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PROOF OF THE PROJECTIVE LICHNEROWICZ-OBATA CONJECTURE

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Abstract

We prove that if a connected Lie group action on a complete Riemannian manifold preserves the geodesics (considered as unparameterized curves), then the metric has constant positive sectional curvature, or the group acts by affine transformations.

1. Introduction

1.1. Results.

Definition 1. Let (M_1^n, g_1) and (M_2^n, g_2) be smooth Riemannian manifolds.

A diffeomorphism $F : M_1^n \to M_2^n$ is called **projective** if it takes the unparameterized geodesic of g_1 to geodesics of g_2. A projective diffeomorphism of a Riemannian manifold is called a **projective transformation**.

A diffeomorphism $F : M_1^n \to M_2^n$ is called **affine** if it takes the Levi-Civita connection of g_1 to the Levi-Civita connection of g_2. An affine diffeomorphism of a Riemannian manifold is called an **affine transformation.**

Theorem 1 (Projective Lichnerowicz Conjecture)**.** *Let a connected Lie group G act on a complete connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by affine transformations, or g has constant positive sectional curvature.*

The Lie groups of affine transformations of complete Riemannian manifolds are well understood (see, for example, [**40**]). Suppose a connected Lie group acts on a simply-connected complete Riemannian manifold (M^n, g) by affine transformations. Then, there exists a Riemannian decomposition

$$(M^n, g) = (M_1^{n_1}, g_1) + (\mathbb{R}^{n_2}, g_{\text{euclidean}})$$

of the manifold into the direct sum of a Riemannian manifold $(M_1^{n_1}, g_1)$ and Euclidean space $(\mathbb{R}^{n_2}, g_{\text{euclidean}})$ such that the group acts componentwise. More specifically, it acts on $(M_1^{n_1}, g_1)$ and $(\mathbb{R}^{n_2}, g_{\text{euclidean}})$ by

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isometries and by compositions of linear transformations and parallel translations, respectively. In particular, every connected Lie group of affine transformations of a closed manifold consists of isometries [**86**]. Thus, a direct consequence of Theorem 1 is the following

Corollary 1 (Projective Obata Conjecture)**.** *Let a connected Lie group G act on a closed connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by isometries, or g has constant positive sectional curvature.*

Any connected simply-connected Riemannian manifold of constant positive sectional curvature is a round sphere. All projective transformations of the round sphere are known (essentially, since Beltrami [**4**]); in this view, Theorem 1 closes the theory of non-isometric infinitesimal projective transformations of complete manifolds.

1.2. History. The theory of projective transformations has a long and fascinating history. The first non-trivial examples of projective transformations were discovered by Beltrami [**4**]. We describe their natural multi-dimensional generalization. Consider the sphere

$$S^n \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_{n+1}) \in R^{n+1} : \ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

with the restriction of the Euclidean metric. Next, consider the mapping $a : S^n \to S^n$ given by $a : v \mapsto \frac{A(v)}{\|A(v)\|}$, where A is an arbitrary non-degenerate linear transformation of R^{n+1}.

The mapping is clearly a diffeomorphism taking geodesics to geodesics. Indeed, the geodesics of g are great circles (the intersections of planes that go through the origin with the sphere). Since A is linear, it takes planes to planes. Since the normalization $w \mapsto \frac{w}{\|w\|}$ takes punctured planes to their intersections with the sphere, a takes great circles to great circles. Thus, a is a projective transformation. Evidently, if A is not proportional to an orthogonal transformation, a is not affine.

Beltrami investigated some examples of projective transformations. One of the first important papers on smooth families of projective transformations is due to Lie (see [**43**]). He formulated the problem of finding metrics (on surfaces) whose groups of projective transformations are bigger than the groups of isometries (the Lie Problem according to Fubini), and solved it assuming that the groups are big enough. For complete manifolds, this problem was formulated in Schouten [**67**].

The local theory of projective transformations was well understood thanks to efforts of several mathematicians including Dini [**17**], Schur [**68**], Levi-Civita [**39**], Fubini [**21**], Eisenhart [**19**], Cartan [**12**], Weyl [**82**], Solodovnikov [**71**], Sinjukov [**69**], Aminova [**2**, **3**], Mikes [**61**] und Shandra [**70**]. We will recall their results in Theorems 7,8,9,10.

The basic philosophical idea behind these results can be described as follows (see, for example, [**81**]): the Universe can be explained by its infinitesimal structure, and this infinitesimal structure is invariant with respect to a group of transformations.

Weyl studied projective transformations at the tensor level and found a number of tensor reformulations. He constructed the so-called projective Weyl tensor W [**82**], which is invariant with respect to projective transformations. We will recall the definition of W in Section 2.3 and use it in Section 3.5. E. Cartan [**12**], T. Y. Thomas [**76**], J. Douglas [**18**] and A. Lichnerowicz et al [**42**] studied groups of projective transformations at the level of affine connections, sprays and natural Hamiltonian systems. They introduced the so-called projective connection and Thomas projective parameters, which are invariant with respect to projective transformations.

Theorem 1 and Corollary 1 are known in mathematical folklore as the Lichnerowicz and Obata conjectures, respectively, although Lichnerowicz and Obata never formulated them explicitly. They were formulated as "well known classical conjectures" in several papers (see, for example, [**63**, **84**, **24**]).

Perhaps the name "Lichnerowicz-Obata conjecture" appeared because of the similarity with the conformal Lichnerowicz conjecture (proved by Obata [**64**], Alekseevskii [**1**], Ferrand [**20**] and Schoen [**66**]).

Recall that in the time of Lichnerowicz and Obata, projective and conformal transformations were studied by the same people and methods (see, for example, [**14**, **87**]). A reason for this is that the tensor equations for conformal and projective infinitesimal transformations are very similar.

Projective transformations were extremely popular objects of study in the 50s–80s. One of the reasons for that is their possible applications in physics (see, for example, [**65**, **16**]). One may consult the surveys [**61**] (more geometric one) and [**3**] (from the viewpoint of physics), which contain more than 500 references.

Most results on projective transformations require additional geometric assumptions written as tensor equations. For example, Corollary 1 was proven under the assumption that the metric is Einstein [**13**], Kähler [**13**], Ricci-flat [**62**], has negatively definite Ricci curvature [**83**] or has constant scalar curvature [**84**].

An important result which does not require additional tensor assumptions is due to Solodovnikov [**75**]. He proved the Lichnerowicz conjecture under the assumptions that

- the dimension of the manifold is greater than two and
- that all objects (the metric, the manifold, the projective transformations) are real-analytic.

The statement itself is in [**75**], but the technique was mostly developed in [**71**, **72**, **73**, **74**]. In Section 2.3 we will review the results of Solodovnikov (mostly from [**71**]) that we use to prove Theorem 1. We will also use certain results from [**71**] in Section 3.5.

Both of the assumptions are important for Solodovnikov's methods. His technique is based on a very accurate analysis of the behavior of the curvature tensor under projective transformations and completely fails in dimension two (compare Theorem 9 and Examples 1,2 from [**58**]). In addition, real analyticity is also important for his methods. Indeed, all of his global statements are based on it.

In dimension 2, Theorem 1 was announced in [**55**, **57**] and proved in [**58**]. The technique for proving Theorem 1 in dimension two is not applicable for dimensions greater than two. It probably should be mentioned that our proof of the Lichnerowicz-Obata conjecture in dimensions greater than two does not work in dimension two. The reason is that Theorems 8, 9, 10 are wrong in dimension two, and therefore all the results of Sections 3.2, 3.4, 3.5 are not applicable in the proof of Theorem 16. Note that Theorem 16 is still true in dimension two, see [**58**], but its proof in dimension two uses essentially different methods. It is based on Kolokoltsov's and Igarashi-Kiyohara-Sugahara's description of quadratically integrable geodesics flows on complete surfaces, see [**33**, **34**, **26**, **29**, **30**, **9**].

The new techniques which made it possible to prove the Lichnerowicz-Obata conjecture were introduced in [**45**, **77**, **78**, **49**, **48**]. The main observation is that the existence of projective diffeomorphisms allows one to construct commuting integrals for the geodesic flow (see Theorem 5 in Section 2.2). This technique has been used quite successfully for describing the topology of closed manifold admitting non-homothetic projective diffeomorphisms [**47**, **52**, **50**, **53**, **54**, **36**, **59**].

1.3. Counterexamples to Theorem 1 if one of the assumptions is omitted. All assumptions in Theorem 1 are important.

If the Lie group is not connected, a counterexample to Theorem 1 is possible only if the group is discrete. In this case, a counterexample exists already in dimension two. Consider the torus $T^2 := \mathbb{R}^2/_{\mathbb{Z}^2}$ with the standard coordinates $x, y \in (\mathbb{R} \bmod 1)$ and a positive smooth nonconstant function $f : (\mathbb{R} \bmod 1) \to \mathbb{R}$ such that the metric

$$ds^2 := \left(f(x) - \frac{1}{f(y)}\right)\left(\sqrt{f(x)}\, dx^2 + \frac{1}{\sqrt{f(y)}}\, dy^2\right)$$

is positive definite. Then, the diffeomorphism $F : T^2 \to T^2$ given by $F(x, y) := (y, x)$ takes the original metric to the metric

$$\left(f(y) - \frac{1}{f(x)}\right)\left(\frac{\sqrt{f(x)}}{f(x)}\, dx^2 + \frac{f(y)}{\sqrt{f(y)}}\, dy^2\right).$$

Hence, it is a projective transformation by Levi-Civita's Theorem 7.

If the manifold is not complete, a counterexample is as follows (essentially, it was constructed in [**43**]). The method to construct this example and generalisations of this example to all dimensions is explained at the end of Section 4.1.

The metric (x, y are coordinates on $\mathbb{R}^2$, C is a constant) is given by

$$\frac{\exp(Cx)}{(\exp(x)+\exp(-x))^2}dx^2 + \frac{\exp(-x)}{(\exp(x)+\exp(-x))}dy^2.$$

The projective vector field (= vector field whose flow takes geodesics to geodesics) is $v := (1, \frac{1}{2}Cy)$. It is easy to check that the vector field is complete, i.e., it generates projective transformations, and that it is neither a Killing nor an affine vector field.

If we allow the manifold to have more than one connected component, one can construct non-interesting counterexamples as follows. The first component is the round sphere, where the group GL acts by projective transformations as in Beltrami's example. The other components, on which the group GL acts identically, are manifolds of nonconstant curvature.

If the manifold is one-dimensional, every diffeomorphism is a projective transformation and only homotheties are affine transformations.

The next example shows that, even on a complete manifold, the existence of a non-affine projective vector field does not imply that the metric has constant sectional curvature. The example comes from the following observation. If the geodesic flow of a two-dimensional metric g admits three independent integrals, such that one of them is linear in velocities, the second is quadratic in velocities, and the third is the energy integral, then one can construct two projective vector fields. Indeed, by [**45**, **46**], the existence of the quadratic integral allows us to construct a metric $\bar{g}$ projectively equivalent to g. The linear integral gives us a Killing vector field v for g, which allows us to construct a Killing vector field $\bar{v}$ for $\bar{g}$, see [**31**, **78**]. Since $\bar{v}$ preserved the geodesics of $\bar{g}$, it preserved the geodesics of g, i.e. it is a projective vector field for g. The assumption that the integrals are independent insures that the vector fields $\bar{v}$ and v are linearly independent.

The first examples of the metrics such that their geodesic flows admit such three integrals are known since Königs [**35**]. We will give only one example. Other examples can be constructed by the algorithm described above by using explicit formulas for the Königs metrics, which could be found for example in [**27**, **28**].

The metric is (γ is a positive constant, x, y are standard coordinates on $\mathbb{R}^2$)

$$(x^2 + y^2 + \gamma)(dx^2 + dy^2).$$

The independent integrals (in the standard coordinates (x, y, p_x, p_y) on $T^*\mathbb{R}^2$)) are

$$\begin{aligned} H &:= \frac{{p_x}^2 + {p_y}^2}{x^2 + y^2 + \gamma}, \\ F_1 &:= x p_y - y p_x \\ F_2 &:= \frac{{p_x}^2 y^2 - \left(x^2 + \gamma\right) {p_y}^2}{x^2 + y^2 + \gamma} \\ F_3 &:= xyH - p_x p_y. \end{aligned}$$

And the projective vector fields are

$$\begin{aligned} v_1 &:= (y, -x) \\ v_2 &:= \left((x^2 + \gamma)y, y^2 x)\right) \\ v_3 &:= \left((x(y^2 - x^2 - \gamma), y(y^2 + \gamma - x^2)\right). \end{aligned}$$

(The first vector field is the Killing vector field corresponding to the integral F_1.)

Note that the vector fields v_2, v_3 are not complete and generate no diffeomorphism of $\mathbb{R}^2$. In fact, they can not be complete by Theorem 1.

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2. Preliminaries: BM-structures, integrability, and Solodovnikov's $V(K)$ spaces

The goal of this section is to introduce the classical as well as new tools, which will be used in the proof of Theorem 1. In Sections 2.1, 2.2, we introduce the notion of "BM-structure" and explain its relations to projective transformations and integrability; these are new instruments of the proof. In Section 2.3, we formulate in a convenient form classical results of Beltrami, Weyl, Levi-Civita, Fubini, de Vries and Solodovnikov. We will extensively use these results in Sections 3 and 4.

2.1. BM-structure. Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$.

Definition 2. A **BM-structure** on (M^n, g) is a smooth self-adjoint $(1,1)$-tensor L such that for every point $x \in M^n$ and vectors $u, v, w \in T_x M^n$ the following equation holds:

$$g((\nabla_u L)v, w) = \frac{1}{2} g(v, u) \cdot d\mathrm{trace}_L(w) + \frac{1}{2} g(w, u) \cdot d\mathrm{trace}_L(v), \tag{1}$$

where trace_L is the trace of L.

The set of all BM-structures on (M^n, g) is denoted by $\mathcal{B}(M^n, g)$. It is a linear vector space, whose dimension is at least one, since the identity tensor $\mathrm{Id} \stackrel{\mathrm{def}}{=} \mathrm{diag}(1, 1, 1, \ldots, 1)$ is always a BM-structure.

