C_0}^{n+2}) = H^1(T_{C_0}^{m+1})$$

are all zero. As Arbarello-Cornalba-Griffiths-Harris showed twenty years ago, this implies Petri’s conjecture.

We shall use Dolbeault cohomology throughout this paper. In particular, the sheaf  $\mathfrak{D}_n(L_0)$  has both a left and a right  $\mathcal{O}_{C_0}$ -module structure and we define

$$A^{0,i}(\mathfrak{D}_n(L_0)) := A_{C_0}^{0,i} \otimes_{\mathcal{O}_{C_0}} \mathfrak{D}_n(L_0)$$

where  $A^{0,i}$  is the sheaf of  $C^\infty$ - $(0, i)$ -forms. Also the context will hopefully eliminate any confusion between two standard notation used in this paper, namely the notation  $L$  and  $L_0$  for line bundles and the notation

$$L_\tau^k = \underbrace{L_\tau \circ \dots \circ L_\tau}_{k\text{-times}}$$



where  $L_\tau$  denotes Lie differentiation with respect to a vector field  $\tau$ .

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## 2. Deformations of manifolds and differential operators

### 2.1 Review of formal Kuranishi theory

We begin with a brief review of the Newlander-Nirenberg-Kuranishi theory of deformations of complex structures (see [10], [9], II.1 of [8], or [4]). Let

$$(3) \quad M \xrightarrow{\pi} \Delta = \{t \in \mathbb{C} : |t| < 1\}$$

be a deformation of a complex manifold  $M_0$  of dimension  $m$ . Since we are doing formal deformation theory, all calculations will actually take place over the formal neighborhood of 0 in  $\Delta$ . However, convergence will not be an issue in anything that we do since we will always be working from a situation in which we are *given* a geometric deformation and deriving consequences in the category of formal deformations.

**Definition 2.1.** A  $C^\infty$ -diffeomorphism

$$F_\sigma = (\sigma, \pi) : M \rightarrow M_0 \times \Delta$$

will be called a trivialization of the deformation  $M/\Delta$  if

$$\sigma|_{M_0} = \textit{identity}$$

and

$$\sigma^{-1}(x_0)$$

is an analytic disk for each  $x_0 \in M_0$ .

The next four lemmas are standard from formal Kuranishi theory:

**Lemma 2.1.** *Let*

$$T_{M_0}^*$$

*denote the complexification of the real cotangent bundle of  $M_0$ . Given any trivialization  $F_\sigma$ , the holomorphic cotangent bundle of  $M_t$  under the  $C^\infty$ -isomorphisms*

$$M_t \cong M_0$$

*induced by  $F_\sigma$  corresponds to a subbundle*

$$T_t^{1,0} \subseteq T_{M_0}^*.$$

*If*

$$\pi^{1,0} + \pi^{0,1} : T_{M_0}^* \rightarrow T_{M_0}^{1,0} \oplus T_{M_0}^{0,1}$$

*are the two projections, the restriction*

$$\pi^{1,0} : T_t^{1,0} \rightarrow T_{M_0}^{1,0}$$

*is an isomorphism for small  $t$  so that the composition*

$$T_{M_0}^{1,0} \xrightarrow{(\pi^{1,0})^{-1}} T_t^{1,0} \xrightarrow{\pi^{0,1}} T_{M_0}^{0,1},$$

*gives a  $C^\infty$ -mapping*

$$\xi(t) : T_{M_0}^{1,0} \rightarrow T_{M_0}^{0,1}$$

*which determines the deformation of (almost) complex structure.*

Thus, at least formally, we can write

$$\xi(t) = \sum_{i>0} \xi_i t^i$$

with each  $\xi_i \in A_{M_0}^{0,1}(T_{1,0})$ , that is, each  $\xi_i$  is a  $(0,1)$ -form with coefficients in the holomorphic tangent bundle  $T_{1,0}$  of  $M_0$ .

**Lemma 2.2.** *Every relative complex-valued  $C^\infty$ -differential form  $\omega$  on  $M/\Delta$  of type  $(0,q)$  corresponds on a (formal) neighborhood of  $M_0$  to a form*

$$\pi^{0,q} \left( (F_\sigma^{-1})^* (\omega) \right) = \sum_{i,j=0}^{\infty} \omega_{i,j} t^i \bar{t}^j$$

*on*

$$M_0 \times \Delta$$

*and so, working modulo  $\bar{t}$ , gives a holomorphic family*

$$\omega_\sigma := \sum_{i=0}^{\infty} \omega_{i,0} t^i.$$

of  $C^\infty$ -forms. This correspondence is a formal isomorphism

$$(\ )_\sigma^q : \frac{A_{M/\Delta}^{0,q}}{\{\bar{t}\}} \rightarrow A_{M_0}^{0,q} \otimes \mathbb{C}[[t]].$$

If we have two different trivializations  $\sigma$  and  $\sigma'$ , we have a formal isomorphism

$$G_{\sigma'\sigma}^q = (\ )_{\sigma'}^q \circ ((\ )_\sigma^q)^{-1}.$$

**Lemma 2.3.** For any  $C^\infty$ -function  $f$  on  $M$  write

$$f \circ F_\sigma^{-1} = \sum_{i,j=0}^{\infty} f_{i,j} t^i \bar{t}^j$$

and define as above

$$f_\sigma = \sum_{i,j=0}^{\infty} f_{i,0} t^i.$$

Further define

$$\begin{aligned} \bar{D}_\sigma(f_\sigma) &:= \left( \bar{\partial}_{M_0} - \sum_{j=1}^{\infty} t^j \xi_j \right) (f_\sigma) \\ &= \sum_{i=0}^{\infty} \bar{\partial}_{M_0} f_{i,\sigma} t^i - \sum_{i=0, j=1}^{\infty, \infty} \xi_j (f_{i,\sigma}) t^{i+j}. \end{aligned}$$

Then

$$(\ )_\sigma^1 \circ \bar{\partial}_M = \bar{D}_\sigma \circ (\ )_\sigma^0,$$

and

$$\bar{D}_\sigma = G_{\sigma\sigma'}^1 \circ \bar{D}_{\sigma'} \circ G_{\sigma'\sigma}^0.$$

Also

$$f_\sigma \circ F_\sigma$$

is holomorphic on  $M$  if and only if

$$\bar{D}_\sigma(f_\sigma) = 0.$$

We next ask which sequences  $\xi_j \in A^{0,1}(T_{1,0})$  come from a trivialization of a deformation (3). Before answering this question, we need to make precise the various actions of an element  $\xi \in A^{0,k}(T_{1,0})$  on  $\sum A^{p,q}(M_0)$ . For any  $\xi$  we write the action via contraction as

$$\langle \xi | \rangle,$$

and ‘‘Lie differentiation’’ as

$$L_\xi := \langle \xi | \cdot \rangle \circ d + (-1)^k d \circ \langle \xi | \cdot \rangle.$$

The sign is so chosen that, writing any element of  $A^{0,k}(T_{1,0})$  locally as a sum of terms

$$\xi = \bar{\eta} \otimes \chi$$

for some closed  $(0, k)$ -form  $\bar{\eta}$  and  $\chi \in A^{0,0}(T_{1,0})$ . Then

$$L_\xi = \bar{\eta} \otimes L_\chi.$$

(Warning: Since, as an operator on  $A^{0,q}(M_0)$ ,  $L_{f\xi} = fL_\xi$ , one has

$$[\bar{\partial}, L_\xi] = L_{\bar{\partial}\xi} : A^{0,q}(M_0) \rightarrow A^{0,q+k+1}(M_0)$$

however the identity does not hold as an operator on  $A^{p,q}(M_0)$  for  $p > 0$ .)

Also we compute

$$\begin{aligned} L_\xi L_{\xi'} - (-1)^{\deg \bar{\eta} \cdot \deg \bar{\eta}'} L_{\xi'} L_\xi &= (\bar{\eta} \otimes L_\chi) (\bar{\eta}' \otimes L_{\chi'}) - (-1)^{\deg \bar{\eta} \cdot \deg \bar{\eta}'} (\bar{\eta}' \otimes L_{\chi'}) (\bar{\eta} \otimes L_\chi) \\ &= \bar{\eta} \bar{\eta}' (L_\chi L_{\chi'} - L_{\chi'} L_\chi) \\ &= \bar{\eta} \bar{\eta}' L_{[\chi, \chi']}. \end{aligned}$$

So, using this local presentation for

$$\xi \in A^{0,j}(T_{1,0}), \xi' \in A^{0,k}(T_{1,0}),$$

we can define

$$[\xi, \xi'] = \bar{\eta} \bar{\eta}' [\chi, \chi'] \in A^{0,j+k}(T_{1,0}).$$

**Lemma 2.4.** *The almost complex structures given on a coordinate neighborhood  $W_0$  in  $M_0$  by the the  $(0, 1)$ -tangent distributions*

$$\left( \frac{\partial}{\partial v_{W_0}^k} - \sum_{i=1}^{\infty} \sum_l h_{i,k}^l t^i \frac{\partial}{\partial v_{W_0}^l} \right)$$

are integrable, that is, come from a deformation/trivialization of  $M_0$  as in Definition (2.1), if and only if, for

$$\xi = \sum_{i=1}^{\infty} \sum_{k,l} \overline{dv_{W_0}^k} \wedge h_{i,k}^l t^i \frac{\partial}{\partial v_{W_0}^l},$$

we have

$$\bar{\partial}\xi = \frac{1}{2} [\xi, \xi].$$

**Proposition 2.5.** *Two trivializations  $F_\sigma$  and  $F_{\sigma'}$  of the same deformation (3) are related by a holomorphic automorphism  $\varphi$  of  $M/\Delta$ , that is, there is a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M \\ \downarrow \sigma & & \downarrow \sigma' \\ M_0 & = & M_0 \end{array},$$

if and only if

$$\bar{D}_\sigma = \bar{D}_{\sigma'}.$$

*Proof.* One implication is immediate from the definitions of  $\bar{D}_\sigma$  and  $\bar{D}_{\sigma'}$ . For the other, the equality

$$\xi_\sigma = \xi_{\sigma'}$$

implies that the differential of the  $C^\infty$ -automorphism

$$\varphi := (\sigma, \pi)^{-1} \circ (\sigma', \pi) : M \rightarrow M$$

preserves the  $(1,0)$ -subspace of the (complexified) tangent space and therefore  $\varphi$  is holomorphic. q.e.d.

## 2.2 Gauge transformations

We begin now with a deformation

$$M/\Delta$$

of  $M_0$  and let

$$F_\sigma : M \xrightarrow{(\sigma, \pi)} M_0 \times \Delta$$

be a trivialization with associated Kuranishi data

$$\xi_\sigma.$$

Suppose that we have a one-real-parameter group of diffeomorphisms

$$\Phi_s : M_0 \times \Delta \rightarrow M_0 \times \Delta$$

defined over  $\Delta$  such that

$$F_s := \Phi_s \circ F_\sigma : M \xrightarrow{(\sigma_s, \pi)} M_0 \times \Delta$$

is a trivialization for each (sufficiently small)  $s$  and, for each  $x_0 \in M_0$ ,

$$\Phi_s|_{\{x_0\} \times \Delta}$$

is a real-analytic family of complex-analytic embeddings of  $\Delta$  in  $M_0 \times \Delta$ . Then, as for example in §2 of [4], there is then associated a vector field

$$\beta + \bar{\beta}$$

where

$$\beta = \sum_{j>0} \beta_j t^j$$

and each  $\beta_j$  is a  $C^\infty$ -vector field of type  $(1,0)$  on  $M_0$ , such that, for

$$g = g_0 + g_1 t + \dots,$$

on  $M_0 \times \Delta$  we have

$$(4) \quad g \circ \Phi_s = e^{L_{s\beta + \bar{s}\bar{\beta}}}(g).$$

We let

$$F_\beta := F_1 = \Phi_1 \circ F_\sigma : M \rightarrow M_0 \times \Delta.$$

Then by (4) we have for any  $C^\infty$ -function  $g$  on  $M$  that

$$(5) \quad g_\beta = e^{L_\beta}(g_\sigma).$$

If  $\xi_s$  denotes the Kuranishi data for the trivialization  $F_s$ , then by direct computation

$$\frac{\partial \xi_s}{\partial s} = [\bar{\partial}, \beta] + [\beta, \xi_s].$$

(See for example Lemma 2.10 of [4].) On the other hand, if we define

$$(6) \quad \varsigma_\beta := \frac{e^{[\beta, \cdot]} - 1}{[\beta, \cdot]}([\bar{\partial}, \beta])$$

and the action

$$(7) \quad \xi_\beta := \beta \cdot (\xi) = e^{[\beta, \cdot]}(\xi) + \varsigma_\beta,$$

one also has by direct computation that

$$\frac{\partial \xi_{s\beta}}{\partial s} = [\bar{\partial}, \beta] + [\beta, \xi_{s\beta}].$$

(See for example §3 of [4]. Compare with §3 of [5].) The conclusion is that  $\xi_{s\beta}$  is the Kuranishi data for the trivialization  $F_s$  for all  $s$  and so, in particular

$$\xi_\beta$$

is the Kuranishi data for the trivialization  $F_1 = F_\beta$ .

So the group of vector fields  $\beta$  acts on the Kuranishi data associated to the deformation  $M/\Delta$ . This action corresponds to the change of the given trivialization by a  $C^\infty$ -automorphism

$$(8) \quad \Phi_\beta : M_0 \times \Delta \rightarrow M_0 \times \Delta$$

defined over  $\Delta$ .

**Lemma 2.6.** *i)*

$$(e^{L_\beta}) (\bar{\partial} - \xi) (e^{-L_\beta}) = \bar{\partial} - \xi_\beta.$$

*ii) Given a function*

$$f_\beta = \sum_i f_{\beta,i} t^i$$

on  $M_0 \times \Delta$ , the function

$$f_\beta \circ F_\beta$$

is holomorphic on  $M$  if and only if

$$(\bar{\partial} - \xi) (e^{-L_\beta} (f_\beta)) = 0.$$

*Proof.* *i)* This assertion is implicit in (5) but, as a check, we will do it by direct computation.

$$(e^{L_\beta}) (\bar{\partial} - \xi) (e^{-L_\beta}) = \bar{\partial} + [e^{L_\beta}, \bar{\partial}] (e^{-L_\beta}) - e^{[\beta, \cdot]} (\xi).$$

If we can show the identity

$$(9) \quad [\bar{\partial}, e^{L_\beta}] = \varsigma_\beta \circ e^{L_\beta},$$

the lemma will follow from Lemma 2.3 since, by definition,

$$\xi_\beta = e^{[\beta, \cdot]} (\xi) + \varsigma_\beta.$$

To see (9) we prove by induction that

$$[\bar{\partial}, \beta^{n+1}] = \sum_{i=0}^n \binom{n+1}{i} ([\beta, \cdot])^{n-i} [\bar{\partial}, \beta] \beta^i.$$

Inductively

$$\begin{aligned}
[\bar{\partial}, \beta^{n+1}] &= [\bar{\partial}, \beta] \cdot \beta^n + \beta \cdot [\bar{\partial}, \beta^n] \\
&= [\bar{\partial}, \beta] \cdot \beta^n + \beta \left( \binom{n}{0} [\beta, ]^{n-1} \cdot [\bar{\partial}, \beta] \right. \\
&\quad \left. + \dots + \binom{n}{n-1} [\bar{\partial}, \beta] \cdot \beta^{n-1} \right) \\
&= \left( \binom{n}{n} [\bar{\partial}, \beta] \cdot \beta^n \right. \\
&\quad \left. + \left( \binom{n}{0} ([\beta, ]^n [\bar{\partial}, \beta] + ([\beta, ]^{n-1} [\bar{\partial}, \beta] \beta) \right. \right. \\
&\quad \left. \left. + \dots + \binom{n}{n-1} ([\beta, ] [\bar{\partial}, \beta] \beta^{n-1} + [\bar{\partial}, \beta] \beta^n) \right) \right).
\end{aligned}$$