The equation (1) appeared independently in the theory of projectively equivalent metrics (see [**2**] or Chapter 3 in [**69**]), and in the theory of integrable geodesic flows (see [**25**], [**15**], or [**37**]). In this and in the next sections we describe important cross-relations between these three notions.

Definition 3. Let g, $\bar{g}$ be Riemannian metrics on M^n. They are **projectively equivalent**, if they have the same (unparameterized) geodesics.

The relation between BM-structures and projectively equivalent metrics is given by

Theorem 2 ([**69**, **10**])**.** *Let g be a Riemannian metric. Suppose L is a self-adjoint positive-definite $(1,1)$-tensor. Consider the metric $\bar{g}$ defined by*

$$\bar{g}(\xi, \eta) = \frac{1}{\det(L)} g(L^{-1}(\xi), \eta) \tag{2}$$

for every tangent vector ξ and η with the common foot point.

Then, the metrics g and $\bar{g}$ are projectively equivalent, if and only if L is a BM-structure on (M^n, g).

An equivalent form of this theorem is

Corollary 2 ([**10**])**.** *Let g, $\bar{g}$ be Riemannian metrics on M^n.*

Then, they are projectively equivalent, if and only if the tensor L defined by

$$L^i_j \stackrel{\mathrm{def}}{=} \left(\frac{\det(\bar{g})}{\det(g)}\right)^{\frac{1}{n+1}} \sum_{\alpha=1}^{n} \bar{g}^{i\alpha} g_{\alpha j} \tag{3}$$

is a BM-structure on (M^n, g).

A one-parameter group of projective transformations of (M^n, g) gives us a one parameter family of BM-structures, whose derivative is also a BM-structure:

Theorem 3 (Infinitesimal version of Theorem 2)**.** *Let F_t, where $t \in \mathbb{R}$, be a smooth one-parameter family of projective transformations of (M^n, g). Consider the $(1,1)$-tensor A given by*

$$A^i_j \stackrel{\mathrm{def}}{=} \sum_{\alpha=1}^{n} g^{i\alpha} \left(\mathcal{L}g\right)_{\alpha j}, \tag{4}$$

where $\mathcal{L}g$ denotes the Lie derivative with respect to F_t (so that $(\mathcal{L}g)_{\alpha j} = -\frac{d}{dt}\left((F_t^* g)_{\alpha j}\right)_{|t=0}$) and g^{ij} is the inverse of g_{ij}. Then, $A - \frac{1}{n+1}\mathrm{trace}_A \cdot \mathrm{Id}$ is a BM-structure on (M^n, g).

Proof. For every t, let us denote by g_t the pull-back $F_t^* g$. Fix a point $x \in M^n$ and a coordinate system in $T_x M^n$. Then, we can view g_t and g as matrices. Clearly, $g_0 = g$. Since F_t consists of projective transformations, by Corollary 2 for every $t \in \mathbb{R}$ the tensor

$$L_t \stackrel{\mathrm{def}}{=} \left(\frac{\det(g_t)}{\det(g)}\right)^{\frac{1}{n+1}} g_t^{-1} g$$

satisfies the equation

$$g((\nabla_u L_t)v, w) = \frac{1}{2} g(v,u) \cdot d\mathrm{trace}_{L_t}(w) + \frac{1}{2} g(w,u) \cdot d\mathrm{trace}_{L_t}(v) \tag{5}$$

for every $u, v, w \in T_x M^n$. Differentiating this equation by t and substituting $t = 0$, we obtain

$$\begin{aligned} & g\left(\left(\nabla_u \left(\frac{d}{dt} L_t\right)_{|t=0}\right) v, w\right) \\ & = \frac{1}{2} g(v,u) \cdot d\mathrm{trace}_{\left(\frac{d}{dt}L_t\right)_{|t=0}}(w) + \frac{1}{2} g(w,u) \cdot d\mathrm{trace}_{\left(\frac{d}{dt}L_t\right)_{|t=0}}(v), \end{aligned}$$

so that $\left(\frac{d}{dt}L_t\right)_{|t=0}$ is a BM-structure on M^n. Now, let us calculate it.

$$\begin{aligned} \left(\frac{d}{dt} L_t\right)_{|t=0} &= \left(\frac{\det(g_t)}{\det(g)}\right)^{\frac{1}{n+1}}_{|t=0} \left(\frac{d}{dt} g_t^{-1}\right)_{|t=0} g \\ &\quad + \left(\frac{d}{dt}\left(\frac{\det(g_t)}{\det(g)}\right)^{\frac{1}{n+1}}\right)_{|t=0} \left(g_t^{-1} g\right)_{|t=0} \\ &= \left(\frac{d}{dt} g_t^{-1}\right)_{|t=0} g + \left(\frac{d}{dt}\left(\frac{\det(g_t)}{\det(g)}\right)^{\frac{1}{n+1}}\right)_{|t=0} \mathrm{Id} \\ &= -\left(g_t^{-1}\right)_{|t=0} \left(\frac{d}{dt} g_t\right)_{|t=0} \left(g_t^{-1}\right)_{|t=0} g \\ &\quad + \frac{1}{n+1}\left(\frac{d}{dt}\frac{\det(g_t)}{\det(g)}\right)_{|t=0} \mathrm{Id} \\ &= g^{-1}\mathcal{L}g - \frac{1}{n+1}\mathrm{trace}_{g^{-1}\mathcal{L}g}\mathrm{Id}. \end{aligned}$$

Thus, $L := A - \frac{1}{n+1}\mathrm{trace}_A \mathrm{Id}$ is a BM-structure. Theorem 3 is proven. For use in future we recall one more property of BM-structures:

Theorem 4 ([**10**],[**51**]). *The Nijenhuis torsion of a BM-structure vanishes.*

2.2. Integrals for geodesic flows of metrics admitting BM-structure. Objects similar to BM-structures on Riemannian manifolds appear quite often in the theory of integrable systems (see, for example [**5**, **6**, **7**, **25**, **15**]). The relation between BM-structures and integrable geodesic flows is observed on the level of projective equivalence in [**45**], on the level of projective transformations in [**79**] and is as follows:

Let L be a self-adjoint $(1,1)$-tensor on (M^n, g). Consider the family S_t of $(1,1)$-tensors

$$S_t \stackrel{\text{def}}{=} \det(L - t\ \mathrm{Id})\,(L - t\ \mathrm{Id})^{-1}, \quad t \in \mathbb{R}. \tag{6}$$

Remark 1. Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor S_t is well-defined for every t. Moreover, S_t is a polynomial in t of degree $n-1$ with coefficients being (1,1)-tensors.

We will identify the tangent and cotangent bundles of M^n by g. This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n.

Theorem 5 ([**77**, **45**, **78**, **79**])**.** *If L is a BM-structure, then, for every $t_1, t_2 \in \mathbb{R}$, the functions*

$$I_{t_i} : TM^n \to \mathbb{R}, \quad I_{t_i}(v) \stackrel{\text{def}}{=} g(S_{t_i}(v), v) \tag{7}$$

are commuting integrals for the geodesic flow of g.

Since L is self-adjoint, its eigenvalues are real. Denote by $\lambda_1(x) \le \dots \le \lambda_n(x)$ the eigenvalues of L at each $x \in M^n$. λ_i are continuous functions on the manifold. They are smooth at typical points, see Definition 4 below.

Corollary 3. *Let (M^n, g) be a Riemannian manifold such that every two points can be connected by a geodesic. Suppose L is a BM-structure on (M^n, g). Then, for every $i \in \{1, \dots, n-1\}$, for every $x, y \in M^n$, the following statements hold:*

1) $\lambda_i(x) \le \lambda_{i+1}(y)$.
2) *If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

A slightly different version of this corollary was proven in [**54**, **78**]. The proof will be recalled in Section 3.1, to be used in what follows.

At every point $x \in M^n$, denote by $N_L(x)$ the number of different eigenvalues of the BM-structure L at x.

Definition 4. A point $x \in M^n$ will be called **typical** with respect to the BM-structure L, if

$$N_L(x) = \max_{y \in M^n} N_L(y).$$

Corollary 4. *Let L be a BM-structure on a connected Riemannian manifold (M^n, g). Then, almost every point of M^n is typical with respect to L.*

Proof. Consider points $x, y \in M^n$ such that x is typical. Our goal is to prove that almost every point in a small ball around y is typical as well. Consider a path $\gamma \in M^n$ connecting x and y. For every point $z \in \gamma$, there exists $\epsilon_z > 0$ such that the open ball with center in z and radius ϵ_z is convex. Since γ is compact, the union of a finite number of such balls contains the whole path γ. Therefore, there exists a finite sequence of convex balls $B_1, B_2, \dots, B_m$ such that

- B_1 contains x.
- B_m contains y.
- For every $i = 1, \dots, m-1$, the intersection $B_i \bigcap B_{i+1}$ is not empty.

Since the balls are convex, every two points of every ball can be connected by a geodesic. Using that for a fixed i the set $\{x \in M^n : \lambda_i(x) < \lambda_{i+1}(x)\}$ is evidently open, by Corollary 3, almost every point of B_1 is typical. Then, there exists a typical point in the ball B_2. Hence, almost all points of B_2 are typical. Applying this argument $m-2$ times, we obtain that almost all points of B_m are typical. Corollary 4 is proven.

2.3. Projective Weyl tensor, Beltrami Theorem, Levi-Civita's Theorem and Solodovnikov's $V(K)$-metrics. Let g be a Riemannian metric on M^n of dimension n. Let R^i_{jkl} and R_{ij} be the curvature and the Ricci tensors of g. The tensor

$$W^i_{jkl} := R^i_{jkl} - \frac{1}{n-1}\left(\delta^i_l\, R_{jk} - \delta^i_k\, R_{jl}\right) \tag{8}$$

is called the *projective Weyl* tensor.

Theorem 6 ([**82**]). *Consider Riemannian metrics g and $\bar g$ on M^n. Then, the following statements hold:*

1) *If the metrics are projectively equivalent, then their projective Weyl tensors coincide.*
2) *Assume $n \geq 3$. The projective Weyl tensor of g vanishes if and only if the sectional curvature of g is constant.*

Corollary 5 (Beltrami Theorem [**44**, **68**, **60**]). *If there exists a projective diffeomorphism between two Riemannian manifolds, and if one of them has constant sectional curvature, then so does the other.*

Formally speaking, Corollary 5 follows from Theorem 6 for dimensions greater than two only. For dimension two, Corollary 5 was proven by Beltrami himself in [**4**].

Corollary 6. *Let $F : M_1^n \to M_2^n$ be a projective diffeomorphism between complete Riemannian manifolds (M_i^n, g_i), $i = 1, 2$, of dimension $n \geq 2$. Then, if g_1 has constant negative sectional curvature, F is a homothety. Moreover, if g_1 is flat, F is affine.*

Corollary 6 is a mathematical folklore. Unfortunately, we did not find a classical reference for it. If g_1 is flat, Corollary 6 can be found in every good textbook on linear algebra. If the curvature of g_1 is negative, under the assumption that the dimension is two, Corollary 6 was proven in [**11**]. The case of arbitrary dimension trivially follows from the two-dimensional case, since in every two-dimensional direction there exists a totally geodesic complete submanifold.

In view of Theorem 2, the next theorem is equivalent to the classical Levi-Civita's Theorem from [**39**].

Theorem 7 (Levi-Civita's Theorem)**.** *The following statements hold:*

1) *Let L be a BM-structure on (M^n, g). Let $x \in M^n$ be typical. Then, there exists a coordinate system $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ in a neighborhood $U(x)$ containing x, where $\bar{x}_i = (x_i^1, \dots, x_i^{k_i})$, $(1 \le i \le m)$, such that L is diagonal*

$$\mathrm{diag}(\underbrace{\phi_1, \dots, \phi_1}_{k_1}, \dots, \underbrace{\phi_m, \dots, \phi_m}_{k_m}), \tag{9}$$

and the quadratic form of the metric g is given by

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) = \ & P_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \ P_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ & + P_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{10}$$

where $A_i(\bar{x}_i, \dot{\bar{x}}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar{x}}_i$ with coefficients depending on $\bar{x}_i$,

$$P_i \stackrel{\text{def}}{=} (\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \cdots (\phi_m - \phi_i),$$

and $\phi_1 < \phi_2 < \ldots < \phi_m$ are smooth functions such that

$$\phi_i = \begin{cases} \phi_i(\bar{x}_i), & \text{if} \quad k_i = 1, \\ \text{constant}, & \text{if} \quad k_i > 1. \end{cases}$$

2) *Let g be a Riemannian metric and L be a $(1,1)$-tensor. If in a neighborhood $U \subset M^n$ there exist coordinates $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ such that L and g are given by formulas* (9), (10), *then the restriction of L to U is a BM-structure for the restriction of g to U.*

Remark 2. In Levi-Civita's coordinates from Theorem 7, the metric $\bar{g}$ given by (2) has the form

$$\begin{aligned} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) = \ & \rho_1 P_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho_2 P_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ & + \rho_m P_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{11}$$

where

$$\rho_i = \frac{1}{\phi_1^{k_1} \cdots \phi_m^{k_m}} \frac{1}{\phi_i}.$$

The metrics g and $\bar{g}$ are affine equivalent (i.e., they have the same Levi-Civita connections) if and only if all functions ϕ_i are constant.

Let p be a typical point with respect to the BM-structure L. Fix $i \in 1, \dots, m$ and a small neighborhood U of p. At every point of U, consider the eigenspace V_i with the eigenvalue ϕ_i. If the neighborhood is small enough, it contains only typical points and V_i is a distribution. By Theorem 4, it is integrable. Denote by $M_i(p)$ the integral manifold containing p.

Levi-Civita's Theorem says that the eigenvalues ϕ_j, $j \neq i$, are constant on $M_i(p)$, and that the restriction of g to $M_i(p)$ is proportional to the restriction of g to $M_i(q)$, if it is possible to connect q and p by a line orthogonal to M_i and containing only typical points. Actually, in view of [**23**], the first observation follows already from Theorem 4. We will need the second observation later and formulate it in the form of the following

Corollary 7. *Let L be a BM-structure for a connected Riemannian manifold (M^n, g). Suppose the curve $\gamma : [0,1] \to M^n$ contains only typical points and is orthogonal to $M_i(p)$ at every point $p \in Image(\gamma)$. Let the multiplicity of the eigenvalue ϕ_i at every point of the curve be greater than one. Then, the restriction of the metric to $M_i(\gamma(0))$ is proportional to the restriction of the metric to $M_i(\gamma(1))$ (i.e., there exists a diffeomorphism of a small neighborhood $U_i(\gamma(0)) \subset M_i(\gamma(0))$ to a small neighborhood $U_i(\gamma(1)) \subset M_i(\gamma(1))$ taking the restriction of the metric g to $M_i(\gamma(0))$ to a metric proportional to the restriction of the metric g to $M_i(\gamma(1))$).*

Definition 5. Let (M^n, g) be a Riemannian manifold. We say that the metric g has a **warped decomposition** near $x \in M^n$ if a neighborhood U^n of x can be split in the direct product of disks $D^{k_0} \times \cdots \times D^{k_m}$, $k_0 + \cdots + k_m = n$, such that the metric g has the form

$$g_0 + \sigma_1 g_1 + \sigma_2 g_2 + \cdots + \sigma_m g_m, \tag{12}$$

where the ith metric g_i is a Riemannian metric on the corresponding disk D^{k_i}, and functions σ_i are functions on the disk D^{k_0}. The metric

$$g_0 + \sigma_1 dy_1^2 + \sigma_2 dy_2^2 + \cdots + \sigma_m dy_m^2 \tag{13}$$

on $D^{k_0} \times \mathbb{R}^m$ is called the **adjusted metric**.

We will always assume that k_0 is at least 1. Adjusted metric has a very clear geometric sense. Take a point $p = (p_0, \dots, p_m) \in D^{k_0} \times \cdots \times D^{k_m}$. At every disk D^{k_i}, $i = 1, \dots, m$ consider a geodesic segment $\gamma_i \in D^{k_i}$ passing through p_i. Consider now the product

$$M_A := D^{k_0} \times \gamma_1 \times \gamma_2 \times \cdots \times \gamma_m$$

as a submanifold of $D^{k_0} \times \cdots \times D^{k_m}$. As easily follows from Definition 5,

- M_A is a totally geodesic submanifold.
- The restriction of the metric (12) to M_A is (isometric to) the adjusted metric.

Comparing formulas (10,12), we see that if L has at least one simple eigenvalue at a typical point, Levi-Civita's Theorem gives us a warped decomposition near every typical point of M^n: the metric g_0 collects all $P_i A_i$ from (10) such that ϕ_i has multiplicity one, the metrics $g_1, \dots, g_m$ coincide with A_j for multiple ϕ_j, and $\sigma_j = P_j$.

Definition 6 ([**71**, **73**, **74**, **70**])**.** Let K be a constant. A metric g is called a $V(K)$**-metric** near $x \in M^n$ ($n \geq 3$), if there exist coordinates in a neighborhood of x such that g has the Levi-Civita form (10) for which the adjusted metric has constant sectional curvature K.

The definition above is independent of the choice of the Levi-Civita's form for g:

Theorem 8 ([**71**, **72**, **73**, **74**])**.** *Suppose g is a $V(K)$-metric near $x \in M^n$. Assume $n \geq 3$. The following statements hold:*

1) *If there exists another presentation of g (near x) in the form* (10), *then the sectional curvature of the adjusted metric constructed for this other decomposition is constant and is equal to K.*
2) *Consider the metric* (10). *For every $i = 1, \dots, m$, denote*

$$\frac{g(\mathrm{grad}\,(P_i), \mathrm{grad}\,(P_i))}{4\,P_i} + KP_i \tag{14}$$

by K_i. Then, the metric (12) *has constant sectional curvature if and only if for every $i \in 0, \dots, m$ such that $k_i > 1$ the metric A_i has constant sectional curvature K_i. More precisely, if the metric* (12) *is a $V(K)$-metric, if $k_1 > 1$ and if the metric A_1 has constant sectional curvature K_1, then the metric $g_0 + P_1 A_1$ has constant sectional curvature K.*
3) *For a fixed presentation of g in the Levi-Civita form* (10), *for every i such that $k_i > 1$, K_i is a constant.*

The first statement of Theorem 8 is proven in §3, §7 of [**71**]. In the form sufficient for our paper, it appeared already in [**80**]; although it is hidden there. The second and the third statements can be found, for example, in §8 of [**71**]. The relation between $V(K)$-metrics and BM-structures is given by

Theorem 9 ([**73**, **74**])**.** *Let (D^n, g) be a disk of dimension $n \geq 3$ with two BM-structures L_1 and L_2 such that every point of the disk is typical with respect to both structures and the BM-structures* Id, L_1, L_2 *are linearly independent. Assume that at least one eigenvalue of L_1 is simple.*

Then, g is a $V(K)$-metric near every point.

A partial case of this theorem is

Theorem 10 (Fubini's Theorem [**21**, **22**]). *Let (D^n, g) be a disk of dimension $n \geq 3$ with two BM-structures L_1 and L_2 such that $N_{L_1} = N_{L_2} = n$ at every point. If the BM-structures* Id, L_1, L_2 *are linearly independent, then g has constant curvature.*

3. Global theory of projectively equivalent metrics

The goal of this section is to provide necessary tools for Section 4.

3.1. The eigenvalues of a BM-structure are globally ordered. Within this section we assume that (M^n, g) is a Riemannian manifold such that every two points can be connected by a geodesic. Our goal is to prove Corollary 3. We need the following technical lemma. For every fixed $v = (\xi_1, \xi_2, \ldots, \xi_n) \in T_xM^n$, the function (7) is a polynomial in t. Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x,v) \leq t_2(x,v) \leq \cdots \leq t_{n-1}(x,v).$$

Lemma 1 ([**54**, **56**]). *The following holds for every $i \in \{1, \ldots, n-1\}$:*

1) *For every $v \in T_xM^n$,*

$$\lambda_i(x) \leq t_i(x,v) \leq \lambda_{i+1}(x).$$

In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$, then $t_i(x,v) = \lambda_i(x) = \lambda_{i+1}(x)$.

2) *If $\lambda_i(x) < \lambda_{i+1}(x)$, then for every $\tau \in \mathbb{R}$ the Lebesgue measure of the set*

$$V_\tau \subset T_xM^n, \quad V_\tau \stackrel{\text{def}}{=} \{v \in T_xM^n : t_i(x,v) = \tau\},$$

is zero.

Proof. By definition, the tensor L is self-adjoint with respect to g. Then, for every $x \in M^n$, there exist "diagonal" coordinates in T_xM^n such that the metric g is given by the diagonal matrix $\mathrm{diag}(1, 1, \ldots, 1)$ and the tensor L is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then, the tensor (6) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(\Pi_1(t), \Pi_2(t), \ldots, \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\text{def}}{=} (\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_{i-1} - t)(\lambda_{i+1} - t)\cdots(\lambda_{n-1} - t)(\lambda_n - t).$$

Hence, for every $v = (\xi_1, \ldots, \xi_n) \in T_xM^n$, the polynomial $I_t(x,v)$ is given by

$$I_t = \xi_1^2\Pi_1(t) + \xi_2^2\Pi_2(t) + \cdots + \xi_n^2\Pi_n(t). \tag{15}$$

Evidently, the coefficients of the polynomial I_t depend continuously on the eigenvalues λ_i and on the components ξ_i. Then, it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that ξ_i are non-zero. For every $\alpha \neq i$, we evidently have $\Pi_\alpha(\lambda_i) \equiv 0$. Then,

$$I_{\lambda_i} = \sum_{\alpha=1}^{n} \Pi_\alpha(\lambda_i)\xi_\alpha^2 = \Pi_i(\lambda_i)\xi_i^2.$$

Hence $I_{\lambda_i}(x,v)$ and $I_{\lambda_{i+1}}(x,v)$ have different signs. Hence, the open interval $]\lambda_i, \lambda_{i+1}[$ contains a root of the polynomial $I_t(x,v)$. The degree of the polynomial I_t is equal $n-1$; we have $n-1$ disjoint intervals; every interval contains at least one root so that all roots are real and the ith root lies between λ_i and λ_{i+1}. The first statement of Lemma 1 is proven.

Let us prove the second statement of Lemma 1. Assume $\lambda_i < \lambda_{i+1}$. Suppose first $\lambda_i < \tau < \lambda_{i+1}$. Then, the set

$$V_\tau \stackrel{\text{def}}{=} \{v \in T_xM^n : t_i(x,v) = \tau\},$$

consists of the points v such that the function $I_\tau(x,v) \stackrel{\text{def}}{=} (I_t(x,v))_{|t=\tau}$ is zero. Then, V_τ is a nontrivial quadric in $T_xM^n \equiv \mathbb{R}^n$ and, hence, has zero measure.

Now suppose τ is an endpoint of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can assume $\tau = \lambda_i$. Let k be the multiplicity of the eigenvalue λ_i. Then, every coefficient $\Pi_\alpha(t)$ of the quadratic form (15) has the factor $(\lambda_i - t)^{k-1}$. Hence,

$$\hat{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is a polynomial in t and $\hat{I}_\tau$ is a nontrivial quadratic form. Evidently, for every point $v \in V_\tau$, we have $\hat{I}_\tau(v) = 0$ so that the set V_τ is a subset of a nontrivial quadric in T_xM^n and, hence, has zero measure. Lemma 1 is proven.

Proof of Corollary 3. The first statement of Corollary 3 follows immediately from the first statement of Lemma 1: Let us join the points $x, y \in M^n$ by a geodesic $\gamma : \mathbb{R} \to M^n$, $\gamma(0) = x$, $\gamma(1) = y$. Consider the one-parametric family of integrals $I_t(x,v)$ and the roots

$$t_1(x,v) \leq t_2(x,v) \leq \cdots \leq t_{n-1}(x,v).$$

By Theorem 5, every root t_i is constant on every orbit $(\gamma, \dot\gamma)$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)).$$

Using Lemma 1, we obtain

$$\lambda_i(\gamma(0)) \leq t_i(\gamma(0), \dot\gamma(0)), \quad \text{and} \quad t_i(\gamma(1), \dot\gamma(1)) \leq \lambda_{i+1}(\gamma(1)).$$

Thus $\lambda_i(\gamma(0)) \leq \lambda_{i+1}(\gamma(1))$ and the first statement of Corollary 3 is proven.

Let us prove the second statement of Corollary 3. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for every point y of some subset $V \subset M^n$. Then, λ_i is a constant on V (i.e., λ_i is independent of $y \in V$). Indeed, by the first statement of Corollary 3,

$$\lambda_i(y_0) \leq \lambda_{i+1}(y_1) \text{ and } \lambda_i(y_1) \leq \lambda_{i+1}(y_0),$$

so that $\lambda_i(y_0) = \lambda_i(y_1) = \lambda_{i+1}(y_1) = \lambda_{i+1}(y_0)$ for every $y_0, y_1 \in V$.

We denote this constant by τ. Let us join the point x with every point of V by all possible geodesics. Consider the set $V_\tau \subset T_xM^n$ of the initial velocity vectors (at the point x) of these geodesics.

By the first statement of Lemma 1, for every geodesic γ passing through at least one point of V, the value $t_i(\gamma, \dot\gamma)$ is equal to τ. By the second statement of Lemma 1, the measure of V_τ is zero. Since the set V lies in the image of the exponential mapping of V_τ, the measure of the set V is also zero. Corollary 3 is proven.

3.2. Local theory: behavior of BM-structure near non-typical points. Within this section we assume that L is a BM-structure on a connected (M^n, g). As in Section 2.2, we denote by $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$ the eigenvalues of L, and by $N_L(x)$ the number of different eigenvalues of L at $x \in M^n$.

Theorem 11. *Suppose the eigenvalue λ_1 is not constant, the eigenvalue λ_2 is constant and $N_L = 2$ in a typical point. Let p be a non-typical point. Then, the following statements hold:*

1) *The spheres of small radius with center in p are orthogonal to the eigenvector of L corresponding to λ_1, and tangent to the eigenspace of L corresponding to λ_2. In particular, the points q such that $\lambda_1(q) = \lambda_2$ are isolated.*
2) *For every sphere of small radius with center in p, the restriction of g to the sphere has constant sectional curvature.*

Proof. Since λ_1 is not constant, it is a simple eigenvalue in every typical point. Since $N_L = 2$, the roots $\lambda_2, \lambda_3, \ldots, \lambda_n$ coincide at every point and are constant. We denote this constant by λ. By Lemma 1, at every point $(x, \xi) \in T_xM^n$, the number λ is a root of multiplicity at least $n-2$ of the polynomial $I_t(x, \xi)$. Then,

$$I'_t(x,\xi) := \frac{I_t(x,\xi)}{(\lambda - t)^{n-2}}$$

is a linear function in t and, for every fixed t, is an integral of the geodesic flow of g. Denote by $\tilde I : TM \to \mathbb{R}$ the function

$$\tilde I(x,\xi) := I'_\lambda(x,\xi) := \left(I'_t(x,\xi)\right)_{|t=\lambda}.$$

Since λ is a constant, the function $\tilde{I}$ is an integral of the geodesic flow of g. At every tangent space T_xM^n, consider the coordinates such that the metric is given by $\mathrm{diag}(1, \dots, 1)$ and L is given by $\mathrm{diag}(\lambda_1, \lambda, \dots, \lambda)$. By direct calculations we see that the restriction of $\tilde{I}$ to T_xM^n is given by (we assume $\xi = (\xi_1, \xi_2, \dots, \xi_n)$)

$$\tilde{I}_{|T_xM^n}(\xi) = (\lambda_1(x) - \lambda)(\xi_2^2 + \cdots + \xi_n^2).$$

Thus, for every geodesic γ passing through p, the value of $\tilde{I}(\gamma(\tau), \dot{\gamma}(\tau))$ is zero. Then, for every typical point of such a geodesic, since $\lambda_1 < \lambda$, the components $\xi_2, \dots, \xi_n$ of the velocity vector vanish. Finally, the velocity vector is an eigenvector of L with the eigenvalue λ_1.

Then, the points where $\lambda_1 = \lambda$ are isolated: otherwise we can pick two such points p_1 and p_2 lying in a ball with radius less than the radius of injectivity. Then, for almost every point q of the ball, the geodesics connecting q with p_1 and p_2 intersect transversally at q. Then, the point q is non-typical; otherwise the eigenspace of λ_1 contains the velocity vectors of geodesics and is not one-dimensional. Finally, almost every point of the ball is not typical, which contradicts Corollary 4. Thus, the points where $\lambda_1 = \lambda$ are isolated.