Now use the identity

$$\left( \binom{n-1}{r} + \binom{n-1}{r-1} \right) = \binom{n}{r}.$$

Thus

$$\begin{aligned}
\left[ \bar{\partial}, \left( \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \right) \right] &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=0}^{n-1} \binom{n}{i} ([\beta, ])^{n-1-i} [\bar{\partial}, \beta] \beta^i \\
&= \sum_{k=0, i=0}^{\infty, \infty} \frac{1}{(k+1)!} ([\beta, ])^k [\bar{\partial}, \beta] \frac{\beta^i}{i!} \\
&= \xi_{\beta} \circ \left( \sum_{i=0}^{\infty} \frac{\beta^i}{i!} \right).
\end{aligned}$$

ii)

$$\begin{aligned}
0 &= [\bar{\partial}, e^{L_{\beta}} \circ e^{-L_{\beta}}] \\
&= [\bar{\partial}, e^{L_{\beta}}] \circ e^{-L_{\beta}} + e^{L_{\beta}} \circ [\bar{\partial}, e^{-L_{\beta}}] \\
&= \varsigma_{\beta} + e^{L_{\beta}} \circ \varsigma_{-\beta} \circ e^{-L_{\beta}},
\end{aligned}$$

so that

$$e^{L_{\beta}} \circ (\varsigma_{-\beta} - \xi) \circ e^{-L_{\beta}} = -(\varsigma_{\beta} + (e^{L_{\beta}} \circ \xi \circ e^{-L_{\beta}})).$$

q.e.d.



Suppose now that we have two trivializations

$$\begin{aligned} F_\sigma &: M \rightarrow M_0 \times \Delta, \\ F_{\sigma'} &: M \rightarrow M_0 \times \Delta \end{aligned}$$

of a given deformation

$$M/\Delta.$$

Then

$$F_{\sigma'} \circ F_\sigma^{-1}$$

is a  $C^\infty$ -diffeomorphism of  $M_0 \times \Delta$  and so can be realized as the value at  $s = 1$  of a one-parameter group of diffeomorphisms which restrict to an analytic family of analytic embeddings of  $\{x_0\} \times \Delta$  for each  $x_0 \in M_0$ . Thus referring to the notation of Lemma 2.3 above we have that there is a  $C^\infty$ -vector field  $\kappa$  of type  $(1, 0)$  such that

$$(10) \quad \begin{aligned} g_\sigma &= e^{L-\kappa}(g_{\sigma'}) \\ \overline{D}_{\sigma'} &= e^{L\kappa} \circ \overline{D}_\sigma \circ e^{L-\kappa}. \end{aligned}$$

### 2.3 Schiffer-type deformations

We now consider a special class of deformations of  $M_0$ , those for which the change of complex structure can be localized at a union  $A_0$  of codimension-one subvarieties on  $M_0$ . We let

$$(11) \quad \beta \in A_{M_0}^{0,0} \left( T_{M_0}^{1,0} \right) \otimes t\mathbb{C}[[t]]$$

be a vector field which is

- i) meromorphic in an analytic neighborhood  $(U_0 \times \Delta)$  of the set  $(A_0 \times \Delta)$  on  $(M_0 \times \Delta)$ ,
- ii)  $C^\infty$  on  $(M_0 - A_0) \times \Delta$ .

Using Lemma 2.6 for the case in which we first take

$$(12) \quad F_\sigma : ((M_0 - A_0) \times \Delta) \rightarrow ((M_0 - A_0) \times \Delta)$$

in 2.2 as the identity map, we define a deformation  $M_\beta/\Delta$  of  $M_0$  by the integrable Kuranishi data

$$(13) \quad \xi_\beta := \varsigma_\beta.$$

Notice that  $\xi_\beta = 0$  in a neighborhood of  $A_0 \times \Delta$  so  $\xi_\beta$  corresponds to a trivialization

$$F_\beta : M_\beta \xrightarrow{(\sigma_\beta, \pi)} M_0 \times \Delta$$

with

$$F_\beta : (\sigma_\beta)^{-1}(U_0) \rightarrow U_0 \times \Delta$$

an analytic isomorphism. Denote

$$(14) \quad \bar{D}_\beta := \bar{D}_{\sigma_\beta} = \bar{\partial} - \varsigma_\beta.$$

We call  $A_0$  the *center* of the Schiffer-type deformation.

Let

$$A_\beta := (\sigma_\beta)^{-1}(A_0) \subseteq M_\beta.$$

From 2.2, Lemma 2.6 and the above we conclude:

**Lemma 2.7.**

$$f_\beta \circ F_\beta$$

is analytic on  $M_\beta$  if and only if

$$\bar{\partial}_{M_0}(e^{-L_\beta}(f_\beta)) = 0.$$

In fact, for any divisor  $B_0$  supported on  $A_0$ ,  $B_0$  has a unique extension to a divisor

$$B_\beta$$

on  $M_\beta$  which is supported on  $A_\beta$ . We denote by

$$C_{B_0}$$

the vector space of functions  $f_0$  which are  $C^\infty$  on  $(M_0 - A_0)$  and meromorphic on  $U_0$  and for which

$$B_0 + \text{div}(f_0)$$

is effective on  $U_0$ . Then:

**Lemma 2.8.** *i) A meromorphic function  $f$  on  $M_\beta$  with*

$$B_\beta + \text{div}(f)$$

*effective is a formal sum*

$$f_\beta := f_{\sigma_\beta} = \sum_{i=0}^{\infty} f_{\beta,i} t^i$$

*such that each  $f_{\beta,i} \in C_{B_0}$  and*

$$\left( \bar{\partial} - \sum_{j=1}^{\infty} \xi_{\beta,j} t^j \right) \left( \sum_{i=0}^{\infty} f_{\beta,i} t^i \right) = 0.$$

ii) The meromorphic functions on  $M_\beta$  with

$$B_\beta + \text{div}(f)$$

effective are given by the kernel of the mapping

$$e^{L_\beta} : H^0(\mathcal{O}_{M_0}(\infty \cdot A_0)) \otimes \mathbb{C}[[t]] \rightarrow H^0\left(\frac{\mathcal{O}_{M_0}(\infty \cdot A_0)}{\mathcal{O}_{M_0}(B_0)}\right) \otimes \mathbb{C}[[t]].$$

iii) If

$$i : A_0 \rightarrow M_0$$

is the inclusion map and  $R$  denote the image of the map

$$(i^{-1}\mathcal{O}_{M_0}(B_0) \otimes t\mathbb{C}[[t]]) \xrightarrow{\bar{\partial} \circ e^{-L_\beta}} (H^1(\mathcal{O}_{M_0}(B_0)) \otimes \mathbb{C}[[t]]),$$

then  $f_0 \in H^0(\mathcal{O}_{M_0}(B_0))$  extends to a global section of  $\mathcal{O}_{M_\beta}(B_\beta)$  if and only if

$$[\bar{\partial}, e^{-L_\beta}](f_0) \in R.$$

*Proof.* i) The assertion is immediate from Lemma 2.3.

ii) Again by Lemma 2.6i) occurs exactly when  $f_\beta$  lies in

$$C_{B_0} \otimes \mathbb{C}[[t]] \cap \text{image}\left(H^0(\mathcal{O}_{M_0}(\infty \cdot A_0)) \otimes \mathbb{C}[[t]] \xrightarrow{e^{L_\beta}} C_{\infty \cdot A_0} \otimes \mathbb{C}[[t]]\right).$$

iii) follows from Lemma 2.7 and the cohomology exact sequence associated to the short exact sequence

$$0 \rightarrow \mathcal{O}_{M_0}(B_0) \rightarrow \mathcal{O}_{M_0}(\infty \cdot A_0) \rightarrow \frac{\mathcal{O}_{M_0}(\infty \cdot A_0)}{\mathcal{O}_{M_0}(B_0)} \rightarrow 0.$$

q.e.d.

## 2.4 Gauge transformation on Schiffer-type trivializations

Next suppose we wish to change our trivialization

$$F_\beta : M_\beta \xrightarrow{(\sigma_\beta, \pi)} M_0 \times \Delta$$

by an allowable  $C^\infty$ -automorphism

$$\begin{array}{ccc} M_\beta & \xrightarrow{F_\beta} & M_0 \times \Delta \\ = & & \downarrow \Phi \\ M_\beta & \xrightarrow{G} & M_0 \times \Delta \end{array}$$

defined over  $\Delta$ . That is

1. such that  $\Phi$  preserves  $A_0 \times \Delta$  as a set
2. is holomorphic on  $U_0 \times \Delta$ .
3.  $\Phi$  restricts to an analytic embedding of each disk  $\{x_0\} \times \Delta$ .

To calculate the Kuranishi data for  $G$ , we proceed as in 2.2. We can assume that  $\Phi = \Phi_1$  for a family  $\Phi_s$  as in 2.2. We can further assume that  $\Phi_s|_{U_0 \times \Delta}$  is a real analytic family of complex analytic maps. Let  $\kappa = \sum_{j=1}^{\infty} \kappa_j t^j$  denote the  $C^\infty$ -vector field of type  $(1, 0)$  such that the family  $\Phi_s$  is associated to

$$s(\kappa + \bar{\kappa}).$$

Then by (10) for  $F_\sigma = F_\beta$  and  $F_{\sigma'} = G$  we have

$$\begin{aligned} g_\beta &= e^{L-\kappa}(g_{\sigma'}) \\ \bar{D}_{\sigma'} &= e^{L\kappa} \circ \bar{D}_\beta \circ e^{L-\kappa}. \end{aligned}$$

Computing, using (7) and (9),

$$\begin{aligned} e^{L\kappa} \circ \bar{D}_\beta \circ e^{L-\kappa} &= e^{L\kappa} \circ (\bar{\partial} - \zeta_\beta) \circ e^{L-\kappa} \\ &= e^{L\kappa} \circ (\bar{\partial} - [\bar{\partial}, e^{L\beta}] \circ e^{L-\beta}) \circ e^{L-\kappa} \\ &= e^{L\kappa} \circ (e^{L\beta} \circ \bar{\partial} \circ e^{L-\beta}) \circ e^{L-\kappa}. \end{aligned}$$

Thus we conclude that  $g_{\sigma'}$  is holomorphic if and only if

$$\bar{\partial}_{M_0}(e^{L-\beta} \circ e^{L-\kappa}(g_{\sigma'})) = 0.$$

**Lemma 2.9.** *For a power series*

$$g = \sum_{i=0}^{\infty} g_i t^i$$

on  $M_0 \times \Delta$ ,  $g \circ G$  is holomorphic on  $M_\beta$  if and only if

$$\bar{\partial}_{M_0}(e^{L-\beta} \circ e^{L-\kappa}(g)) = 0.$$

### 3. Deformations of line bundles and differential operators

#### 3.1 The $\mu$ -maps

Let  $X_0$  be a complex manifold and let  $L_0$  be a holomorphic line bundle on  $X_0$ . Let

$$\mathfrak{D}(L_0), \mathfrak{D}_n(L_0)$$

denote the sheaf of (holomorphic) differential operators, respectively the sheaf of differential operators of order  $\leq n$ , on (sections of) the line bundle  $L_0$ . Whenever

$$H^2(\mathfrak{D}_n(L_0)) = 0$$

we have a natural exact sequence

$$H^1(\mathfrak{D}_n(L_0)) \rightarrow H^1(\mathfrak{D}_{n+1}(L_0)) \rightarrow H^1(S^{n+1}T_{X_0}) \rightarrow 0,$$

where the second last map is induced by the symbol map on differential operators. So there exists natural mappings

$$(15) \quad \tilde{\mu}^n : H^1(\mathfrak{D}_n(L_0)) \rightarrow \text{Hom}(H^0(L_0), H^1(L_0))$$

and

$$(16) \quad \mu^{n+1} : H^1(S^{n+1}T_{X_0}) \rightarrow \frac{\text{Hom}(H^0(L_0), H^1(L_0))}{\text{image } \tilde{\mu}^n}.$$

(In the next chapter we will establish Petri's conjecture on generic curve  $C_0$  by establishing that the mappings (16) are zero for  $n \geq 0$  and  $X_0 = C_0$ .)

Suppose now that we are given a deformation

$$(17) \quad L \xrightarrow{p} X \xrightarrow{\pi} \Delta$$

of the pair  $(L_0, X_0)$ . We consider  $C^\infty$ -sections of  $L$  as  $C^\infty$ -functions on the dual line bundle  $L^\vee$ . These functions  $f$  are characterized by the properties

$$(18) \quad \begin{aligned} \chi(f) &= f, \\ \bar{\chi}(f) &= 0, \end{aligned}$$

where  $\chi$  is the (holomorphic) Euler vector-field associated with the  $\mathbb{C}^*$ -action on  $L^\vee$ .

### 3.2 Trivializations of deformations of line bundles

We next claim that, given a trivialization  $\sigma$  of the deformation  $X/\Delta$  and given a line bundle  $L/X$  we can make compatible trivializations

$$(19) \quad \begin{array}{ccc} L^\vee & \xrightarrow{F_\lambda = (\lambda, \pi \circ q)} & L_0^\vee \times \Delta \\ \downarrow q & & \downarrow (q_0, id.) \\ X & \xrightarrow{F_\sigma = (\sigma, \pi)} & X_0 \times \Delta \\ \downarrow \pi & & \downarrow \\ \Delta & = & \Delta \end{array}$$

of the deformation  $L^\vee/X$  of  $L_0^\vee/X_0$  as in Lemma 2.1 but with the additional property that each fiber of the trivialization respects the structure of holomorphic line bundles, that is, if we denote by  $\tau = \tau_\sigma$  the lifting of  $\frac{\partial}{\partial t}$  induced by the trivialization of  $X/\Delta$ , then  $\tau = \tau_\lambda$  for the deformation  $L^\vee$  of  $L_0^\vee$  is obtained as a lifting of  $\tau_\sigma$  such that

$$(20) \quad [\tau_\lambda, \chi] = 0.$$

To see that this is always possible, let  $\{W\}$  be a covering of  $X$  by coordinate disks and  $\{W_0\}$  the restriction of this covering to  $X_0$ . We construct a  $C^\infty$  partition-of-unity  $\{\rho_{W_0}\}$  subordinate to the induced covering of  $X_0$ . Recall that  $L$  is given with respect to the trivialization  $\sigma$  by holomorphic local patching data

$$\begin{aligned} g^{WW'}(x) &= \sum g_i^{WW'}(x_0) t^i \\ &= g^{W_0W'_0}(x_0) \exp\left(\sum_{j>0} a_j^{WW'}(x_0) t^j\right), \end{aligned}$$

where  $x_0 = \sigma(x)$  and

$$\sum_{j>0} a_j^{WW'}(x_0) t^j = \log \frac{g^{WW'}(x)}{g^{W_0W'_0}(x_0)}.$$

Notice that, if  $V, W$ , and  $W'$  are three open sets of the cover which have non-empty intersection, then, for all  $j > 0$ ,

$$a_j^{VW} + a_j^{WW'} = a_j^{VW'}.$$

Define the mapping

$$L \rightarrow L_0$$

over  $W_0 \times \Delta$  by

$$(21) \quad (x, v) \mapsto \left( x_0, \exp \left( \sum_{W'} \rho_{W'_0}(x_0) \left( \sum_{j>0} a_j^{W'W'}(x_0) t^j \right) \right) \cdot v \right).$$