It is known (Lemma of Gauß), that the geodesics passing through p intersect the spheres of small radius with center in p orthogonally. Since the velocity vectors of such geodesics are eigenvectors of L with eigenvalue λ_1, then the eigenvector with eigenvalue λ_1 is orthogonal to the spheres of small radius with center in p. Since L is self-adjoint, the spheres are tangent to the eigenspaces of λ. The first statement of Theorem 11 is proven.

The second statement of Theorem 11 is trivial, if $n = 2$. In order to prove the second statement for $n \geq 3$, we will use Corollary 7. The curve γ from Corollary 7 plays the geodesic passing through p. We put $i = 2$. By the first statement of Theorem 11, $M_i(x)$ are spheres with center in p. Then, by Corollary 7, for every sufficiently small sphere S_{ϵ_1} and S_{ϵ_2} with center in p, the restriction of g to the first sphere is proportional to the restriction of g to the second sphere. Since for very small values of ϵ the metric in a ϵ-ball is very close to the Euclidean metric, the restriction of g to the ϵ-sphere is close to the round metric of the sphere. Thus, the restriction of g to every (sufficiently small) sphere with center in p has constant sectional curvature. Theorem 11 is proven.

Theorem 12. *Suppose $N_L = 3$ at a typical point and there exists a point where $N_L = 1$. Then, the following statements hold:*

1) *There exist points p_1, p_n such that $\lambda_1(p_1) < \lambda_2(p_1) = \lambda_n(p_1)$ and $\lambda_1(p_n) = \lambda_2(p_n) < \lambda_n(p_n)$.*
2) *The points p such that $N_L(p) = 1$ are isolated.*

Proof. Let us prove the first statement. Suppose $\lambda_1(p_2) = \lambda_2(p_2) = \cdots = \lambda_n(p_2)$ and the number of different eigenvalues of L at a typical point equals three. Then, by Corollary 3, the eigenvalues $\lambda_2 = \cdots = \lambda_{n-1}$ are constant. We denote this constant by λ. Take a ball B of small radius with center in p_2. We will prove that this ball has a point p_1 such that $\lambda_1(p_1) < \lambda_2 = \lambda_n(p_1)$; the proof that there exists a point where $\lambda_1 = \lambda_2 < \lambda_n$ is similar. Take $p \in B$ such that $\lambda_1(p) < \lambda$ and $\lambda_1(p)$ is a regular value of the function λ_1. Denote by $\check{M}_1(p)$ the connected component of $\{q \in M^n : \ \lambda_1(q) = \lambda_1(p)\}$ containing the point p. Since $\lambda_1(p)$ is a regular value, $\check{M}_1(p)$ is a submanifold of codimension 1. Then, there exists a point $p_1 \in \check{M}_1(p)$ such that the distance from this point to p_2 is minimal over all points of $\check{M}_1(p)$.

Let us show that $\lambda_1(p_1) < \lambda = \lambda_n(p_1)$. The inequality $\lambda_1(p_1) < \lambda$ is fulfilled by definition, since $p_1 \in \check{M}_1(p)$. Let us prove that $\lambda_n(p_n) = \lambda$.

Consider the shortest geodesic γ connecting p_2 and p_1. We will assume $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Consider the values of the roots $t_1 \le \cdots \le t_{n-1}$ of the polynomial I_t at the points of the geodesic orbit $(\gamma, \dot\gamma)$. Since I_t are integrals, the roots t_i are independent of the point of the orbit. Since the geodesic passes through the point where $\lambda_1 = \cdots = \lambda_n$, by Lemma 1, we have

$$t_1 = \cdots = t_{n-1} = \lambda. \tag{16}$$

Since the distance from p_1 to p_2 is minimal over all points of $\check{M}_1$, the velocity vector $\dot\gamma(0)$ is orthogonal to $\check{M}_1$. By Theorem 4 and [**23**], the sum of eigenspaces of L corresponding to λ and λ_n is tangent to $\check{M}_1$. Hence, the vector $\dot\gamma(0)$ is an eigenvector of L with eigenvalue λ_1.

At the tangent space $T_{p_1}M^n$, choose a coordinate system such that L is diagonal $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and g is Euclidean $\mathrm{diag}(1, \dots, 1)$. In this coordinate system, $I_t(\xi)$ is given by (we assume $\xi = (\xi_1, \dots, \xi_n)$)

$$\begin{aligned}(\lambda - t)^{n-3}\Big((\lambda_n - t)(\lambda - t)\xi_1^2 \\ + (\lambda_n - t)(\lambda_1 - t)(\xi_2^2 + \cdots + \xi_{n-1}^2) + (\lambda_1 - t)(\lambda - t)\xi_n^2\Big).\end{aligned} \tag{17}$$

Since $\dot\gamma(0)$ is an eigenvector of L with eigenvalue λ_1, the last $n-1$ components of $\dot\gamma(0)$ vanish, so that $t_{n-1} = \lambda_n$. Comparing this with (16), we see that $\lambda_n(p_1) = \lambda$. The first statement of Theorem 12 is proven.

Now let us prove the second statement. We suppose that in a small convex ball $B \subset M^n$ there exist four points p', p'', p''', p'''' with $N_L = 1$, and will find a contradiction. By Corollary 4, almost every point p of the ball is typical. Clearly, for almost every typical point p of the ball the geodesics connecting the point with p', p'', p''', p'''' intersect mutually-transversally in p. Since these geodesics pass through points where $\lambda_1 = \cdots = \lambda_n$, by Lemma 1, the roots $t_1, t_2, \dots, t_{n-1}$ on the geodesics are all equal to λ. If the point p is typical, the restriction of I_t to T_pM^n has the form (17). Then, if $(\xi'_1,\dots,\xi'_n)$, $(\xi''_1,\dots,\xi''_n)$, $(\xi'''_1,\dots,\xi'''_n)$, $(\xi''''_1,\dots,\xi''''_n)$ are the coordinates of velocity vectors of the geodesics at p, the sums $((\xi'_2)^2+\cdots+(\xi'_{n-1})^2)$, $((\xi''_2)^2+\cdots+(\xi''_{n-1})^2)$, $((\xi'''_2)^2+\cdots+(\xi'''_{n-1})^2)$, $((\xi''''_2)^2+\cdots+(\xi''''_{n-1})^2)$ vanish and the following system of equations holds:

$$\left\{\begin{array}{rcl} (\lambda_n-\lambda)(\xi'_1)^2+(\lambda_1-\lambda)(\xi'_n)^2 &=& 0\\ (\lambda_n-\lambda)(\xi''_1)^2+(\lambda_1-\lambda)(\xi''_n)^2 &=& 0\\ (\lambda_n-\lambda)(\xi'''_1)^2+(\lambda_1-\lambda)(\xi'''_n)^2 &=& 0\\ (\lambda_n-\lambda)(\xi''''_1)^2+(\lambda_1-\lambda)(\xi''''_n)^2 &=& 0. \end{array}\right.$$

Thus, at least two of the vectors $(\xi'_1,\dots,\xi'_n)$ and $(\xi''_1,\dots,\xi''_n)$ are proportional. Then, there exists a pair of geodesics that do not intersect transversally at p. The contradiction shows that the points where $N_L=1$ are isolated. Theorem 12 is proven.

3.3. Splitting Lemma.

Definition 7. A **local-product structure** on M^n is the triple (h,B_r,B_{n-r}), where h is a Riemannian metric and B_r, B_{n-r} are transversal foliations of dimensions r and $n-r$, respectively (it is assumed that $1\le r<n$), such that every point $p\in M^n$ has a neighborhood $U(p)$ with coordinates

$$(\bar x,\bar y)=\big((x_1,x_2,\dots,x_r),(y_{r+1},y_{r+2},\dots,y_n)\big)$$

such that the x-coordinates are constant on every leaf of the foliation $B_{n-r}\cap U(p)$, the y-coordinates are constant on every leaf of the foliation $B_r\cap U(p)$, and the metric h is block-diagonal such that the first $(r\times r)$ block depends on the x-coordinates and the last $((n-r)\times(n-r))$ block depends on the y-coordinates.

A model example of manifolds with local-product structure is the direct product of two Riemannian manifolds (M_1^r,g_1) and (M_2^{n-r},g_2). In this case, the leaves of the foliation B_r are the products of M_1^r and the points of M_2^{n-r}, the leaves of the foliation B_{n-r} are the products of the points of M_1^r and M_2^{n-r}, and the metric h is the product metric g_1+g_2.

Below we assume that

(a) L is a BM-structure for a connected (M^n,g).
(b) There exists r, $1\le r<n$, such that $\lambda_r<\lambda_{r+1}$ at every point of M^n.

We will show that (under the assumptions (a,b)) we can naturally define a local-product structure (h,B_r,B_{n-r}) such that the (tangent spaces to) leaves of B_r and B_{n-r} are invariant with respect to L, and such that the restrictions $L_{|B_r}$, $L_{|B_{n-r}}$ are BM-structures for the metrics $h_{|B_r}$, $h_{|B_{n-r}}$, respectively.

At every point $x \in M^n$, denote by V_x^r the subspaces of $T_x M^n$ spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_1, \dots, \lambda_r$. Similarly, denote by V_x^{n-r} the subspaces of $T_x M^n$ spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_{r+1}, \dots, \lambda_n$. By assumption, for every i, j such that $i \le r < j$, we have $\lambda_i \ne \lambda_j$ so that V_x^r and V_x^{n-r} are two smooth distributions on M^n. By Theorem 4, the distributions are integrable so that they define two transversal foliations B_r and B_{n-r} of dimensions r and $n-r$, respectively.

By construction, the distributions V_r and V_{n-r} are invariant with respect to L. Let us denote by L_r, L_{n-r} the restrictions of L to V_r and V_{n-r}, respectively. We will denote by χ_r, χ_{n-r} the characteristic polynomials of L_r, L_{n-r}, respectively. Consider the (1,1)-tensor

$$C \stackrel{\text{def}}{=} ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

and the metric h given by the relation

$$h(u, v) \stackrel{\text{def}}{=} g(C^{-1}(u), v)$$

for every vector u, v. (In the tensor notations, the metrics h and g are related by $g_{ij} = h_{i\alpha} C_j^\alpha$.)

Lemma 2 (Splitting Lemma)**.** *The following statements hold:*

1) *The triple (h, B_r, B_{n-r}) is a local-product structure on M^n.*
2) *For every leaf of B_r, the restriction of L to it is a BM-structure for the restriction of h to it. For every leaf of B_{n-r}, the restriction of L to it is a BM-structure for the restriction of h to it.*

Proof. First of all, h is a well-defined Riemannian metric. Indeed, take an arbitrary point $x \in M^n$. At the tangent space to this point, we can find a coordinate system such that the tensor L and the metric g are diagonal. In this coordinate system, the characteristic polynomials χ_r, χ_{n-r} are given by

$$\begin{array}{lcr} (-1)^r \chi_r & = & (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_r) \\ \chi_{n-r} & = & (\lambda_{r+1}-t)(\lambda_{r+2}-t)\cdots(\lambda_n-t). \end{array} \tag{18}$$

Then, the (1,1)-tensor $C = ((-1)^r \chi_r(L) + \chi_{n-r}(L))$ is given by the diagonal matrices

$$\begin{aligned} \operatorname{diag}\Bigg(& \prod_{j=r+1}^{n} (\lambda_j - \lambda_1), \dots, \\ & \prod_{j=r+1}^{n} (\lambda_j - \lambda_r), \prod_{j=1}^{r} (\lambda_{r+1} - \lambda_j), \dots, \prod_{j=1}^{r} (\lambda_n - \lambda_j) \Bigg). \end{aligned} \tag{19}$$

We see that the tensor is diagonal and that all diagonal components are positive. Then, the tensor C^{-1} is well-defined and h is a Riemannian metric.

By construction, B_r and B_{n-r} are well-defined transversal foliations of dimensions r and $n-r$. In order to prove Lemma 2, we need to verify that, locally, the triple (h, B_r, B_{n-r}) is as in Definition 7, that the restriction of L to a leaf is a BM-structure for the restriction of h to the leaf.

It is sufficient to verify these two statements at almost every point of M^n. Indeed, since the foliations and the metric are globally-given and smooth, if the restrictions of g and L to a leaf satisfy Definitions 7, 2 at almost every point, then they satisfy Definitions 7, 2 at every point.

Thus, by Corollary 4, it is sufficient to prove the Splitting Lemma near every typical point. Consider Levi-Civita's coordinates $\bar{x}_1, \dots, \bar{x}_m$ from Theorem 7 near a typical point. As in Levi-Civita's Theorem, we denote by $\phi_1 < \dots < \phi_m$ the different eigenvalues of L. In Levi-Civita's coordinates, the matrix of L is diagonal

$$\operatorname{diag}\left(\underbrace{\phi_1, \dots, \phi_1}_{k_1}, \dots, \underbrace{\phi_m, \dots, \phi_m}_{k_m}\right) = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

Consider s such that $\phi_s = \lambda_r$ (clearly, $k_1 + \dots + k_s = r$). Then, by constructions of the foliations B_r and B_{n-r}, the coordinates $\bar{x}_1, \dots, \bar{x}_s$ are constant on every leaf of the foliation B_{n-r}, the coordinates $\bar{x}_{s+1}, \dots, \bar{x}_m$ are constant on every leaf of the foliation B_r. The coordinates $\bar{x}_1, \dots, \bar{x}_s$ will play the role of x-coordinates from Definition 7, and the coordinates $\bar{x}_{s+1}, \dots, \bar{x}_m$ will play the role of y-coordinates from Definition 7.

Using (19), we see that, in Levi-Civita's coordinates, C is given by

$$\begin{aligned}\operatorname{diag}\Bigg(&\underbrace{\prod_{j=s+1}^{m}(\phi_j - \phi_1)^{k_j}, \dots, \prod_{j=s+1}^{m}(\phi_j - \phi_1)^{k_j}}_{k_1}, \dots, \\ &\underbrace{\prod_{j=s+1}^{m}(\phi_j - \phi_s)^{k_j}, \dots, \prod_{j=s+1}^{m}(\phi_j - \phi_s)^{k_j}}_{k_s}, \\ &\underbrace{\prod_{j=1}^{s}(\phi_{s+1} - \phi_j)^{k_j}, \dots, \prod_{j=1}^{s}(\phi_{s+1} - \phi_j)^{k_j}}_{k_{s+1}}, \dots, \\ &\underbrace{\prod_{j=1}^{s}(\phi_m - \phi_j)^{k_j}, \dots, \prod_{j=1}^{s}(\phi_m - \phi_j)^{k_j}}_{k_m}\Bigg).\end{aligned}$$

Thus, h is given by

$$h(\dot{\bar{x}},\dot{\bar{x}}) = \tilde{P}_1 A_1(\bar{x}_1,\dot{\bar{x}}_1) + \cdots + \tilde{P}_s A_s(\bar{x}_s,\dot{\bar{x}}_s) + \tilde{P}_{s+1} A_{s+1}(\bar{x}_{s+1},\dot{\bar{x}}_{s+1}) + \cdots + \tilde{P}_m A_m(\bar{x}_m,\dot{\bar{x}}_m), \tag{20}$$

where the functions $\tilde{P}_i$ are as follows: for $i \le s$, they are given by

$$\tilde{P}_i \stackrel{\text{def}}{=} (\phi_i - \phi_1)\cdots(\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i)\cdots(\phi_s - \phi_i)\prod_{\substack{j=1\\ j\ne i}}^{s} |\phi_i - \phi_j|^{1-k_j}.$$

For $i > s$, the functions $\tilde{P}_i$ are given by

$$\tilde{P}_i \stackrel{\text{def}}{=} (\phi_i - \phi_{s+1})\cdots(\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i)\cdots(\phi_m - \phi_i)\prod_{\substack{j=s+1\\ j\ne i}}^{m} |\phi_i - \phi_j|^{1-k_j}.$$

Clearly, $|\phi_i - \phi_j|^{k_j-1}$ can depend on the variables $\bar{x}_i$ only. Then, the products

$$\prod_{\substack{j=1\\ j\ne i}}^{s} |\phi_i - \phi_j|^{1-k_j}\ , \qquad \prod_{\substack{j=s+1\\ j\ne i}}^{m} |\phi_i - \phi_j|^{1-k_j}$$

can be hidden in A_i, i.e., instead of A_i we consider

$$\tilde{A}_i \stackrel{\text{def}}{=} \prod_{\substack{j=1\\ j\ne i}}^{s} |\phi_i - \phi_j|^{1-k_j}\ A_i \quad \text{ for } i \le s \text{ and}$$

$$\tilde{A}_i \stackrel{\text{def}}{=} \prod_{\substack{j=s+1\\ j\ne i}}^{m} |\phi_i - \phi_j|^{1-k_j}\ A_i \quad \text{for } i > s.$$

Finally, the restriction of the metric to the leaves of B_r has the form from Levi-Civita's Theorem. Hence, the restriction of L is a BM-structure for it. We see that the leaves of B_r are orthogonal to leaves of B_{n-r}, and that the restriction of h to B_r (B_{n-r}, respectively) is precisely the first row of (20) (second row of (20), respectively) and depends on the coordinates $\bar{x}_1,\dots,\bar{x}_s$ ($\bar{x}_{s+1},\dots,\bar{x}_m$, respectively) only. Lemma 2 is proven.

Let p be a typical point with respect to the BM-structure L. Fix $i \in 1,\dots,n$. At every point of M^n, consider the eigenspace V_i with the eigenvalue λ_i. V_i is a distribution near p. Denote by $M_i(p)$ its integral manifold containing p.

Remark 3. The following statements hold:

1) If $\lambda_i(p)$ is multiple, the restriction of g to $M_i(p)$ is proportional to the restriction of h to $M_i(p)$.

2) The restriction of L to B_r does not depend on the coordinates $y_{r+1}, \dots, y_n$ (which are coordinates $\bar{x}_{s+1}, \dots, \bar{x}_m$ in the notations in proof of Lemma 2). The restriction of L to B_{n-r} does not depend on the coordinates $x_1, \dots, x_r$ (which are coordinates $\bar{x}_1, \dots, \bar{x}_s$ in the notations in proof of Lemma 2).

Combining Lemma 2 with Theorem 11, we obtain

Corollary 8. *Let L be BM-structure on connected (M^n, g). Suppose there exist $i \in 1, \dots, n$ and $p \in M^n$ such that:*

- *λ_i is multiple (with multiplicity $k \geq 2$) at a typical point.*
- *$\lambda_{i-1}(p) = \lambda_{i+k-1}(p) < \lambda_{i+k}(p)$.*
- *The eigenvalue λ_{i-1} is not constant.*

Then, for every typical point $q \in M^n$ which is sufficiently close to p, $M_i(q)$ is diffeomorphic to the sphere and the restriction of g to $M_i(q)$ has constant sectional curvature.

Indeed, take a small neighborhood of p and apply Splitting Lemma 2 two times: for $r = i + k - 1$ and for $r = i - 2$. We obtain a metric h such that locally, near p, the manifold with this metric is the Riemannian product of three disks with BM-structures, and BM-structure is the direct sum of these BM-structures. The second component of such decomposition satisfies the assumption of Theorem 11; applying Theorem 11 and Remark 3 we obtain what we need.

Arguing as above, combining Lemma 2 with Theorem 12, we obtain

Corollary 9. *Let L be a BM-structure on connected (M^n, g). Suppose the eigenvalue λ_i has multiplicity k at a typical point. Suppose there exists a point where the multiplicity of λ_i is greater than k. Then, there exists a point where the multiplicity of λ_i is precisely $k + 1$.*

Combining Lemma 2 with Corollary 4, we obtain

Corollary 10. *Let L be a BM-structure on connected (M^n, g). Suppose the eigenvalue λ_i has multiplicity $k_i \geq 2$ at a typical point and multiplicity $k_i + d$ at a point $p \in M^n$. Then, there exists a point $q \in M^n$ in a small neighborhood of p such that the eigenvalue λ_i has multiplicity $k_i + d$ in p, and such that*

$$N_L(q) = \max_{x \in M^n} (N_L(x)) - d.$$

We saw that under hypotheses of Theorems 11, 12, the set of typical points is connected. As it was shown in [**47**], in dimension 2 the set of typical points is connected as well. Combining these observations with Lemma 2, we obtain

Corollary 11. *Let L be a BM-structure on connected (M^n, g) of dimension $n \geq 2$. Then, the set of typical points of L is connected.*

3.4. If ϕ_i is not isolated, A_i has constant sectional curvature. In this section we assume that L is a BM-structure on a connected complete Riemannian manifold (M^n, g). As usual, we denote by $\lambda_1(x) \le \cdots \le \lambda_n(x)$ the eigenvalues of L at $x \in M^n$.

Definition 8. An eigenvalue λ_i is called **isolated**, if, for all points $p_1, p_2 \in M^n$ and for every $i, j \in \{1, \dots, n\}$, the equality $\lambda_i(p_1) = \lambda_j(p_1)$ implies $\lambda_i(p_2) = \lambda_j(p_2)$.

As in Section 3.3, at every point $p \in M^n$, we denote by V_i the eigenspace of L with the eigenvalue $\lambda_i(p)$. If p is typical, V_i is a distribution near p; by Theorem 4, it is integrable. We denote by $M_i(p)$ the connected component (containing p) of the intersection of the integral manifold with a small neighborhood of p.

Theorem 13. *Suppose λ_i is a non-isolated eigenvalue such that its multiplicity at a typical point is greater than one. Then, for every typical point p, the restriction of g to $M_i(p)$ has constant sectional curvature.*

It could be easier to understand this theorem using the language of Levi-Civita's Theorem 7: denote by $\phi_1 < \phi_2 < \cdots < \phi_m$ the different eigenvalues of L at a typical point. Theorem 13 says that if ϕ_i (of multiplicity ≥ 2) is non-isolated, then A_i from Levi-Civita's Theorem has constant sectional curvature.

Proof of Theorem 13. Let $k_i > 1$ be the multiplicity of λ_i at a typical point. Then, λ_i is constant. Take a typical point p. We assume that λ_i is not isolated; without loss of generality, we can suppose $\lambda_{i-1}(p_1) = \lambda_{i+k_i-1}(p_1)$ for some point p_1. By Corollary 9, without loss of generality, we can assume $\lambda_{i-1}(p_1) = \lambda_i(p_1) < \lambda_{i+k_i}(p_1)$. By Corollary 10, we can also assume that $N_L(p_1) = N_L(p) - 1$, so that multiplicity of $\lambda_i(p_1)$ is precisely $k_i + 1$.

Consider a geodesic segment $\gamma : [0, 1] \to M^n$ connecting p_1 and p, $\gamma(0) = p$ and $\gamma(1) = p_1$. Since it is sufficient to prove Theorem 13 at almost every typical point, without loss of generality, we can assume that p_1 is the only nontypical point of the geodesic segment $\gamma(\tau)$, $\tau \in [0, 1]$. More precisely, take $j \notin i-2, i, \dots, i+k_i-1$. If there exists a point $p_2 \in M^n$ such that $\lambda_j(p_2) < \lambda_{j+1}(p_2)$, then, by assumptions, $\lambda_j(p_1) < \lambda_{j+1}(p_1)$. Hence, by Lemma 1, for almost every $\xi \in T_{\gamma(1)}M^n$, we have $t_j(\gamma(1), \xi) \ne \max_{x \in M^n}(\lambda_j(x))$. Thus, almost all geodesics starting from p_1 do not contain points where $\lambda_j = \lambda_{j+1}$. Finally, for almost every $p \in M^n$, the geodesics connecting p and p_1 contain no points where $\lambda_j = \lambda_{j+1}$.

Take the point $q := \gamma(1 - \epsilon)$ of the segment, where $\epsilon > 0$ is small enough. By Corollary 8, the restriction of g to $M_i(q)$ has constant sectional curvature.

Let us prove that the geodesic segment $\gamma(\tau),\ \tau \in [0, 1-\epsilon]$ is orthogonal to $M_i(\gamma(\tau))$ at every point. Consider the function

$$\tilde I : TM^n \to \mathbb{R};\ \tilde I(x,\xi) := \left(\frac{I_t(x,\xi)}{(\lambda_i - t)^{k_i-1}}\right)_{|t=\lambda_i}.$$

Since the multiplicity of λ_i at every point is at least k_i, by Lemma 1, the function $\left(\frac{I_t(x,\xi)}{(\lambda_i-t)^{k_i-1}}\right)$ is polynomial in t of degree $n-k_i$. Since I_t is an integral, for every fixed t, the function $\left(\frac{I_t(x,\xi)}{(\lambda_i-t)^{k_i-1}}\right)$ is an integral for the geodesic flow of g. Thus, $\tilde I$ is an integral.

At the tangent space to every point of geodesic γ, consider a coordinate system such that $L = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $g = \mathrm{diag}(1, \dots, 1)$. In these coordinates, $I_t(\xi)$ is given by (15). Then, the integral $\tilde I(\xi)$ is the sum (we assume $\xi = (\xi_1, \dots, \xi_n)$)

$$\left(\sum_{\alpha=i}^{i+k_i-1}\left(\xi_\alpha^2 \prod_{\substack{\beta=1\\ \beta\ne i,i+1,\dots,i+k_i-1}}^{n} (\lambda_\beta - \lambda_i)\right)\right) \tag{21}$$

$$+\left(\sum_{\substack{\alpha=1\\ \alpha\ne i,i+1,\dots,i+k_i-1}}^{n}\left(\xi_\alpha^2 \prod_{\substack{\beta=1\\ \beta\ne i+1,\dots,i+k_i-1\\ \beta\ne\alpha}}^{n} (\lambda_\beta - \lambda_i)\right)\right). \tag{22}$$

Since the geodesic passes through the point where $\lambda_{i-1} = \lambda_i = \cdots = \lambda_{i+k_i-1}$, all products in the formulas above contain the factor $\lambda_i - \lambda_i$, and, therefore, vanish, so that $\tilde I(\gamma(0), \dot\gamma(0)) = 0$. Since $\tilde I$ is an integral, $\tilde I(\gamma(\tau), \dot\gamma(\tau)) = 0$ for every τ. Let us show that it implies that the geodesic is orthogonal to M_i at every typical point; in particular, at points lying on the segment $\gamma(\tau),\ \tau \in [0, 1[$.

Clearly, every term in the sum (22) contains the factor $\lambda_i - \lambda_i$, and, therefore, vanishes. Then, the integral $\tilde I$ is equal to (21).

At a typical point, we have

$$\lambda_1 \le \cdots \le \lambda_{i-1} < \lambda_i = \cdots = \lambda_{i+k_i-1} < \lambda_{k_i} \le \cdots \le \lambda_n.$$

Then, the coefficient

$$\prod_{\substack{\beta=1\\ \beta\ne i,i+1,\dots,i+k_i-1}}^{n} (\lambda_\beta - \lambda_i)$$

is nonzero. Then, all components ξ_α, $\alpha \in i, \dots, i+k_i-1$ vanish. Thus, γ is orthogonal to M_i at every typical point.

Finally, by Corollary 7, the restriction of g to $M_i(p)$ is proportional to the restriction of g to $M_i(q)$ and, hence, has constant sectional curvature. The theorem is proven.

3.5. If g is $V(K)$-metric, if $\phi_\aleph$ is not isolated, and if the sectional curvature of $A_\aleph$ is constant, then it is equal to $K_\aleph$. Within this section we assume that $L \neq \text{const} \cdot \text{Id}$ is a BM-structure on a connected Riemannian manifold (M^n, g) of dimension $n \geq 3$. We denote by m the number of different eigenvalues of L at a typical point. For every typical point $x \in M^n$, consider the Levi-Civita coordinates $(\bar{x}_1, \dots, \bar{x}_m)$ such that the metric has the form (10). We assume that there exists i such that $k_i = 1$. Recall that the functions ϕ_i are the eigenvalues of L: in the Levi-Civita coordinates,

$$L = \text{diag}(\lambda_1, \dots, \lambda_n) = \text{diag}(\underbrace{\phi_1, \dots, \phi_1}_{k_1}, \dots, \underbrace{\phi_m, \dots, \phi_m}_{k_m}).$$

Consider $\aleph \in \{1, \dots, m\}$ such that $k_\aleph \geq 2$. We put $r := k_1 + \dots + k_{\aleph-1}$. Let us assume that the eigenvalue $\phi_\aleph = \lambda_{r+1}$ is not isolated; that means that there exists a point $p_1 \in M^n$ such that $\lambda_r(p_1) = \lambda_{r+1}$, or $\lambda_{r+k_\aleph+1}(p_1) = \lambda_{r+1}$.