This map is well defined since, over  $V \cap W$  we have

$$g^{VW}(x) = g^{V_0W_0}(x_0) \exp \left( \sum_{j>0} a_j^{VW}(x_0) t^j \right),$$

and so

$$\begin{aligned} & g^{VW}(x) \cdot \exp \left( \sum_{W'} \rho_{W'_0}(x_0) \left( \sum_{j>0} a_j^{W'W'}(x_0) t^j \right) \right) \\ &= g^{V_0W_0}(x_0) \exp \left( \sum_{j>0} a_j^{VW}(x_0) t^j \right) \\ & \quad \cdot \exp \left( \sum_{W'} \rho_{W'_0}(x_0) \left( \sum_{j>0} a_j^{W'W'}(x_0) t^j \right) \right) \\ &= g^{V_0W_0}(x_0) \exp \left( \sum_{W'} \rho_{W'_0}(x_0) \sum_{j>0} \left( a_j^{VW} + a_j^{W'W'} \right) (x_0) t^j \right) \\ &= g^{V_0W_0}(x_0) \exp \left( \sum_{W'} \rho_{W'_0}(x_0) \sum_{j>0} a_j^{VW'}(x_0) t^j \right). \end{aligned}$$

Referring to Lemma 2.3 our deformation/trivialization (19) is given by

$$\xi_j \in A^{0,1} \left( T_{L_0^\vee} \right)$$

for which

$$(22) \quad L_\chi \xi_j = L_{\bar{\chi}} \xi_j = 0.$$

We call a trivialization satisfying (19)-(22) a *trivialization of line bundles*. We say that the trivializations  $\lambda$  of  $L^\vee/\Delta$  and  $\sigma$  of  $X/\Delta$  are *compatible* if they make the diagram (19) commutative. By an elementary computation in local coordinates, sections

$$\xi_i \in A_{L_0^\vee}^{0,1} \otimes T_{L_0^\vee}$$

associated to a trivialization of line bundles lie in a subspace

$$A \subseteq A_{L_0^\vee}^{0,1} \otimes T_{L_0^\vee}$$

comprising the the middle term of an exact sequence

$$(23) \quad 0 \rightarrow q_0^{-1} \left( A_{X_0}^{0,1} \right) \otimes_{\mathbb{C}} \mathbb{C}\chi \rightarrow A \rightarrow q_0^{-1} \left( A_{X_0}^{0,1} \otimes T_{X_0} \right) \rightarrow 0,$$

that is,

$$(24) \quad A = A_{X_0}^{0,1}(\mathfrak{D}_1(L_0)).$$

Notice that the first form

$$\xi_1 \in A_{X_0}^{0,1}(\mathfrak{D}_1(L_0))$$

must be  $\bar{\partial}$ -closed by the integrability conditions in Lemma 2.4. Its cohomology class in

$$H^1(\mathfrak{D}_1(L_0))$$

is the first-order deformation of the pair  $(X_0, L_0)$  given by (17) (see [1]). Its symbol is just the element of  $H^1(T_{X_0})$  giving the Kodaira-Spencer class for the compatible first-order deformation of the manifold  $X_0$ .

**Lemma 3.1.** *i) If  $X_0$  is a Riemann surface  $C_0$ , the space of all (formal) deformation/trivializations of the pair (curve, line bundle) taken modulo holomorphic isomorphisms over  $\Delta$ , is naturally the space of power series in  $t$  with coefficients  $\xi_i \in A_{C_0}^{0,1}(\mathfrak{D}_1(L_0))$ .*

*ii) In general, a (formal) holomorphic section of  $L$  is a power series*

$$s = \sum_i t^i s_i$$

*with coefficients  $s_i$  which are  $C^\infty$ -sections of  $L_0$  such that*

$$\sum_{i=0}^{\infty} (\bar{\partial}s_i) t^i - \sum_{i=0, j=1}^{\infty, \infty} \xi_j(s_i) t^{i+j} = 0.$$

*iii) Suppose*

$$f \in H^0(L)$$

*has divisor  $D$  such that*

$$D_0 = D \cdot X_0$$

*is smooth and reduced. Then there is a trivialization*

$$F_\sigma : X \rightarrow X_0 \times \Delta$$

*such that*

$$\sigma^{-1}(D_0) = D,$$

*and a unique  $\sigma$ -compatible trivialization*

$$F_\lambda : L^\vee \rightarrow L_0^\vee \times \Delta$$



such that

$$f = f_0 \circ \lambda$$

where

$$f_0 = f|_{X_0}.$$

We call the trivialization  $F_\lambda$  adapted to the section  $f$ .

*Proof.* i) By (23) and Lemma 2.4 all integrability conditions vanish automatically.

ii) is immediate from Lemma 2.3.

iii) Let  $N$  be a tubular neighborhood of  $D_0$  in  $X$ . On  $N$  use a partition-of-unity argument as in §5 of [3] to construct a  $C^\infty$ -retraction

$$v : N \rightarrow N \cap D_0$$

such that each fiber is an analytic polydisk. Cover  $N$  as above by coordinate disks  $\{W\}$ . For each  $W_0 = W \cap X_0$  which meets  $D_0$  construct a *holomorphic* projection

$$v^{-1}(W_0 \cap D_0) \rightarrow W_0,$$

which takes

$$(W \cap D) \rightarrow (W_0 \cap D_0).$$

Again as in §5 of [3], use a  $C^\infty$ -partition-of-unity argument to “average” these local projections to obtain a projection

$$\varkappa : N \rightarrow N \cap X_0$$

such that

$$v \circ \varkappa = v$$

and such that, for each  $x_0 \in D_0$ ,

$$\varkappa|_{v^{-1}(x_0)}$$

is holomorphic.  $\varkappa$  gives a projection  $\sigma$  in some neighborhood  $D$  such that

$$D = \sigma^{-1}(D_0).$$

Extend by a partition of unity argument to obtain  $\sigma : X \rightarrow X_0$  and the corresponding trivialization  $F_\sigma = (\sigma, \pi)$ .

Now let

$$L_0 = \mathcal{O}_{X_0}(D_0),$$

and suppose  $D$  is given by local defining functions. Then, on each slice

$$v^{-1}(x_0),$$

$x_0 \in D_0$ , the invertible holomorphic functions

$$\frac{z_W}{z_{W_0} \circ \sigma}$$

fit together to give an invertible  $C^\infty$ -function on  $W \subset N$  so that

$$h_W := \frac{z_W \circ F_\sigma^{-1}}{z_{W_0}}$$

is an invertible  $C^\infty$ -function on  $W_0 \times \Delta$ . If  $W \not\subseteq N$  put

$$h_W = 1.$$

So for patching data

$$\begin{aligned} g^{WW'}(x) &= \sum g_i^{WW'}(x_0) t^i \\ &= g^{W_0 W'_0}(x_0) \exp\left(\sum_{j>0} a_j^{WW'}(x_0) t^j\right) \end{aligned}$$

we have

$$\sum_{j>0} a_j^{WW'}(x_0) t^j = \log h_{W'} - \log h_W.$$

The  $\sigma$ -compatible trivialization  $F_\lambda$  constructed in (21) is given in this case by

$$(x, v) \mapsto \left(x_0, \exp\left(\sum_{W'} \rho_{W'_0}(x_0) (\log h_{W'} - \log h_W)\right) \cdot v\right).$$

So, under this trivialization,  $z_W$  corresponds to the section of  $L_0^\vee \times \Delta$  given over  $(x_0, t) \in W_0 \times \Delta$  by  $(v, t)$  where

$$\begin{aligned} v &= \frac{z_{W_0}}{z_W \circ F_\sigma^{-1}} \exp\left(\sum_{W'} \rho_{W'_0}(x_0) (\log h_{W'})\right) \cdot (z_W \circ F_\sigma^{-1}) \\ &= \exp\left(\sum_{W'} \rho_{W'_0}(x_0) (\log h_{W'})\right) \cdot z_{W_0}. \end{aligned}$$

Now replace  $\lambda$  with

$$\frac{\lambda}{\exp\left(\sum_{W'} \rho_{W'_0}(x_0) (\log h_{W'})\right) \cdot \lambda}.$$

q.e.d.