Let us assume in addition that in a neighborhood of every typical point, the following holds:

1) The sectional curvature of $A_\aleph$ is constant,
2) g is a $V(K)$ metric.

As we saw in Section 3.4, the assumption 1 follows from the previous assumptions, if the metric is complete. As we saw in Section 2.3, the assumption 2 is automatically fulfilled, if the space of all BM-structures is more than two-dimensional.

Theorem 14. *Under the above assumptions, the sectional curvature of $A_\aleph$ is equal to $K_\aleph$.*

Recall that the definition of $K_\aleph$ is in the second statement of Theorem 8.

Proof of Theorem 14. Let us denote by $\bar{K}_\aleph$ the sectional curvature of the metric $A_\aleph$. By assumption, it is constant in a neighborhood of every typical point. Since the set of typical points is connected by Corollary 11, $\bar{K}_\aleph$ is independent of the typical point. Similarly, since $K_\aleph$ is locally-constant by Theorem 8, $K_\aleph$ is independent of the typical point. Thus, it is sufficient to find a point where $\bar{K}_\aleph = K_\aleph$.

Without loss of generality, we can suppose that there exists $p_1 \in M^n$ such that $\lambda_r(p_1) = \lambda_{r+1}$.

By Corollaries 10 and 11, without loss of generality we can assume that the multiplicity of λ_{r+1} is $k_\aleph + 1$ in p_1, and that $N_L(p_1) = m - 1$. Take a typical point p in a small neighborhood of p_1.

Then, by Corollary 8, the submanifold $M_{r+1}(p)$ is homeomorphic to the sphere. Since it is compact, there exists a set of local coordinate charts on it such that there exist constants const > 0 and CONST such that, in every chart $(x_{\aleph}^1, \dots, x_{\aleph}^{k_{\aleph}})$, for every $\alpha, \beta \in \{1, \dots, k_{\aleph}\}$, the entry $(A_{\aleph})_{\beta\beta}$ is greater than const and the absolute value of the entry $(A_{\aleph})_{\alpha\beta}$ is less than CONST, i.e., $A_{\aleph}(\frac{\partial}{\partial x_{\aleph}^{\beta}}, \frac{\partial}{\partial x_{\aleph}^{\beta}}) > \text{const}$; $|A_{\aleph}(\frac{\partial}{\partial x_{\aleph}^{\alpha}}, \frac{\partial}{\partial x_{\aleph}^{\beta}})| < \text{CONST}$.

By shifting these local coordinates along the vector fields $\frac{\partial}{\partial x_i^j}$, where $i \neq \aleph$, for every typical point p' in a neighborhood of p_1, we obtain coordinate charts on $M_{r+1}(p')$ such that for every α, β, $(A_{\aleph})_{\beta\beta} > \text{const}$, $|(A_{\aleph})_{\alpha\beta}| < \text{CONST}$.

Let us calculate the projective Weyl tensor W for g in these local coordinate charts. We will be interested in the components (actually, in one component) of W corresponding to the coordinates $\bar{x}_{\aleph}$. In what follows we reserve the Greek letter α, β for the coordinates from $\bar{x}_{\aleph}$, so that, for example, $g_{\alpha\beta}$ will mean the component of the metric staying on the intersection of column number $r+\beta$ and row number $r+\alpha$.

As we will see below, the formulas will include only the components of $A_{\aleph}$. To simplify the notations, we will not write subindex $\aleph$ near $A_{\aleph}$, so for example, $g_{\alpha\beta}$ is equal to $P_{\aleph} A_{\alpha\beta}$.

Take $\alpha \neq \beta$. Let us calculate the component $W^{\alpha}_{\beta\beta\alpha}$. In order to do it by formula (8), we need to know $R^{\alpha}_{\beta\beta\alpha}$ and $R_{\beta\beta}$. It is not easy to calculate them: a straightforward way is to calculate $R^{\alpha}_{\beta\beta\alpha}$ and $R_{\beta\beta}$ for the metric (10), then combine the result with assumption 2 (which could be written as a system of partial differential equations) and with assumption 1 (which is a system of algebraic equations). This was done in §8 of [**71**], see formula (8.14) and what goes after it there. Rewriting the results of Solodovnikov in our notations, we obtain

$$\begin{aligned} R^{\alpha}_{\beta\beta\alpha} &= \left(\bar{K}_{\aleph} - (K_{\aleph} - K\, P_{\aleph})\right) A_{\beta\beta}, \\ R_{\beta\beta} &= \left((k_{\aleph}-1)\,\bar{K}_{\aleph} + K\,(n-1)\,P_{\aleph} - (k_{\aleph}-1)\,K_{\aleph}\right)\, A_{\beta\beta}. \end{aligned}$$

Substituting these expressions in the formula for projective Weyl tensor (8), we obtain

$$W^{\alpha}_{\beta\beta\alpha} = (\bar{K}_{\aleph} - K_{\aleph})\frac{n-k_{\aleph}}{n-1} A_{\beta\beta}.$$

We see that, if $\bar{K}_{\aleph} \neq K_{\aleph}$, the component $W^{\alpha}_{\beta\beta\alpha}$ is bounded from zero.

But if we consider a sequence of typical points converging to p_1, the component $W^{\alpha}_{\beta\beta\alpha}$ converges to zero. Indeed, by definition $W^{\alpha}_{\beta\beta\alpha} = W(dx_{\aleph}^{\alpha}, \frac{\partial}{\partial x_{\aleph}^{\beta}}, \frac{\partial}{\partial x_{\aleph}^{\beta}}, \frac{\partial}{\partial x_{\aleph}^{\alpha}})$, and the length of $\frac{\partial}{\partial x_{\aleph}^{\beta}}$ goes to zero, if the point goes to p_1. Finally, $\bar{K}_{\aleph} = K_{\aleph}$. The theorem is proven.

4. Proof of Theorem 1

If the dimension of $\mathcal{B}(M^n, g)$ is one, Theorem 1 is trivial: every projective transformation is a homothety. In Section 4.1, we prove Theorem 1 under the additional assumption that the dimension of $\mathcal{B}(M^n, g)$ is two (Theorem 15). In Section 4.2, we prove Theorem 1 under the additional assumption that the dimension of M^n and the dimension of $\mathcal{B}(M^n, g)$ is at least three (Theorem 16). The last case, namely the existence of non-affine projective transformations on complete two-dimenstion manifolds such that the dimension of $\mathcal{B}(M^n, g)$ is at least three, was treated in [**58**]. It was proved that they are possible if and only if the metric has constant positive sectional curvature.

4.1. If the space $\mathcal{B}(M^n, g)$ has dimension two. Suppose g and $\bar g$ are projectively equivalent. The next lemma shows that the spaces $\mathcal{B}(M^n, g)$ and $\mathcal{B}(M^n, \bar g)$ are canonically isomorphic:

Lemma 3. *Let L be the BM-structure* (3) *constructed for the projectively equivalent metrics g and $\bar g$. Suppose L_1 is one more BM-structure for g. Then, $L^{-1}L_1$ is a BM-structure for $\bar g$.*

Corollary 12. *If $\mathcal{B}(M^n,\ g)$ is two-dimensional, every projective transformation takes typical points to typical points.*

Proof of Lemma 3. It is sufficient to prove the statement locally. Let us fix a coordinate system and think about tensors as about matrices. For every sufficiently big constant α, the tensor $L + \alpha \cdot \mathrm{Id}$ is positive defined. Then, by Theorem 2,

$$\bar{\bar g} := \frac{1}{\det(L_1 + \alpha \cdot \mathrm{Id})} \cdot g \cdot (L_1 + \alpha \cdot \mathrm{Id})^{-1}$$

is a Riemannian metric projectively equivalent to g. Then, it is projectively equivalent to $\bar g$. Direct calculation of the tensor (3) for the metrics $\bar g$, $\bar{\bar g}$ gives us that $L^{-1}(L_1 + \alpha \cdot \mathrm{Id})$ is a BM-structure for $\bar g$. Since it is true for all big α, and since $\mathcal{B}(M^n, \bar g)$ is a linear space, $L^{-1}L_1$ is a BM-structure for $\bar g$. Lemma 3 is proven.

Definition 9. A vector field v is called **projective** if its flow takes geodesics to geodesics.

A smooth one-parameter family of projective transformations $F_t : M^n \to M^n$ immediately gives us a projective vector field $\left(\frac{d}{dt}F_t\right)_{|t=0}$. A projective vector field gives us a one parameter family of projective transformations if and only if it is complete.

Theorem 15. *Suppose (M^n, g) of dimension $n \geq 2$ is complete and connected. Let v be a complete projective vector field. Assume in addition that the dimension of $\mathcal{B}(M^n, g)$ is precisely two. Then, the flow of the vector field acts by affine transformations, or the metric has constant positive sectional curvature.*

Proof. Denote by F_t the flow of the vector field v. If it contains not only affine transformations, there exists $t_0 \in \mathbb{R}$ such that the pull-back $\bar{g} := F_{t_0}^*(g)$ is projectively equivalent to g, and is not affine equivalent to g.

Consider the BM-structure $L \in \mathcal{B}(M^n, g)$ given by (3). Take a typical point p such that v does not vanish at p . Since $\bar{g}$ is not affine equivalent to g, without loss of generality we can assume that at least one eigenvalue is not constant near p. By Levi-Civita's Theorem 7, there exists a coordinate system $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ in a neighborhood of p such that the tensor L and the metrics $g, \bar{g}$ have the form (9, 10, 11), respectively. In particular, all these objects are block-diagonal with the parameters of the blocks $(k_1, \dots, k_m)$. Note that the nonconstant eigenvalue of L has multiplicity one.

Let the vector v be $(\bar{v}_1, \dots, \bar{v}_m)$ in this coordinate system. Then, by Theorem 3 and Lemma 3, and using that the space $\mathcal{B}(M^n, g)$ is two-dimensional, we obtain that the Lie derivatives $\mathcal{L}_v g$ and $\mathcal{L}_v \bar{g}$ are block-diagonal as well. In the coordinate system $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$, the metric $\bar{g}$ is given by the matrix $\frac{1}{\det(L)} g L^{-1}$. Then, the Lie derivative $\mathcal{L}_v \bar{g}$ is

$$\left(\mathcal{L}_v \frac{1}{\det(L)}\right) g L^{-1} + \frac{1}{\det(L)} \left(\mathcal{L}_v g\right) L^{-1} + \frac{1}{\det(L)} g \left(\mathcal{L}_v L^{-1}\right).$$

We see that the first two terms of the sum are block-diagonal; then, the third term must be block-diagonal as well so that $\mathcal{L}_v L$ is block-diagonal. Let us calculate the element of $\mathcal{L}_v L$ which is on the intersection of x_i^α-row and x_j^β-column. If $i \neq j$, it is equal to

$$\pm(\phi_i - \phi_j) \frac{\partial v_i^\alpha}{\partial x_j^\beta}.$$

Thus,

$$\frac{\partial v_i^\alpha}{\partial x_j^\beta} \equiv 0.$$

Finally, the block $\bar{v}_i$ of the projective vector field depends on the variables $\bar{x}_i$ only.

Using this, let us calculate the tensor $g^{-1}\mathcal{L}_v g$ from Theorem 3 for the metric g. We see that (in matrix notations), the matrix of g is AP, where

$$P := \operatorname{diag}(\underbrace{P_1, \dots, P_1}_{k_1}, \dots, \underbrace{P_m, \dots, P_m}_{k_m})$$
$$\text{and} \quad A := \text{block-diagonal}(A_1, \dots, A_m).$$

We understand A as a (0,2)-tensor and P as a (1,1)-tensor. Then,

$$g^{-1}\mathcal{L}_v g = P^{-1}A^{-1}(\mathcal{L}_v A)P + P^{-1}\mathcal{L}_v P.$$

Since the entries of A_i and of $\bar{v}_i$ depend on the coordinates $\bar{x}_i$ only, $\mathcal{L}_v A$ is block-diagonal so that the first term on the right hand side is equal to $A^{-1}\mathcal{L}_v A$. Thus,

$$g^{-1}\mathcal{L}_v g = A^{-1}\mathcal{L}_v A + P^{-1}\mathcal{L}_v P. \tag{23}$$

By direct calculations, we obtain:

$$P^{-1}\mathcal{L}_v P$$
$$= \operatorname{diag}\left(\underbrace{\sum_{i=2,\dots,m}\frac{\phi_1' v_1^1}{\phi_i-\phi_1}+\frac{\phi_i' v_i^1}{\phi_1-\phi_i},\dots,\sum_{i=2,\dots,m}\frac{\phi_1' v_1^1}{\phi_i-\phi_1}+\frac{\phi_i' v_i^1}{\phi_1-\phi_i}}_{k_1},\right.$$
$$\left.\dots,\underbrace{\sum_{i=1,\dots,m-1}\frac{\phi_m' v_m^1}{\phi_i-\phi_m}+\frac{\phi_i' v_i^1}{\phi_m-\phi_i},\dots,\sum_{i=1,\dots,m-1}\frac{\phi_m' v_m^1}{\phi_i-\phi_m}+\frac{\phi_i' v_i^1}{\phi_m-\phi_i}}_{k_m}\right).$$

By Theorem 3, since the space $\mathcal{B}(M^n, g)$ is two-dimensional, $g^{-1}\mathcal{L}_v g$ equals

$$\alpha L + \beta\cdot\mathrm{Id} + \mathrm{trace}_{\alpha L+\beta\cdot\mathrm{Id}}\cdot\mathrm{Id},$$

where $\alpha, \beta \in \mathbb{R}$. Then, the first entry of the block number j gives us the following equation:

$$a_j(\bar{x}_j) + \sum_{\substack{i=1,\dots,m\\ i\neq j}}\frac{\phi_j' v_j^1}{\phi_i-\phi_j}+\frac{\phi_i' v_i^1}{\phi_j-\phi_i} = \alpha\phi_j + \beta + \mathrm{trace}_{\alpha L+\beta\cdot\mathrm{Id}}, \tag{24}$$

where $a_j(\bar{x}_j)$ collects the terms coming from $A^{-1}\mathcal{L}_v A$, and, hence, depends on the variables $\bar{x}_j$ only.