### 3.3 Schiffer-type deformations of line bundles

We next wish to consider a very special type of line bundle deformation. Our aim is to be able to apply Lemmas 2.7-2.8 to a case in which  $M_0 = L_0^\vee$  is the total space of a line bundle and the holomorphic functions under consideration are the holomorphic sections of  $L_0$ . Let  $X_\beta/\Delta$  be a Schiffer-type deformation as in 2.2. That is, referring to (5), suppose that  $X_\beta/\Delta$  is given by Kuranishi data

$$\xi_{X_0}^\beta = \frac{e^{L_\beta} - 1}{L_\beta} ([\bar{\partial}, L_\beta])$$

for some divisor

$$A_0 \subseteq X_0.$$

Let

$$A_\beta$$

denote the extension of  $A_0$  to a divisor on  $X_\beta$ .

Let  $L/X_\beta$  be a deformation of  $L_0/X_0$ . By Lemma 2.9 and (21) there are compatible trivializations

$$\begin{aligned} F_\beta & : X_\beta \rightarrow X_0 \times \Delta, \\ F_\lambda & : L^\vee \rightarrow L_0^\vee \times \Delta. \end{aligned}$$

We need that  $F_\lambda = F_{\tilde{\beta}}$  for some lifting  $\tilde{\beta}$  of  $\beta$  to a vector field on  $L_0^\vee \times \Delta$  for which

$$[\tilde{\beta}, \chi] = 0.$$

**Lemma 3.2.** *i) Suppose that  $L_0$  is trivial over a neighborhood of  $A_0$  and that the mapping*

$$H^0 \left( \frac{\mathcal{O}_{X_0}(\infty \cdot A_0)}{\mathcal{O}_{X_0}} \right) \rightarrow H^1(\mathcal{O}_{X_0})$$

*induced by the exact sequence*

$$0 \rightarrow \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X_0}(\infty \cdot A_0) \rightarrow \frac{\mathcal{O}_{X_0}(\infty \cdot A_0)}{\mathcal{O}_{X_0}} \rightarrow 0$$

*is surjective. Then there is a lifting*

$$\tilde{\beta}$$

of  $\beta$  to a vector field on  $L_0^\vee \times \Delta$  which is meromorphic above

$$A_0$$

and otherwise  $C^\infty$  such that  $F_{\tilde{\beta}}$  is a trivialization of  $L^\vee$ .

ii) Referring to i), suppose that

$$L = \mathcal{O}_{X_\beta}(D)$$

and

$$\Phi \circ F_\beta(D) = D_0 \times \Delta,$$

where  $D_0$  is the zero-scheme associated to a holomorphic section

$$f_0 : L_0^\vee \rightarrow \mathbb{C}.$$

Suppose further that  $\Phi$  is holomorphic in a neighborhood of  $A_0 \times \Delta$ . Then there is a lifting  $\tilde{\Phi}$  of  $\Phi$  so that the section

$$f_0 \circ \tilde{\Phi} \circ F_{\tilde{\beta}}$$

is a holomorphic section of  $L$ .

*Proof.* i) Since  $L$  is trivial near  $A_0$ , we can lift  $\beta$  to a vector field  $\tilde{\beta}$  commuting with  $\chi$  and meromorphic near  $A_0$  by a patching argument as in 3.2. Any two liftings differ by a vector field

$$a\chi = \sum_{j>0} a_j \chi t^j,$$

where the  $a_j$  are functions on  $X_0$  which are meromorphic near  $A_0$  and  $C^\infty$  elsewhere. Given that modulo  $t^n$

$$(25) \quad L^\vee = L_{\tilde{\beta}}^\vee$$

we use the surjectivity hypothesis in the statement of the lemma to choose  $a_{n+1}$  and achieve (25) modulo  $t^{n+1}$ .

ii) The deformation  $X_\beta$  is trivial in a neighborhood of  $A_0 \times \Delta$ , so we can choose a lifting  $\tilde{\Phi}'$  of  $\Phi$  which is holomorphic near  $A_0 \times \Delta$  and extend by a partition-of-unity argument. Referring to Lemma 3.1iii),  $\tilde{\Phi}' \circ F_{\tilde{\beta}} = (\sigma', \pi)$  and the adapted trivialization  $(\sigma, \pi)$  are related by

$$\sigma = e^b \sigma'$$

for some  $C^\infty$ -function  $b$  on  $X_0 \times \Delta$ . Now set

$$\tilde{\Phi} = e^b \tilde{\Phi}'.$$

q.e.d.

### 3.4 Differential operators and basepoint-free systems

Suppose now that

$$H^0(L_0)$$

is basepoint-free. Let  $\mathbb{P}_0 = \mathbb{P}(H^0(L_0))$  and let

$$\begin{aligned} \nu &: \mathbb{P}_0 \times X_0 \rightarrow \mathbb{P}_0, \\ \rho &: \mathbb{P}_0 \times X_0 \rightarrow X_0 \end{aligned}$$

be the two projections. Let

$$\tilde{L}_0(1) = \nu^* \mathcal{O}_{\mathbb{P}_0}(1) \otimes \rho^* L_0.$$

Then by the Leray spectral sequence there are natural isomorphisms

$$(26) \quad \begin{aligned} \rho_* \tilde{L}_0(1) &= L_0 \otimes H^0(L_0)^\vee, \\ H^k(\tilde{L}_0(1)) &= H^k(L_0) \otimes H^0(L_0)^\vee. \end{aligned}$$

There is a tautological section

$$(27) \quad \tilde{f}_0 \in H^0(\tilde{L}_0(1)) = H^0(L_0) \otimes H^0(L_0)^\vee = \text{End}(H^0(L_0))$$

given by the identity map on  $H^0(L_0)$ . Furthermore

$$(28) \quad \rho_* \left( \tilde{f}_0 \right)$$

is given by the tautological homomorphism

$$H^0(L_0) \otimes \mathcal{O}_{X_0} \rightarrow L_0.$$

Also one easily shows by induction using the Euler sequence that

$$H^i(\mathfrak{D}_n(\mathcal{O}_{\mathbb{P}_0}(1))) = 0$$

for all  $i > 0$ , so also

$$R^i \rho_* \mathfrak{D}_n(\tilde{L}_0(1)) = 0$$

and

$$H^1(\mathfrak{D}_n(\tilde{L}_0(1))) = H^1(\rho_* \mathfrak{D}_n(\tilde{L}_0(1))).$$

There is a natural map

$$h : \rho_* \mathfrak{D}_n(\tilde{L}_0(1)) \rightarrow \mathfrak{D}_n(\rho_* \tilde{L}_0(1))$$

and

$$\begin{aligned}\mathfrak{D}_n(\rho_*\tilde{L}_0(1)) &= \mathfrak{D}_n(L_0 \otimes H^0(L_0)^\vee) \\ &= \mathfrak{D}_n(L_0) \otimes \text{End}(H^0(L_0)).\end{aligned}$$

Now via the trace map we have a canonical splitting

$$\text{End}(H^0(L_0)) = \mathbb{C} \cdot 1 \oplus \text{End}^0(H^0(L_0)),$$

where  $\text{End}^0$  denotes trace-zero endomorphisms. Notice that

$$\mathfrak{D}'_n := h\left(\rho_*\mathfrak{D}_n(\tilde{L}_0(1))\right) = \mathfrak{D}_n(L_0) \otimes 1 \oplus \mathfrak{D}_{n-1}(L_0) \otimes \text{End}^0(H^0(L_0))$$

so we have that

$$\mathfrak{D}'_0 = \mathcal{O}_{X_0}$$

and we have the exact sequence

$$(29) \quad \begin{array}{c} 0 \rightarrow \mathfrak{D}'_n \rightarrow \mathfrak{D}'_{n+1} \\ \xrightarrow{\text{symbol}} (S^{n+1}(T_{X_0}) \otimes 1) \oplus (S^n(T_{X_0}) \otimes \text{End}^0(H^0(L_0))) \rightarrow 0 \end{array}$$

is exact.

### 3.5 Extendable linear systems

Suppose now that the assumptions of 3.4 and Lemma 3.2 continue to hold and that, for some line bundle extension

$$L/X_\beta$$

of  $L_0/X_0$ ,

$$\pi_*(L)$$

is locally free over  $\Delta$ . Let

$$\mathbb{P}_{\cong} = \mathbb{P}_\Delta(\pi_*(L)).$$

Let

$$\begin{aligned}D_0 &\subseteq \mathbb{P}_0 \times X_0, \\ D &\subseteq \mathbb{P}_\Delta \times_\Delta X_\beta\end{aligned}$$

be the incidence divisors for the respective linear systems. The section  $\tilde{f}_0$  of  $\tilde{L}_0(1)$  defined in (27) has divisor  $D_0$ . For each holomorphic trivialization

$$(30) \quad \begin{aligned} T : \mathbb{P}_\Delta &\rightarrow \mathbb{P}_0 \times \Delta, \\ \tilde{T} : \mathcal{O}_{\mathbb{P}_\Delta}(-1) &\rightarrow \mathcal{O}_{\mathbb{P}_0}(-1) \times \Delta, \end{aligned}$$

let

$$\begin{aligned} F_T &: \mathbb{P}(\tilde{L}^\vee(-1)) \rightarrow \mathbb{P}(\tilde{L}_0^\vee(-1)) \times \Delta, \\ \tilde{F}_T &: \tilde{L}^\vee(-1) \rightarrow \tilde{L}_0^\vee(-1) \times \Delta \end{aligned}$$

be the trivializations induced by  $\tilde{T}$  and the trivializations

$$\begin{aligned} F_\beta &: X_\beta \rightarrow X_0 \times \Delta, \\ F_{\tilde{\beta}} &: L^\vee \rightarrow L_0^\vee \times \Delta \end{aligned}$$

defined in Lemma 3.2. We denote the infinitesimal automorphism of

$$\tilde{L}_0^\vee(-1) \times \Delta = (\rho^* L_0^\vee \otimes \nu^* \mathcal{O}_{\mathbb{P}_0}(-1)) \times \Delta$$

induced by  $\tilde{\beta}$  on  $\rho^* L_0^\vee$  and the identity on  $\nu^* \mathcal{O}_{\mathbb{P}_0}(-1)$  as

$$\tilde{\beta}_T.$$

If we change  $\tilde{T}$  in (30) to  $\tilde{T}'$ , then

$$(31) \quad e^{-L_{\tilde{\beta}_{T'}}} = e^{-L_{\tilde{\beta}_T}} (\tilde{T}' \circ \tilde{T}^{-1})^*.$$

Next, since

$$F_T(D) \cdot (\mathbb{P}_0 \times \{x_0\})$$

is a hyperplane for each  $x_0 \in X_0$  by basepoint-freeness, we can build a  $C^\infty$ -diffeomorphism

$$\Phi_T : (\mathbb{P}_0 \times X_0) \times \Delta \rightarrow (\mathbb{P}_0 \times X_0) \times \Delta$$

such that

$$\Phi_T(F_T(D)) = D_0 \times \Delta,$$

and such that the diagram

$$\begin{array}{ccc} \mathbb{P}_\Delta \times_\Delta X_\beta & \xrightarrow{\Phi_T \circ F_T} & (\mathbb{P}_0 \times X_0) \times \Delta \\ \downarrow & & \downarrow \\ X_\beta & \xrightarrow{F_\beta} & X_0 \times \Delta \end{array}$$

is commutative, where the vertical maps are the standard projections and such that the restriction of  $\Phi_T \circ F_T$  to each fiber of the left-hand projection is a linear automorphism of projective spaces. Further we can suppose that  $\Phi_T$  is holomorphic over  $F_\beta^{-1}(U_0 \times \Delta)$ . Let

$$G_T = \Phi_T \circ F_T.$$

Then, by Lemma 3.1iii), there is a  $G_T$ -compatible trivialization

$$\tilde{G}_T = \tilde{\Phi}_T \circ \tilde{F}_T : \tilde{L}^\vee(-1) \rightarrow \tilde{L}_0^\vee(-1) \times \Delta$$

such that

$$\tilde{f} = \tilde{f}_0 \circ \tilde{G}$$

is a holomorphic section of  $\tilde{L}^\vee(1)$  with divisor  $D$ . Since the restriction of  $\tilde{\Phi}_T$  to fibers of  $\rho$  are assumed to be a holomorphic automorphism of  $\mathcal{O}(-1)$  on the corresponding projective spaces, the infinitesimal automorphism of  $\tilde{L}_0^\vee(-1) \times \Delta$  is given by

$$\tilde{\gamma}_T \in \rho^{-1} A_{X_0}^0 \left( \mathfrak{gl} \left( (L_0^\vee)^{\oplus(r+1)} \right) \right) \otimes \mathbb{C}[[t]].$$

Thus, by Lemma 2.9,

$$(32) \quad \left[ \bar{\partial}, e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}} \right] (\tilde{f}_0) = 0.$$

Notice that, if

$$e^{\tilde{A}} : \mathcal{O}_{\mathbb{P}_0}(-1) \times \Delta \rightarrow \mathcal{O}_{\mathbb{P}_0}(-1) \times \Delta$$

is any holomorphic automorphism, we also have

$$(34) \quad \left[ \bar{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}} \right] (\tilde{f}_0) = 0.$$

Next, using (29) and (34), we need to analyze the elements

$$\begin{aligned} \left[ \bar{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}} \right] &\in \sum_{n>0} H^1(\mathfrak{D}'_n) t^n \\ &= \sum_{n>0} H^1(\mathfrak{D}_n(L_0)) \otimes \text{End}(H^0(L_0)) t^n. \end{aligned}$$

Applying this element to  $\rho_* \tilde{f}_0$ , by (34) we obtain that

$$(35) \quad \begin{aligned} \rho_* \left[ \bar{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}} \right] (\rho_* \tilde{f}_0) \\ = 0 \in \sum_{n>0} \text{Hom}(H^0(L_0), H^1(L_0)) t^n. \end{aligned}$$



**Theorem 3.3.** *Suppose that all assumptions of 3.3-3.5, in particular, the hypotheses of Lemma 3.2, hold. Suppose further that, by varying of  $\beta$  in 3.3 in such a way that all these assumptions continue to hold, the coefficients to  $t^{n+1}$  in all expressions*

$$[\bar{\partial}, e^{-L\beta}]$$

generate  $H^1(S^{n+1}(T_{X_0}))$  for each  $n \geq 0$ . (For example we allow the divisor  $A_0$  to move.) Then the maps

$$\mu^{n+1} : H^1(S^{n+1}T_{X_0}) \rightarrow \frac{Hom(H^0(L_0), H^1(L_0))}{image \tilde{\mu}^n}$$

are zero for all  $n \geq 0$ .

*Proof.* Let

$$\rho_* \left[ \bar{\partial}, e^{\tilde{A}} \circ e^{-L\tilde{\beta}_T} \circ e^{-L\tilde{\gamma}_T} \right]_n$$

denote the coefficient of  $t^n$ . Notice that the operators in  $e^{\tilde{A}}$  and  $e^{-L\tilde{\gamma}_T}$  are 0-th order operators so that, referring to (29), there is an element

$$\sigma_n^0 \in S^n(T_{X_0}) \otimes End^0(H^0(L_0))$$

such that

$$symbol \left( \left( \rho_* \left[ \bar{\partial}, e^{-L\tilde{\beta}_T} \circ e^{-L\tilde{\gamma}_T} \right] \right)_{n+1} \right)$$

is given by a formula

$$(36) \quad (\bar{\partial}\beta_1^n \otimes 1) \oplus \sigma \in S^{n+1}(T_{X_0}) \oplus (S^n(T_{X_0}) \otimes End^0(H^0(L_0))),$$

where

$$\beta = \sum_{j>0} \beta_j t^j.$$

Let

$$A_1 \in End^0(H^0(L_0))$$

be such that

$$e^{\tilde{A}} = 1 + A_1 t + \dots$$

Then

$$(37) \quad symbol \left( \rho_* \left[ \bar{\partial}, e^{\tilde{A}} \circ e^{-L\tilde{\beta}_T} \circ e^{-L\tilde{\gamma}_T} \right]_{n+1} - \rho_* \left[ \bar{\partial}, e^{-L\tilde{\beta}_T} \circ e^{-L\tilde{\gamma}_T} \right]_{n+1} \right) \\ = 0 + (\bar{\partial}\beta_1^n \otimes A_1).$$

Using (36) and (37) and the hypothesis that the elements

$$\bar{\partial}\beta_1^{n+1}$$

generate

$$H^1(S^{n+1}T_{X_0}),$$

by varying  $\beta$  and  $A_1$  we have that the elements

$$\text{symbol} \left( \rho_* \left[ \bar{\partial}, e^{\tilde{A}} \circ e^{-L\tilde{\beta}_T} \circ e^{-L\tilde{\gamma}_T} \right]_{n+1} \right)$$

generate

$$S^{n+1}(T_{X_0}) \oplus (S^n(T_{X_0}) \otimes \text{End}^0(H^0(L_0)))$$

for each  $n \geq 0$ .