Our next goal is to show that

- Only one ϕ_j can be nonconstant. The behavior of nonconstant ϕ_j on the orbit passing through p is given by

  $$\phi_j(F_t(p)) := \frac{b}{2\alpha} + \frac{\sqrt{D}}{\alpha}\tanh(\sqrt{D}(t+d)),$$

  where $D := b^2/4 + \alpha c$, where b, c and d are certain (universal along the orbit) constants.
- The constant eigenvalues ϕ_s are roots of the polynomial

  $$-\alpha\phi^2 + b\phi + c.$$

  In particular, m is at most 3.

Take $s \neq j \in 1, \ldots, m$. We see that the terms in (24) depending on the variables $\bar{x}_s$ are

$$\Phi_j := \frac{\phi_j' v_j^1}{\phi_s - \phi_j} + \frac{\phi_s' v_s^1}{\phi_j - \phi_s} - k_s \alpha \phi_s.$$

Thus, Φ_j depends on the variables $\bar{x}_j$ only. Similarly,

$$\Phi_s := \frac{\phi_s' v_s^1}{\phi_j - \phi_s} + \frac{\phi_j' v_j^1}{\phi_s - \phi_j} - k_j \alpha \phi_j$$

depends on the variables $\bar{x}_s$ only. Using that $\Phi_j - \Phi_s$ is equal to $\alpha(k_j\phi_j - k_s\phi_s)$, we obtain that (for an appropriate constant B) Φ_j must be equal to $\alpha k_j \phi_j + B$. Thus,

$$\frac{\phi_j' v_j^1}{\phi_s - \phi_j} + \frac{\phi_s' v_s^1}{\phi_j - \phi_s} - k_s \alpha \phi_s = \alpha k_j \phi_j + B. \tag{25}$$

Now let us prove that at least one of the functions ϕ_s and ϕ_j is constant near p. Otherwise, $k_s = k_j = 1$, and Equation (25) is equivalent to

$$\phi_j' v_j^1 + \alpha \phi_j^2 - B\phi_j = \phi_s' v_s^1 + \alpha \phi_s^2 - B\phi_s.$$

We see that the terms on the left-hand side depend on the variable x_j^1 only, and the terms on the right-hand side depend on the variable x_s^1 only. Then, there exists a constant c such that

$$\begin{aligned} \phi_j' v_j^1 &= -\alpha \phi_j^2 + b\phi_j + c \\ \phi_s' v_s^1 &= -\alpha \phi_s^2 + b\phi_s + c, \end{aligned}$$

where $b := B$. Since ϕ_j and v_j^1 depend on the variable x_j^1 only, $\phi_j' v_j^1$ at the point $F_t(p)$ is equal to $\dot{\phi}_j := \frac{d}{dt}\phi_j(F_t(p))$, so that $\phi_s(F_t(p))$ and $\phi_j(F_t(p))$ are solutions of the following differential equation:

$$\dot{\phi} = -\alpha\phi^2 + b\phi + c. \tag{26}$$

By Corollary 12, all points $F_t(p)$ are typical. Then, ϕ_s and ϕ_j are solutions of Equation (26) at every point of the orbit passing through p, and the constants α, b, c are universal along the orbit. The equation can be solved (we assume $\alpha \neq 0$). The constant solutions are $\frac{b}{2\alpha} \pm \frac{\sqrt{b^2/4+\alpha c}}{\alpha}$, and the nonconstant solutions are:

1) For $D := b^2/4 + \alpha c < 0$, every nonconstant solution is the function $\frac{b}{2\alpha} + \frac{\sqrt{-D}}{\alpha}\tan(\sqrt{-D}(-t + d_1))$.
2) For $D := b^2/4 + \alpha c > 0$, every nonconstant solution is one of the functions
   a) $\frac{b}{2\alpha} + \frac{\sqrt{D}}{\alpha}\tanh(\sqrt{D}(t + d))$,
   b) $\frac{b}{2\alpha} + \frac{\sqrt{D}}{\alpha}\coth(\sqrt{D}(t + d_3))$.
3) For $D := b^2/4 + \alpha c = 0$, every nonconstant solution is the function $\frac{b}{2\alpha} + \frac{1}{\alpha(t+d_4)}$.

The solutions (1, 2b, 3) explode in finite time. This gives us a contradiction: the metrics g and $\bar{g}$ are smooth and, hence, the eigenvalues of L are finite on every compact set.

If the functions $\phi_s(F_t(p))$ and $\phi_j(F_t(p))$ have the form (2a), then there exist points p_1, p_2, q_1, q_2 of the orbit passing through p, such that $\phi_s(p_1) < \phi_j(p_2)$, $\phi_s(q_1) > \phi_j(q_2)$. This gives a contradiction with Corollary 3.

Thus, only one eigenvalue of L can be nonconstant in a neighborhood of p. Let ϕ_j be nonconstant near p.

Now let us show that a constant eigenvalue is a root of the polynomial $-\alpha\phi^2 + b\phi + c$; in particular, there are no more than two different constant eigenvalues.

Suppose ϕ_s is constant. Then, the derivative ϕ_s' vanishes and (25) reads:

$$\phi_j' v_j^1 = -\alpha\phi_j^2 + (\alpha\phi_s - \alpha k_s\phi_s - B)\phi_j + \alpha\phi_s^2 k_s + \phi_s B.$$

Denoting $\alpha\phi_s - \alpha k_s\phi_s - B$ by b and $\alpha\phi_s^2 k_s$ by c, and arguing as above, we see that ϕ_j is a solution of (26). Hence, the behavior of ϕ_j on the orbit (passing through p) is given by (2a). Clearly, ϕ_s is a root of $-\alpha\phi^2 + b\phi + c$.

Thus, near p only the following three cases are possible:

Case 1: $m = 3$. The eigenvalues ϕ_1, ϕ_3 are constant; the eigenvalue ϕ_2 is not constant.

Case 2a: $m = 2$. The eigenvalue ϕ_1 is constant; the eigenvalue ϕ_2 is not constant.

Case 2b: $m = 2$. The eigenvalue ϕ_2 is constant; the eigenvalue ϕ_1 is not constant.

In all three cases, one can prove that the metric has constant sectional curvature. We will carefully consider the most complicated case, case 1, and sketch the proof for case 2a. The proof for case 2b is similar to the proof for case 2a.

Suppose $m = 3$, the eigenvalues ϕ_1, ϕ_3 are constant and the eigenvalue ϕ_2 is not constant in a neighborhood of p. Without loss of generality, we can assume $A_2(dx_2^1, dx_2^1) = (dx_2^1)^2$. Then, $\alpha L + \beta \cdot \mathrm{Id}$ is

$$\mathrm{diag}\left(\underbrace{\alpha\phi_1 + \beta, \dots, \alpha\phi_1 + \beta}_{k_1}, \alpha\phi_2 + \beta, \underbrace{\alpha\phi_3 + \beta, \dots, \alpha\phi_3 + \beta}_{k_3}\right).$$

Since $g^{-1}\mathcal{L}_v g$ is equal to $\alpha L + \beta \cdot \mathrm{Id} + \mathrm{trace}_{\alpha L+\beta\cdot\mathrm{Id}} \cdot \mathrm{Id}$, Equation (23) gives us the following system:
(27)

$$\begin{cases}(\alpha\phi_1+\beta)(k_1+1)+\alpha\phi_2+\beta+(\alpha\phi_3+\beta)k_3 = a_1(\bar{x}_1)+\frac{\phi_2' v_2^1}{\phi_1-\phi_2}\\ (\alpha\phi_1+\beta)k_1+2(\alpha\phi_2+\beta)+(\alpha\phi_3+\beta)k_3 = -2\frac{\partial v_2^1}{\partial x_2^1}+\frac{\phi_2' v_2^1}{\phi_1-\phi_2}+\frac{\phi_2' v_2^1}{\phi_3-\phi_2}\\ (\alpha\phi_1+\beta)k_1+\alpha\phi_2+\beta+(\alpha\phi_3+\beta)(k_3+1) = a_3(\bar{x}_3)+\frac{\phi_2' v_2^1}{\phi_3-\phi_2}.\end{cases}$$

Here $a_1,\ a_3$ collect the terms coming from $A_1^{-1}\mathcal{L}_v A_1$ and $A_3^{-1}\mathcal{L}_v A_3$, respectively. Using $\phi_2' v_2^1 = \dot{\phi}_2 = -\alpha(\phi_1-\phi_2)(\phi_3-\phi_2)$, we obtain

$$\begin{cases}(\alpha\phi_1+\beta)(k_1+1)+(\alpha\phi_3+\beta)(k_3+1) & = & a_1(\bar{x}_1)\\ -\frac{a_1(\bar{x}_1)}{2} & = & \frac{\partial v_2^1}{\partial x_2^1}\\ (\alpha\phi_1+\beta)(k_1+1)+(\alpha\phi_3+\beta)(k_3+1) & = & a_3(\bar{x}_3).\end{cases} \tag{28}$$

We see that $a_1 = a_3 = \mathit{const}$; we denote this constant by a. Let us prove that $a = 0$. We assume that $a \neq 0$ and will find a contradiction.

Consider Equation (23). We see that the first block of the left-hand side and the first block of the second term at the right-hand side are proportional to $\mathrm{diag}(\underbrace{1,\dots,1}_{k_1})$. Then, $A_1^{-1}\mathcal{L}_v A_1$ is proportional to identity. The coefficient of proportionality is clearly a. Then, $A_1^{-1}\mathcal{L}_v A_1 = a\mathrm{diag}(\underbrace{1,\dots,1}_{k_1})$, and $\mathcal{L}_v A_1 = aA_1$, so that the flow of the vector field $(\bar{v}_1, 0, \underbrace{0,\dots,0}_{k_3})$ acts by homotheties on the restriction of g to the coordinate plaque of the coordinates $\bar{x}_1$. Note that this vector field is the orthogonal projection of v to the coordinate plaque.

Similarly, the vector field $(\underbrace{0,\dots,0}_{k_1}, 0, \bar{v}_3)$ acts by homotheties (with the same coefficient of stretching) on the restriction of g to the coordinate plaque of the coordinates $\bar{x}_3$.

Without loss of generality, we can suppose that $\phi_2(p)$ is a regular value of ϕ_2. Denote by $\check{M}_2$ the connected component of the set $\{q \in M^n :\ \phi_2(q) = \phi_2(p)\}$ containing p. By construction, $\check{M}_2$ is a submanifold of codimension 1, and the derivative of ϕ_2 vanishes at no points of $\check{M}_2$. Then, at every point of $\check{M}_2$, the flow of the orthogonal projection of v to $\check{M}_2$ acts by homotheties. Since M^n is complete, $\check{M}_2$ is complete as well. Then, $\check{M}_2$ with the restriction of the metric g is isometric to the standard Euclidean space $(\mathbb{R}^{n-1}, g_{euclidean})$, and there exists precisely one point where the orthogonal projection of v vanishes, see [**40**] for details. Without loss of generality, we can think that p is the point where the orthogonal projection of v vanishes.

Moreover, at every point of $\check{M}_2$, consider the eigenspaces of L corresponding to ϕ_1 and ϕ_3. By Theorem 4, they are tangent to $\check{M}_2$; by Corollary 3, every point of $\check{M}_2$ is typical so that the eigenspaces corresponding to ϕ_1 and ϕ_3 give us two distributions. These distributions are integrable by Theorem 4. We denote by $M_1(p)$ and $M_3(p)$ their integral manifolds passing through p. Locally, in Levi-Civita's coordinates from Theorem 7, the manifold M_1 is the coordinate plaque of coordinates $\bar{x}_1$ and M_3 is the coordinate plaque of coordinates $\bar{x}_3$. Then, $M_1(p)$ and $M_3(p)$ are invariant with respect to the orthogonal projection of v to $\check{M}_2$.

Consider the orbit of the projective action of $(\mathbb{R}, +)$ containing the point p. Since at the point p the vector v has the form $(\underbrace{0, \dots, 0}_{k_1}, v_2^1, \underbrace{0, \dots, 0}_{k_3})$, and since the components $\bar{v}_1$, $\bar{v}_3$ do not depend on the coordinate $\bar{x}_2$, at every point of the orbit v is an eigenvector of L with the eigenvalue ϕ_2.

Let us show that the length of the orbit is finite at least in one direction. Indeed, the second equation in (28) implies

$$\frac{d}{dt} \log \left(v_2^1(F_t(p)) \right) = -\frac{a}{2}.$$

Its solution is $v_2^1(F_t(p)) = \text{Const} \exp\left(-\frac{a}{2}t\right)$. Then, the length between points $t_1 < t_2$ of the orbit is equal to

$$\int_{t_1}^{t_2} \sqrt{g(v,v)}\, dt = \text{Const} \int_{t_1}^{t_2} \sqrt{(\phi_2 - \phi_1)(\phi_3 - \phi_2)} \exp\left(-\frac{a}{2}t\right) dt.$$

Since $(\phi_2 - \phi_1)(\phi_3 - \phi_2)$ is bounded, the length of the orbit is finite at least in one direction.

Then, since the manifold is complete, the closure of the orbit contains a point $q \in M^n$ such that either $\phi_2(q) = \phi_1$ or $\phi_2(q) = \phi_3$. Without loss of generality, we can assume that $\phi_2(q) = \phi_1$.

Then, without loss of generality, we can assume that p is close enough to q, so that, by Corollary 8, $M_1(p)$ is homeomorphic to the sphere. We got a contradiction with the fact that $M_1(p)$ admits a vector field whose flow acts by homotheties, see [**40**]. Finally, $a = 0$.

Since $a = 0$, the second equation in (28) implies

$$\frac{d}{dt} \log \left(v_2^1(t) \right) = 0,$$

so that locally

$$x_2^1 = \text{Const}\, t. \tag{29}$$

Then, the adjusted metric has the form

$$(1-\tanh(y_2+d))\,(dy_1)^2+C(1-\tanh(y_2+d))(1+\tanh(y_2+d))(dy_2)^2 \\ +(1+\tanh(y_2+d))(dy_3)^2$$

in a certain coordinate system and for certain constants $C,\ d$. By direct calculation, we see that the sectional curvature of the adjusted metric is positive constant. If the dimension of the manifold is three, it implies that the sectional curvature of g is constant.