Thus, by (29) and (34), the map  $\tilde{\nu}^{n+1}$  given by

$$\begin{aligned} H^1(\mathfrak{D}_{n+1}(\tilde{L}_0(1))) &\rightarrow \frac{H^1(\tilde{L}_0(1))}{\text{image}(\tilde{\nu}^n)} \\ D &\mapsto D(\tilde{f}_0) \end{aligned}$$

is zero for all  $n \geq 0$ . Moreover, in particular the image of

$$H^1(\mathfrak{D}_{n+1}(\tilde{L}_0(1))) \rightarrow \sum_{n>0} H^1(\tilde{L}_0(1)) t^n$$

lies inside the image of the map

$$(38) \quad \begin{aligned} \sum_{n>0} H^1(\mathcal{O}_{\mathbb{P}_0 \times X_0}) t^n &\rightarrow \sum_{n>0} H^1(\tilde{L}_0(1)) t^n \\ \alpha &\mapsto \alpha(\tilde{f}_0). \end{aligned}$$

But, under the identifications in 3.4,

$$\rho_*(\tilde{f}_0)$$

is the tautological map

$$H^0(L_0) \otimes \mathcal{O}_{X_0} \rightarrow L_0.$$

So applying  $\rho_*$  (38) becomes

$$\sum_{n>0} H^1(\mathcal{O}_{X_0}) t^n \rightarrow \sum_{n>0} \text{Hom}(H^0(L_0), H^1(L_0)) t^n.$$

Since

$$\rho_* [\bar{\partial}, e^{-L_{\beta_T}}] = [\bar{\partial}, e^{-L_{\beta}}] \otimes 1,$$

we conclude that the map

$$[\bar{\partial}, e^{-L_{\beta}}] \otimes H^0(L_0) \rightarrow \sum_{n>0} Hom(H^0(L_0), H^1(L_0)) t^n$$

takes values in the image of (38). So finally the hypothesis that the elements

$$\bar{\partial}\beta_1^{n+1}$$

generate  $H^1(S^{n+1}(T_{X_0}))$  gives the theorem. q.e.d.

#### 4. Brill-Noether theory

In this last section we give a simple application of Theorem 3.3 to Brill-Noether theory. From now on we assume that  $X_0$  is a *generic* compact Riemann surface  $C_0$ . We choose

$$A_0 = \{x_0\}$$

in §2-3 where  $x_0$  is a general point of  $C_0$ , and let

$$C_{\beta}/\Delta$$

denote the family of Schiffer-type deformation associated to some vector field

$$\beta = \sum_{j>0} \beta_j t^j,$$

where each  $\beta_j$  is meromorphic with poles in some neighborhood  $U_0$  of  $x_0$ . Since  $C_0$  is generic, there exists a line-bundle deformation

$$L/C_{\beta}$$

such that

$$H^0(L) \rightarrow H^0(L_0)$$

is surjective. We wish to apply Theorem 3.3 to conclude that the maps  $\mu^{n+1}$  are all zero for  $n \geq 0$ .

**Lemma 4.1.** *Let  $\beta_1$  range over all vector fields such that the Kodaira-Spencer class*

$$\bar{\partial}\beta_1$$

generates the kernel of the map

$$H^1(T_{C_0}) \rightarrow H^1(T_{C_0}(x_0)).$$

Then the elements

$$\bar{\partial}(\beta_1^{k+1})$$

generate the kernel of the map

$$H^1(S^{k+1}T_{C_0}) \rightarrow H^1(S^{k+1}T_{C_0}((k+1)x_0)).$$

*Proof.* Let  $z$  be a local analytic coordinate for  $C_0$  centered on  $x_0$ . We trivialize our Schiffer-type variation of  $C_0$  so that

$$\beta_1 = \frac{\rho}{z} \frac{\partial}{\partial z},$$

where  $\rho$  is a  $C^\infty$ -function on  $C_0$  such that

- i)  $\rho$  is supported on an arbitrarily small neighborhood of  $x_0$ ,
- ii) in a smaller neighborhood  $U_0$  of  $x_0$ ,

$$\rho = \frac{a_{-1}}{z} + a_0 + \dots + a_k z^k.$$

So

$$\bar{\partial} \left( \frac{\rho}{z} \right)^{k+1} \left( \frac{\partial}{\partial z} \right)^{k+1}$$

represents the symbol of  $[\bar{\partial}, L_{\beta_1}^{k+1}]$ . By varying the choice of the  $a_i$  in the definition of  $\rho$  we can therefore obtain symbols which generate the image of

$$\frac{S^{k+1}T_{C_0}((k+1)x_0)}{S^{k+1}T_{C_0}}$$

in  $H^1(S^{k+1}T_{C_0})$ . q.e.d.

Now if  $x_0$  varies over a dense subset of  $C_0$ , the elements of kernel of

$$H^1(S^{k+1}T_{C_0}) \rightarrow H^1(S^{k+1}T_{C_0}(x_0))$$

generate  $H^1(S^{k+1}T_{C_0})$ . So we conclude by Theorem 3.3:

**Theorem 4.2.** *If  $C_0$  is a curve of general moduli and  $H^0(L_0)$  is basepoint-free, the mapping*

$$\mu^{k+1} : H^1(S^{k+1}T_{C_0}) \rightarrow \frac{Hom(H^0(L_0), H^1(L_0))}{\sum_{k' \leq k} image \tilde{\mu}^{k'}}$$

given in 3.1 must be the zero map for  $k \geq 0$ .

To see that Petri's conjecture follows from Theorem 4.2, we reason as in §9 of [2]. Namely we consider the dual mappings

$$\mu_k : \ker \mu_{k-1} \rightarrow H^0(\omega_{C_0}^{k+1})$$

(inductively defined beginning with the zero map

$$\mu_{-1} : H^0(L_0) \otimes H^0(\omega_{C_0} \otimes L_0^\vee) \rightarrow \{0\}.$$

Petri's conjecture asserts that, for our  $C_0$  of general moduli, the mapping

$$\mu_0 : H^0(L_0) \otimes H^0(\omega_{C_0} \otimes L_0^\vee) \rightarrow H^0(\omega_{C_0}),$$

which is of course simply the multiplication map, is injective. To see that this follows from Theorem 4.2, let  $\{s_i\}$  denote a basis for  $H^0(L_0)$ . Suppose now that

$$\mu_0 \left( \sum s_i \otimes t_i \right) = 0.$$

Then the element

$$\sum (ds_i)t_i \in H^0(\omega_{C_0}^2)$$

is well-defined, giving the mapping  $\mu_1$ , etc. Since, by Theorem 4.2, successive maps  $\mu_k$  are the zero map we have, for any local trivialization of  $L_0$  and local coordinate  $z$  near a general point  $x_0$  on  $C_0$ , the local system of (pointwise) equations

$$\sum_i t_i(x_0) \frac{d^k s_i}{dz^k}(x_0) = 0$$

for all  $k$ , which is clearly impossible unless all the  $t_i(x_0)$  are zero.

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