If the dimension of the manifold is greater than three, in view of Theorems 13,14,8, it is sufficient to show that there exist points $q_1,\ q_3$ such that $\phi_2(q_1)=\phi_1$ and $\phi_2(q_3)=\phi_3$. Take the geodesic γ such that $\gamma(0)=p$ and $\dot\gamma(0)=(\underbrace{0,\dots,0}_{k_1},v_2^1,\underbrace{0,\dots,0}_{k_3})$.

Let us show that at every typical point of the geodesic, in Levi-Civita's coordinates, the $\bar x_1$- and $\bar x_3$-components of the geodesic vanish.

Consider the functions

$$I' := \left(\frac{I_t}{(\phi_1-t)^{k_1-1}}\right)_{|t=\phi_1} : TM^n\to\mathbb{R},$$

$$I'' := \left(\frac{I_t}{(\phi_3-t)^{k_3-1}}\right)_{|t=\phi_3} : TM^n\to\mathbb{R}.$$

They are integrals of the geodesic flow. Since

$$I'(\gamma(0),\dot\gamma(0))=I''(\gamma(0),\dot\gamma(0))=0,$$

at every point τ of the geodesic we have $I'(\gamma(\tau),\dot\gamma(\tau))=I''(\gamma(\tau),\dot\gamma(\tau))$ $=0$. Then, at every typical point of the geodesic, the $\bar x_1$- and $\bar x_3$- components of the velocity vector vanish. Consider the maximal (open) segment of this geodesic containing p and containing only typical points. Let us show that this segment has finite length; that $\phi_1=\phi_2$ at one end of the segment and $\phi_3=\phi_2$ at the other end of the segment.

Using (29), we obtain that $v_2^1=\mathrm{Const}$ near every point of the segment. Then, we can globally parameterize the coordinate x_2^1 near the points of the geodesic segment such that the constant Const is universal along the segment. Then, the length of the segment is given by (we denote by v_2 the projection of v to the segment)

$$\int_{-\infty}^{+\infty}\sqrt{g(v_2,v_2)}\;dt=\mathrm{Const}\int_{-\infty}^{+\infty}\sqrt{(\phi_2-\phi_1)(\phi_3-\phi_2)}\;t\;dt.$$

Since $(\phi_2-\phi_1)(\phi_3-\phi_2)$ decreases exponentially for $t\longrightarrow\pm\infty$, the length of the segment is finite. Clearly, the limit of ϕ_2 is ϕ_1 in one direction and ϕ_3 in the other direction. Finally, there exists a point where $\phi_1=\phi_2$ and a point where $\phi_2=\phi_3$.

Then, all eigenvalues of L are not isolated. Then, by Theorems 13, 14, every A_i has constant sectional curvature K_i, and, by Theorem 8, g has constant sectional curvature. By Corollary 6, the sectional curvature of g is positive. Theorem 15 is proven under the assumptions of case 1.

The proof for cases 2a and 2b is similar; we will sketch the proof for case 2a: First of all, under the assumptions of case 2a, $\alpha L + \beta \cdot \mathrm{Id}$ is

$$\mathrm{diag}\left(\underbrace{\alpha\phi_1 + \beta, \dots, \alpha\phi_1 + \beta}_{k_1}, \alpha\phi_2 + \beta\right).$$

Since $g^{-1}\mathcal{L}_v g$ is equal to $\alpha L + \beta \cdot \mathrm{Id} + \mathrm{trace}_{\alpha L + \beta \cdot \mathrm{Id}}$, Equation (23) gives us the following system:

$$\left\{\begin{array}{rcl} (\alpha\phi_1 + \beta)(k_1 + 1) + \alpha\phi_2 + \beta & = & a_1(\bar{x}_1) + \frac{\phi_2' v_2^1}{\phi_1 - \phi_2} \\ (\alpha\phi_1 + \beta)k_1 + 2(\alpha\phi_2 + \beta) & = & -2\frac{\partial v_2^1}{\partial x_2^1} + \frac{\phi_2' v_2^1}{\phi_1 - \phi_2}. \end{array}\right. \tag{30}$$

As we have proven, ϕ_1 is a root of $-\alpha\phi^2 + b\phi + c$. We denote by ϕ_3 the second root of $-\alpha\phi^2 + b\phi + c$. Arguing as in case 1, we have

$$\left\{\begin{array}{rcl} (\alpha\phi_1 + \beta)(k_1 + 1) + (\alpha\phi_3 + \beta) & = & a_1(\bar{x}_1) \\ \alpha(\phi_2 - \phi_1) + a_1(\bar{x}_1) & = & -2\frac{\partial v_2^1}{\partial x_2^1}. \end{array}\right. \tag{31}$$

Then, $a_1 = Const$. Arguing as in case 1, one can prove that $a_1 = 0$. Then, the second equation of (31) implies

$$\frac{d}{dt}\log\left(v_2^1\right) = \alpha(1 + \tanh(\sqrt{b^2/4 + \alpha c}\ t + d)).$$

Thus, $v_2^1(t) = \frac{\exp(\sqrt{b^2/4+\alpha c}\ t/2 + d/2)}{\sqrt{\cosh(\sqrt{b^2/4+\alpha c}\ t + d)}}$. Then, the adjusted metric is (locally) proportional to

$$(1 - \tanh(\sqrt{b^2/4 + \alpha c}\ y_2 + d))\left(dy_1^2 + \frac{\exp(\sqrt{b^2/4 + \alpha c}\ y_2 + d)}{\cosh(\sqrt{b^2/4 + \alpha c}\ y_2 + d)}\ dy_2^2\right)$$

and, therefore, has constant curvature. If $n = 2$, it implies that g has constant curvature. If $n \geq 3$, similar to the proof for case 1, we can show that there exists a point where $\phi_1 = \phi_2$. Then, by Theorems 13, 14, 8, the sectional curvature of g is constant. By Corollary 6, it is positive. Theorem 15 is proven.

4.2. Proof if $\dim(M^n) \geq 3$; $\dim(\mathcal{B}(M^n, g)) \geq 3$. Assume $\dim(\mathcal{B}(M^n, g)) \geq 3$, where (M^n, g) is a connected complete Riemannian manifold of dimension $n \geq 3$. Instead of proving Theorem 1 under these assumptions, we will prove the following stronger

Theorem 16. *Let (M^n, g) be a connected complete Riemannian manifold of dimension $n \geq 3$. Suppose* $\dim(\mathcal{B}(M^n, g)) \geq 3$.

Then, if a complete Riemannian metric $\bar g$ is projectively equivalent to g, then g has positive constant sectional curvature or $\bar g$ is affine equivalent to g.

Proof. Denote by L the BM-structure from Theorem 2. In view of Remark 2, without loss of generality, we can assume that at least one eigenvalue of L is not constant.

Denote by m the number of different eigenvalues of L in a typical point. The number m does not depend on the typical point. If $m = n$, Theorem 16 follows from Fubini's Theorem 10 and Corollary 6.

Thus, we can assume $m < n$. Denote by m_0 the number of simple eigenvalues of L at a typical point. By Corollary 3, the number m_0 does not depend on the typical point. Then, by Levi-Civita's Theorem 7, the metric g has the following warped decomposition near every typical point p:

$$g = g_0 + \left|\prod_{\substack{i=1\\ i\neq m_0+1}}^{m} (\phi_{m_0+1} - \phi_i)\right| g_{m_0+1} + \cdots + \left|\prod_{\substack{i=1\\ i\neq m}}^{m} (\phi_m - \phi_i)\right| g_m. \tag{32}$$

Here the coordinates are $(\bar y_0, \ldots, \bar y_m)$, where $\bar y_0 = (y_0^1, \ldots, y_0^{m_0})$ and for $i > 1 \quad \bar y_i = (y_i^1, \ldots, y_i^{k_i})$. For $i \geq 0$, every metric g_{m_0+i} depends on the coordinates $\bar y_i$ only. Every function ϕ_i depends on y_0^i for $i \leq m_0$ and is constant for $i > m_0$.

Let us explain the relation between Theorem 7 and the formula above. The term g_0 collects all one-dimensional terms of (10). The coordinates $\bar y_0$ collect all one-dimensional $\bar x_i$ from (10). For $i > m_0$, the coordinate $\bar y_i$ is one of the coordinates $\bar x_j$ with $k_j > 1$. Every metric g_{m_0+i} for $i > 1$ came from one of the multidimensional terms of (10) and is proportional to the corresponding A_j. The functions ϕ_i are eigenvalues of L; they must not be ordered anymore: the indexing can be different from (9). Note that, by Corollary 3, this re-indexing can be done simultaneously in all typical points.

Since the dimension of the space $\mathcal{B}(M^n, g)$ is greater than two, by Theorem 9, g is a $V(K)$ metric near every typical point. By Corollary 11, the set of the typical points is connected, so that the constant K is independent of the typical point.

According to Definition 8, a multiple eigenvalue ϕ_i of L is **isolated** if there exists no nonconstant eigenvalue ϕ_j such that $\phi_j(q) = \phi_i$ at some point $q \in M^n$. If every multiple eigenvalue of L is non-isolated, then, by Theorems 13, 14, 8, g has constant sectional curvature. By Corollary 6, the sectional curvature is positive.

Thus, we can assume that there exist isolated eigenvalues. Without loss of generality, we can assume that (at every typical point) the re-indexing of ϕ_i is made in such a way that the first multiple eigenvalues $\phi_{m_0+1}, \dots, \phi_{m_1}$ are non-isolated and the last multiple eigenvalues $\phi_{m_1+1}, \dots, \phi_m$ are isolated. By assumption, $m_1 < m$.

We will prove that in this case all eigenvalues of L are constant. By Remark 2, it implies that the metrics g, $\bar{g}$ are affine equivalent.

Let us show that K is nonpositive. We suppose that it is positive and will find a contradiction.

At every point q of M^n, denote by $V_0 \subset T_qM^n$ the direct product of the eigenspaces of L corresponding to the eigenvalues $\phi_1, \dots, \phi_{m_1}$. Since the eigenvalues $\phi_{m_1+1}, \dots, \phi_m$ are isolated by the assumptions, the dimension of V_0 is constant, and V_0 is a distribution. By Theorem 4, V_0 is integrable. Take a typical point $p \in M^n$ and denote by M_0 the integral manifold of the distribution containing this point. Since M_0 is totally geodesic, the restriction $g_{|M_0}$ of the metric g to M_0 is complete. By Theorems 13,14,8, the metric $g_{|M_0}$ has constant sectional curvature K, or M_0 is one-dimensional.

Consider the direct product $M_0 \times \mathbb{R}^{m-m_1}$ with the metric

$$g_{|M_0} + \left|\prod_{\substack{i=1\\ i\neq m_1+1}}^{m} (\phi_{m_1+1} - \phi_i)\right| dt^2_{m_1+1} + \cdots + \left|\prod_{\substack{i=1\\ i\neq m}}^{m} (\phi_m - \phi_i)\right| dt^2_m, \tag{33}$$

where $(t_{m_1+1}, \dots, t_m)$ are the standard coordinates on $\mathbb{R}^{m-m_1}$. Since the eigenvalues $\phi_{m_1+1}, \dots, \phi_m$ are isolated, (33) is a well-defined Riemannian metric. Since $g_{|M_0}$ is complete, the metric (33) is complete. Since the sectional curvature of the adjusted metric is K, and since the sectional curvature of $g_{|M_0}$ is K (or M_0 has dimension 1), the sectional curvature of (33) is K as well. If $K > 0$, then the product $M_0 \times \mathbb{R}^{m-m_1}$ must be compact, which contradicts the fact that $\mathbb{R}^{m-m_1}$ is not compact. Finally, K is not positive.

Now let us prove that all eigenvalues of L are constant. Without loss of generality, we can assume that the manifold is simply connected. We will construct a totally geodesic (immersed) submanifold M_A, which is a global analog of the submanifold M_A from Section 2.3. At every point $x \in M^n$, consider $V_{m_1+1}, \dots, V_m \subset T_xM^n$, where V_{m_1+i} is the eigenspace of the eigenvalue ϕ_{m_1+i}. Since the eigenvalues ϕ_{m_1+i} are isolated, $V_{m_1+1}, \dots, V_m$ are distributions. By Theorem 4, they are integrable. Denote by $M_{m_1+1}, M_{m_1+2}, \dots, M_m$ the corresponding integral submanifolds.

Since M^n is simply connected, by [**8**], it is diffeomorphic to the product $M_0 \times M_{m_1+1} \times M_{m_1+2} \times \cdots \times M_m$. Clearly, the metric g on

$$M^n \simeq M_0 \times M_{m_1+1} \times M_{m_1+2} \times \cdots \times M_m$$

has the form

$$g_{|M_0} + \left| \prod_{\substack{i=1\\ i\neq m_1+1}}^{m} (\phi_{m_1+1} - \phi_i) \right| g_{m_1+1} + \cdots + \left| \prod_{\substack{i=1\\ i\neq m}}^{m} (\phi_m - \phi_i) \right| g_m, \tag{34}$$

where every g_k is a metric on M_k. Take a point

$$P = (p_0, p_{m_1+1}, p_{m_1+2}, \ldots, p_m) \in M_0 \times M_{m_1+1} \times M_{m_1+2} \times \cdots \times M_m.$$

On every M_{m_1+k}, $k = 1, \ldots, m - m_1$, pick a geodesic γ_{m_1+k} (in the metric g_{m_1+k}) passing through p_k. Denote by M_A the product

$$M_0 \times \gamma_{m_1+1} \times \cdots \times \gamma_m.$$

M_A is an immersed totally geodesic manifold. Hence, it is complete in the metric $g_{|M_A}$ and in the metric $\bar g_{|M_A}$. Locally, in a neighborhood of every point, M_A coincides with M_A from Section 2.3 constructed for the warped decomposition (34). The restriction of the metric g to M_A is isometric to (33) and, therefore, has nonpositive constant sectional curvature K. Then, by Corollary 6, the restriction of $\bar g$ to M_A is affine equivalent to the restriction of g to M_A. Then, by Remark 2, all ϕ_i are constant. Then, g is affine equivalent to $\bar g$. Theorem 16 is proven.

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